Discount-Optimal Infinite Runs in Priced Timed Automata

Uli Fahrenberg Kim G. Larsen

Department of Computer Science
Aalborg University
Denmark

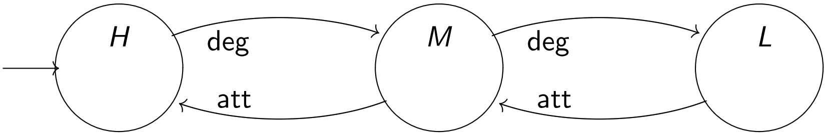
INFINITY 2008

Motivation

- **The general goal:** Find optimal infinite paths in priced timed automata
- Different versions:
  - Bouyer, Brinksma, Larsen, HSCC'04: *Staying alive as cheap as possible*: Given a timed automaton with *two* price functions, *cost* and *reward*, find infinite path with lowest ratio *cost*/*reward*
  - Bouyer, Fahrenberg, Larsen, Markey, Srba, FORMATS'08: *Infinite Runs in Weighted Timed Automata with Energy Constraints*: Given a timed automaton with positive or negative weights and pre-assigned thresholds a, b, find infinite path for which accumulated cost $\geq a$, or $\leq b$, or both
- Present work: All prices are non-negative, but for computing accumulated cost, discounting is applied

$=$ things which happen t time units in the future are taken into account only with a discount λ^t, for some fixed discounting factor λ

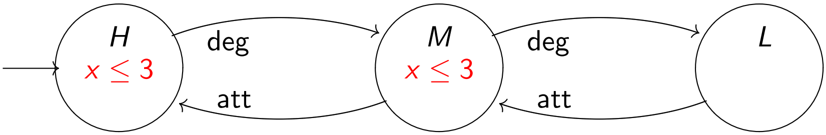
Priced timed automata



- Timed automaton:

  Finite automaton

Priced timed automata



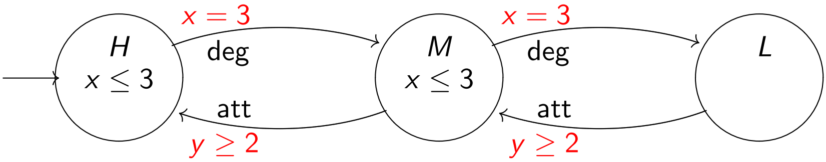
- Timed automaton:

  Finite automaton

  $+$ finite set of clocks $C = \{x, y, \dots\}$

  $+$ location invariants

Priced timed automata



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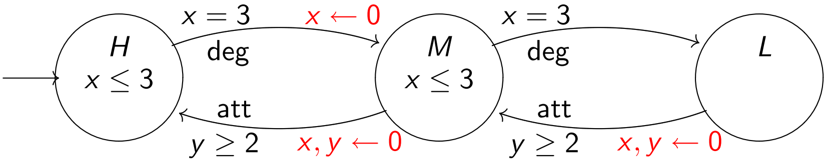
  $+$ edge constraints

Priced timed automata



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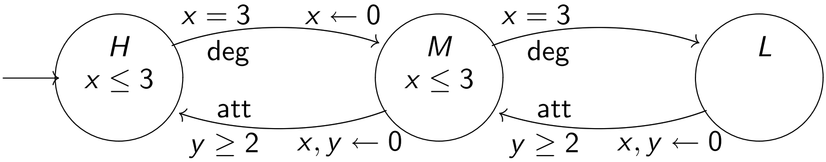
  \+ edge constraints

  \+ edge resets

Priced timed automata



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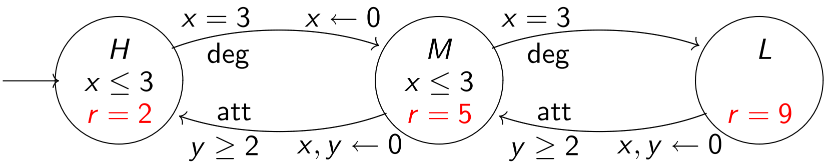
    Finite automaton
  + finite set of clocks $C = \{x, y, \ldots\}$
  + location invariants
  + edge constraints
  + edge resets

Priced timed automata



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  Finite automaton
  - $+$ finite set of clocks $C = \{x, y, \dots\}$
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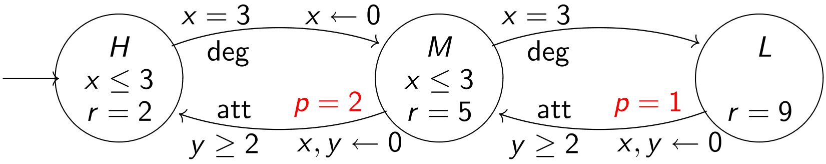
  - $+$ edge resets
- Priced timed automaton:
  - $+$ price rates in locations (cost per time unit)

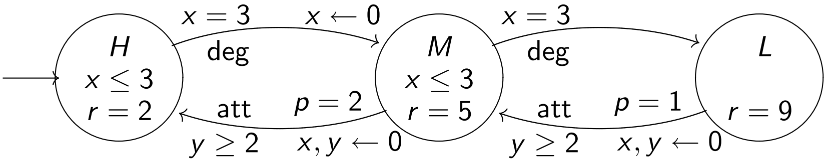
Priced timed automata



- Timed automaton:

