Towards Directed Homology

Ulrich Fahrenberg

Dept. of Mathematical Sciences
Aalborg University
9220 Aalborg East, Denmark

Email: uli@math.auc.dk

The Task

To invent a notion of "Directed Homology"
which can act as a replacement of "usual" homology
in directed topology.

Directed homology should

- be functorial
- respect directed homotopy
- be "easy" to compute
- in dimension 1 be an "abelianization" of the fundamental category

The Potential

Directed homology could be used for

- computing directed homotopy
- computing invariants of directed topological spaces
- computing invariants of higher-dimensional automata (e.g. for proving non-existence of (different kinds of) (bi)simulations)

Directed Topology

- (local) partially ordered spaces (Fajstrup, Raussen)
- d-spaces (Grandis)
- flows (Gaucher)
- cubical complexes (Serre, Pratt, Goubault)

The Big Picture

Usual (cubical) homology: The chain complex

$$\cdots \xrightarrow{\partial} C_{n+1}X \xrightarrow{\partial} C_nX \xrightarrow{\partial} C_{n-1}X \xrightarrow{\partial} \cdots$$

X a topological space, C_iX free abelian groups of singular cubes $\square_n : I^n \to X$, ∂ boundary mappings:

$$\partial \circ \partial = 0$$

Homology:

$$H_nX = \frac{\text{kernel of } \partial : C_nX \to C_{n-1}X}{\text{image of } \partial : C_{n+1}X \to C_nX}$$

that is,

$$H_nX = \frac{\text{group of } n\text{-loops}}{\text{subgroup of } (n+1)\text{-boundaries}}$$

The Big Picture, 2.

Directed homology: We are not only interested in loops, but in paths, more specifically in equivalence of paths with fixed endpoints.

Our "basic objects" are not singular cubes $\square_n : I^n \to X$, but directed cubes (monotone maps) $\vec{\square}_n : \vec{I}^n \to X$. These do not have inverses ("reflections") in general, so our $\vec{C}_i X$ are not groups, but free abelian monoids.

Instead of the boundary mapping ∂ we have two, lower and upper boundary, ∂^- and ∂^+, both homomorphisms and fulfilling

$$\partial^-\partial^- = \partial^-\partial^+ \qquad \partial^+\partial^- = \partial^+\partial^+$$

So altogether we have a globular structure

$$\cdots \overset{\partial^+}{\underset{\partial^-}{\rightrightarrows}} \vec{C}_{n+1}X \overset{\partial^+}{\underset{\partial^-}{\rightrightarrows}} \vec{C}_n X \overset{\partial^+}{\underset{\partial^-}{\rightrightarrows}} \vec{C}_{n-1}X \overset{\partial^+}{\underset{\partial^-}{\rightrightarrows}} \cdots$$

The Big Picture, 3.

So the chain complex

$$\cdots \xrightarrow{\partial} C_{n+1}X \xrightarrow{\partial} C_nX \xrightarrow{\partial} C_{n-1}X \xrightarrow{\partial} \cdots$$

has been replaced by

$$\cdots \underset{\partial^-}{\overset{\partial^+}{\rightrightarrows}} \vec{C}_{n+1}X \underset{\partial^-}{\overset{\partial^+}{\rightrightarrows}} \vec{C}_nX \underset{\partial^-}{\overset{\partial^+}{\rightrightarrows}} \vec{C}_{n-1}X \underset{\partial^-}{\overset{\partial^+}{\rightrightarrows}} \cdots$$

and we can take the "dihomology" of this by saying that $x \sim_n y \in \vec{C}_nX$ iff $\exists \alpha \in C_{n+1}X$ such that $\partial^-\alpha = x$, $\partial^+\alpha = y$ (and taking the symmetric closure of that relation).

But wait! There's more: In $\vec{C}_1X$ we can compose paths: If α goes from x to y, and β goes from y to z, then $\alpha * \beta$ is a path from x to z. The globular structure above should mirror this.

The Big Picture, 4.

So what we really want is a (strict) globular ω-category structure

$$\cdots \underset{e}{\overset{\partial^\alpha}{\rightleftarrows}} \vec{C}_{n+1}X \underset{e}{\overset{\partial^\alpha}{\rightleftarrows}} \vec{C}_n X \underset{e}{\overset{\partial^\alpha}{\rightleftarrows}} \vec{C}_{n-1}X \underset{e}{\overset{\partial^\alpha}{\rightleftarrows}} \cdots$$

with operations $+_m : C_n \times_m C_n \to C_n$ "modeling" concatenation of (sums of) n-cubes along (sums of) m-subcubes.