  Finite automaton
  - \+ finite set of clocks $C = \{x, y, \dots\}$
  - \+ location invariants
  - \+ edge constraints
  - \+ edge resets
- Priced timed automaton:
  - \+ price rates in locations (cost per time unit)
  - \+ prices on transitions

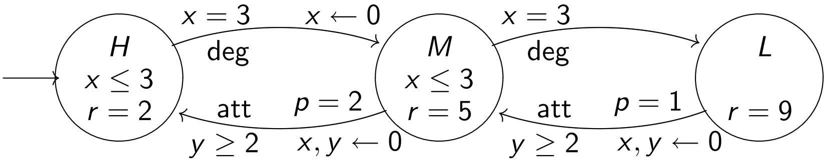
Priced timed automata



- An example run:

loc.	H	d	H	s	M	d	M	s	L	d	L	s	M	d	M	s	H
x	0		3		0		3		3		4		0		2		0
y	0		3		3		6		6		7		0		2		0
t	0		3		3		6		6		7		7		9		9

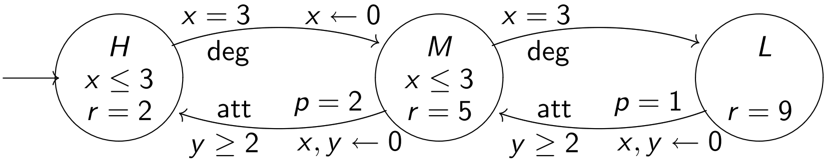
Priced timed automata



- An example run:

loc.	H	d	H	s	M	d	M	s	L	d	L	s	M	d
x	0		3		0		3		3		4		0	
y	0		3		3		6		6		7		0	
t	0		3		3		6		6		7		7	
		$2\int_0^3 \lambda^t dt$		0		$5\int_3^6 \lambda^t dt$		0		$9\int_6^7 \lambda^t dt$		$1\lambda^7$		$5\int_7^9 \lambda^t$

Priced timed automata



- An example run:

loc.	H	d	H	s	M	d	M	s	L	d	L	s	M	d	M	s	H
x	0		3		0		3		3		4		0		2		0
y	0		3		3		6		6		7		0		2		0
t	0		3		3		6		6		7		7		9		9
$\lambda = e^{-1}$		1.9		0		.23		0		.014		.00091		.0039		.00025	

- Total (discounted) price of run: 2.16

Discounted price

Formally: Let A be a priced timed transition system and $\lambda \in]0,1[$.

- The discounted price of a *finite* alternating path $\pi = s_0 \xrightarrow{t_0} s_0' \to s_1 \to \cdots \xrightarrow{t_{n-1}} s_{n-1}' \to s_n$ is

$$P(\pi) = \sum_{i=0}^{n-1} \Big(\int_{T_{i-1}}^{T_i} \lambda^t r(s_i^t) dt + \lambda^{T_i} p(s_i' \to s_{i+1}) \Big)$$

  where $T_i = \sum_{j=0}^{i} t_j$.
- The discounted price of an *infinite* alternating path $\pi = s_0 \xrightarrow{t_0} s_0' \to s_1 \to \cdots$ is the limit

$$P(\pi) = \lim_{n \to \infty} P\big(s_0 \to s_0' \to \cdots \to s_{n-1}' \to s_n \big)$$

  provided that it exists. (!)

Problem and solution

Problem:

- Given: priced timed transition system A, state $s \in A$
- Find: an infinite path from s with lowest discounted price
- (or one that comes arbitrarily close)

Solution:

- For priced timed automata,
- which are *bounded*
- and *time-divergent*,
- and *rational* λ,
- starting in the *initial state*,
- our problem is computable

Proof structure

Given priced timed automaton A:

1. Use corner point abstraction to construct *finite weighted graph* $\mathrm{cp}(A)$ (refinement of *region graph*; Bouyer, Brinksma, Larsen, HSCC'04)
2. Find infinite path $\tilde{\pi}$ with lowest discounted price in $\mathrm{cp}(A)$ using linear programming
(*discounted payoff games*; Andersson, MSc. thesis, Uppsala '06)
3. Find cheapest path lying over $\tilde{\pi}$ in A

This needs a soundness and completeness result:

- Given an infinite path $\tilde{\pi}$ in $\mathrm{cp}(A)$ for which $P(\tilde{\pi})$ converges, then for all $\varepsilon > 0$ there exists an infinite path $\pi \in \mathrm{cp}^{-1}(\tilde{\pi})$ for which $\left|P(\pi) - P(\tilde{\pi})\right| < \varepsilon$.
- Given an infinite path π in A, there exists an infinite path $\tilde{\pi} \in \mathrm{cp}(\pi)$ for which $P(\tilde{\pi}) \leq P(\pi)$.

Proof details

Given an infinite path π in A, there exists an infinite path $\tilde{\pi} \in \text{cp}(\pi)$ for which $P(\tilde{\pi}) \le P(\pi)$.