We shall show how this can be done, and how "directed homology" falls out of a general "homotopy" construction for ω-categories.

What's Wrong in the Paper

1. $\partial^\alpha\partial^- = \partial^\alpha\partial^+$ does not hold.

   Fix: Instead of $\partial^\alpha = \sum_k \delta_k^{k+1+\alpha}$, use the "folding operations" of [Al-Agl et al., 2002]. Basically, these add a load of connection cubes to the ∂^α.

2. (much worse) Addition and the ∂^α mappings do not "fit well". Even if $\partial^+\alpha = \partial^-\beta$, it can happen that $\partial^+(\alpha+\beta) = \partial^+\alpha + \partial^+\beta$ instead of the expected $\partial^+\alpha$.

   In a sense, we need to keep track of what we "want" the boundaries of the n-cubes to be. I.e. our objects of study are not n-cubes, but n-cubes with the lower boundaries specified.

From Semicubical Sets to ω-Categories

Let $X = \{X_n\}$ be a semicubical set, i.e. $\delta_i^\alpha : X_n \to X_{n-1}$, $i = 1, \dots, n$, $\alpha = 0, 1$ satisfy $\delta_i^\alpha \delta_j^\beta = \delta_{j-1}^\beta \delta_i^\alpha$ for $i < j$.

Denote by $\mathbb{N} \cdot X_n$ the free abelian monoid on X_n, extend the δ_i^α to the $\mathbb{N} \cdot X_n$ by declaring them to be homomorphisms (note: no cancellation), and define ∂^-, ∂^+ by

$$\partial^- = \sum_{k=1}^{n} \delta_k^{(k+1) \bmod 2} \qquad \partial^+ = \sum_{k=1}^{n} \delta_k^{k \bmod 2}$$

Lemma: $\partial^+\partial^+ + \partial^-\partial^- = \partial^+\partial^- + \partial^-\partial^+$

From Semicubical Sets to ω-Categories, 2.

Let

$$
\begin{aligned}
C_0X &= \mathbb{N}\cdot X_0\\
C_1X &= \{(\alpha,x,y)\mid \alpha\in\mathbb{N}\cdot X_1, x,y\in\mathbb{N}\cdot X_0, x+\partial^+\alpha = y+\partial^-\alpha\}\\
C_nX &= \{(A,(\alpha,x,y),(\beta,x,y))\mid A\in\mathbb{N}\cdot X_n,\\
&\qquad (\alpha,x,y),(\beta,x,y)\in C_{n-1}X, \alpha+\partial^+A = \beta+\partial^-A\}
\end{aligned}
$$

(because of lemma, $x+\partial^+\beta = y+\partial^-\beta$ "automatically")

$$\mathsf{d}^-(\alpha,x,y) = x \qquad \mathsf{d}^+(\alpha,x,y) = y \qquad \mathsf{e}x = (0,x,x)$$

$$(\alpha,x,y)+_p(\beta,x',y') = \begin{cases}(\alpha+\beta,x,y') & (p=n-1)\\ (\alpha+\beta,x+_p x',y+_p y') & (p<n-1)\end{cases}$$

(defined if $(\mathsf{d}^+)^{n-p}(\alpha,x,y) = (\mathsf{d}^-)^{n-p}(\beta,x',y')$)

From Semicubical Sets to ω-Categories, 3.

This defines a strict globular ω-category CX (which we call the chain ω-category of X), that is,

$$\mathsf{d}^\alpha \mathsf{d}^- = \mathsf{d}^\alpha \mathsf{d}^+ \qquad \mathsf{d}^\alpha \mathsf{e} = \mathsf{id} \qquad \mathsf{e}(x +_p y) = \mathsf{e}x +_p \mathsf{e}y$$

$$\mathsf{d}^-(x +_p y) = \begin{cases} x \\ \mathsf{d}^- x +_p \mathsf{d}^- y \end{cases} \quad \mathsf{d}^+(x +_p y) = \begin{cases} y & (p = n-1) \\ \mathsf{d}^+ x +_p \mathsf{d}^+ y & (p < n-1) \end{cases}$$

$$\mathsf{e}^{n-p}(\mathsf{d}^-)^{n-p} x +_p x = x +_p \mathsf{e}^{n-p}(\mathsf{d}^+)^{n-p} x = x$$

$$x +_p (y +_p z) = (x +_p y) +_p z$$

$$(x +_p y) +_q (z +_p u) = (x +_q z) +_p (y +_q u)$$

From Semicubical Sets to ω-Categories, 4.

Convenient notation: Write

$$C_nX = \{(x_n, \underline{x}_{n-1}, \overline{x}_{n-1}, \dots, \underline{x}_0, \overline{x}_0) \mid \underline{x}_{n-1} + \partial^+ x_n = \overline{x}_{n-1} + \partial^- x_n,$$
$$\underline{x}_i + \partial^+ \underline{x}_{i+1} = \overline{x}_i + \partial^- \underline{x}_{i+1}, \underline{x}_i + \partial^+ \overline{x}_{i+1} = \overline{x}_i + \partial^- \overline{x}_{i+1}\}$$

Then $\mathsf{d}^-(x_n, \dots, \overline{x}_0) = (\underline{x}_{n-1}, \dots, \overline{x}_0)$, $\mathsf{d}^+(x_n, \dots, \overline{x}_0) = (\overline{x}_{n-1}, \dots, \overline{x}_0)$, and

$$(x_n, \dots, \overline{x}_0) +_p (y_n, \dots, \overline{y}_0) = (x_n + y_n, \dots, \underline{x}_p, \overline{y}_p, \dots, \underline{x}_0, \overline{y}_0)$$

So we forgot one operation before:

$$(x_n, \dots, \overline{x}_0) +_{-1} (y_n, \dots, \overline{y}_0) = (x_n + y_n, \dots, \overline{x}_0 + \overline{y}_0)$$

(defined for all pairs of n-cells)

From Semicubical Sets to ω-Categories, Notes

The given construction SCubSet $\to$ GlobCat is functorial: If $f = \{f_n\} : X \to Y$ is a morphism of semicubical sets, then $\widehat{f}$ given by $\widehat{f}(x_n, \dots, \overline{x}_0) = (f_n x_n, \dots, f_0 \overline{x}_0)$ is a morphism of ω-categories.

The operations $+_p$ are currently not commutative, as $x +_p y$ is defined only if $(\mathsf{d}^+)^{n-p} x = (\mathsf{d}^-)^{n-p} y$. Obviously those operations should be commutative.

Also, we believe that our construction can be "twisted" in various ways, to cater for different "restrictions" one might want to put on the to-be-defined notion of directed homology.

The Homotopy Quotient of an ω-Category

In a given ω-category $C = \{C_n, \partial^\alpha, \mathsf{e}, \circ_n\}$, say that $x \sim_n y \in C_n$ if there is a "zigzag" of $(n+1)$-cells connecting x to y.

That is, if $\widehat{C}_{n+1} = C_{n+1} \cup C_{n+1}^{\mathsf{op}}$ denotes the "symmetrization" of C_{n+1}, then $x \sim_n y$ iff there exist $\alpha_1, \dots, \alpha_k \in \widehat{C}_{n+1}$ such that $\partial^- \alpha_1 = x$, $\partial^+ \alpha_i = \partial^- \alpha_{i+1}$, and $\partial^+ \alpha_k = y$. This is an equivalence relation.

Proposition: Assume $x \sim_n y \in C_n$. Then $\mathsf{d}^\alpha x = \mathsf{d}^\alpha y$, and if $x' \sim_n y' \in C_n$ and $p < n$ are such that $x \circ_p x'$ and $y \circ_p y'$ are defined, then $x \circ_m x' \sim_n y \circ_m y'$.

The Homotopy Quotient of an ω-Category, 2.

So if we define $H_n = C_n/{\sim_n}$, $H = \{H_n\}$ has mappings d^α and operations $*_n$ induced by the ones in C; H is a semi-globular ω-category.

Problem: $x \sim_n y$ does not imply $\mathrm{e}x \sim_{n+1} \mathrm{e}y$, so it appears that we have no identities in H. On the other hand we don't use the original identities in the definition of homotopy quotient, but currently we need them if the proof that $x \circ_m x' \sim_n y \circ_m y'$.

Our construction is functorial: If $f : C \to D$ is a morphism of ω-categories, then $\hat{f}$ defined by $\hat{f}[x] = [fx]$ is a morphism of semi-globular ω-categories.

Application: Directed Homology

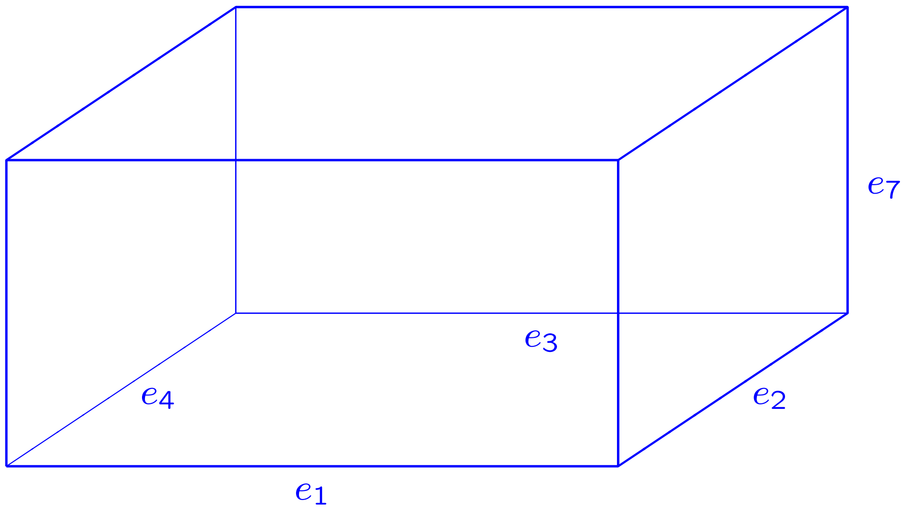
We define the directed homology semi-globular ω-category of a semicubical set X to be the homotopy quotient of its chain ω-category.

So in this case ($C = CX$), the sets $\widehat{C}_n$ are given by

$$\widehat{C}_n X = \{(A, (\alpha, x, y), (\beta, x, y)) \mid A \in \mathbb{Z} \cdot X_n, \\ (\alpha, x, y), (\beta, x, y) \in C_{n-1}X, \alpha + \partial^+ A = \beta + \partial^- A\}$$

Hence $x \sim_n y$ iff there is some $\alpha \in \widehat{C}_{n+1}$ such that $\mathsf{d}^- \alpha = x$, $\mathsf{d}^+ \alpha = y$ (with the d^-, d^+ mappings extended to $\widehat{C}_{n+1}$ the obvious way). This is the symmetric closure of the relation defined in the paper (well, kind of), and it respects (combinatorial) dihomotopy (of dipaths).

Toy Example: The Upside-Down Box



$e_1 + e_2 + e_7 + \partial^-(f_1 + f_2 + f_3 + f_4 + f_5) = e_4 + e_3 + e_7 + \partial^+(f_1 + f_2 + f_3 + f_4 + f_5)$

Hence $e_1 + e_2 + e_7 \sim_1 e_4 + e_3 + e_7$, but also $e_1 + e_2 \sim_1 e_4 + e_3$. So we should also have a "restricted directed homology" definition, e.g. by holding "corners" fixed.

What's Next

Relative dihomology: If $X \subseteq Y$ is a semicubical subset, CX should be a sub-ω-category of CY (details missing). Then, if $C \subseteq D$ is a sub-ω-category, we should make up a notion of "homotopy quotient of D relative to C", resembling the "quotient space D/C", i.e. "identifying things in C".

Exact sequences: Classical homology has various exact sequences, of pairs $X \subseteq Y$, triples $X \subseteq Y \subseteq Z$, etc., making computation of homology feasible. We'd like similar for directed homology. (Which also means that we need to invent a notion of exactness for sequences of (semi-)ω-categories.

Questions to the Audience

Is the construction of "homotopy quotient of ω-categories" known? Does it make sense? Can it be given identities? Otherwise, has there been work done on semi-globular ω-categories?

References

[Al-Agl et al., 2002] Al-Agl, F. A., Brown, R., and Steiner, R. (2002). Multiple categories: the equivalence of a globular and a cubical approach. *Adv. Math.*, 170:71–118. http://arxiv.org/abs/math/0007009.