Write $\pi = (q_0, \nu_0) \to (q_0, \nu_0 + t_0) \xrightarrow[p_0]{} (q_1, \nu_1) \to (q_1, \nu_1 + t_1) \xrightarrow[p_1]{} \cdots$

Then

$$P(\pi) = \sum_{i=0}^{\infty} \Big(\int_{T_{i-1}}^{T_i} \lambda^t r(q_i) dt + \lambda^{T_i} p_i \Big) = \sum_{i=0}^{\infty} \Big(\frac{1}{\ln \lambda} r(q_i) \big(\lambda^{T_i} - \lambda^{T_{i-1}} \big) + p_i \lambda^{T_i} \Big)$$

– a function in variables $T_0, T_1, \dots$.

$\Longrightarrow$ Optimization problem: Minimize $P(\pi) = f(T_0, T_1, \dots)$ under the constraint that $T_0, T_1, \dots$ lie in a specific *zone* defined by π above.

Proof details, 2.

$$P(\pi) = f(T_0, T_1, \dots) = \sum_{i=0}^{\infty} \Big(\frac{1}{\ln \lambda} r(q_i) \big(\lambda^{T_i} - \lambda^{T_{i-1}} \big) + p_i \lambda^{T_i} \Big)$$

Task: Minimize $f(T_0, T_1, \dots)$ under the constraint that $(T_0, T_1, \dots) \in Z$ for a given (bounded, closed) zone Z.

- (can be shown that) f is (weakly) monotonic
- easy to see: monotonic functions over *finite-dimensional* closed zones attain their minimum in a *corner point*

$\Longrightarrow$ for *finite paths*, the corner point abstraction can "see" the path with lowest price (because it goes through corners) $\Longrightarrow$ done

- Need: Generalization of the above to infinite-dimensional zones
- not easy, because infinite-dimensional zones are not compact
- difficult part: show that infimum is attained somewhere

Proof details, 3.

Theorem: $Z \subseteq \mathbb{R}^\infty$ bounded and closed (in the supremum metric)

- $f_1, f_2, \dots$ continuous functions $\text{pr}_i\, Z \to \mathbb{R}_{\geq 0}$ (*non-negative* values!)
- $f(x_1, x_2, \dots) = \sum_{i=1}^\infty f_i(x_i) : Z \to [0, \infty]$ converges for some $x \in Z$

$\Longrightarrow$ exists $z \in Z$ for which $f(z) = \inf_{y \in Z} f(y)$

Proof: Let $x : \mathbb{N} \to Z$ be a sequence for which $\lim f(x_i) = \inf_{y \in Z} f(y)$

Standard argument: Z is compact $\Longrightarrow$ x contains converging subsequence x' $\Longrightarrow$ let $z = \lim x'$ $\Longrightarrow$ done.

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But Z is not compact.

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$x = \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \quad x_7 \quad x_8 \quad x_9$

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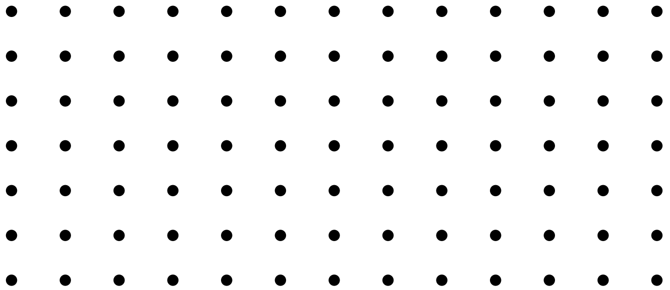

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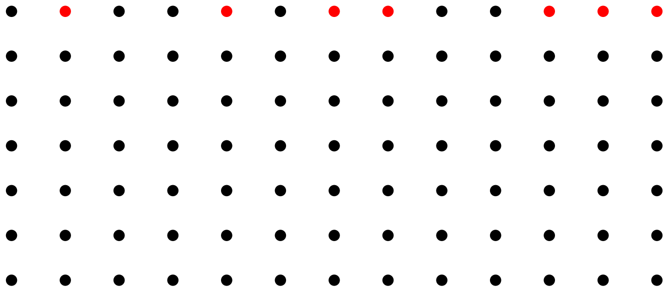

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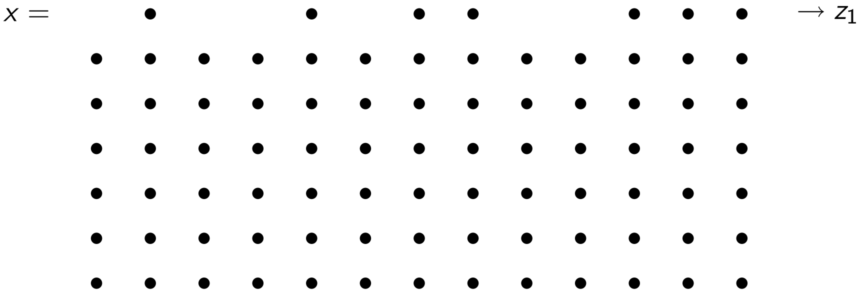

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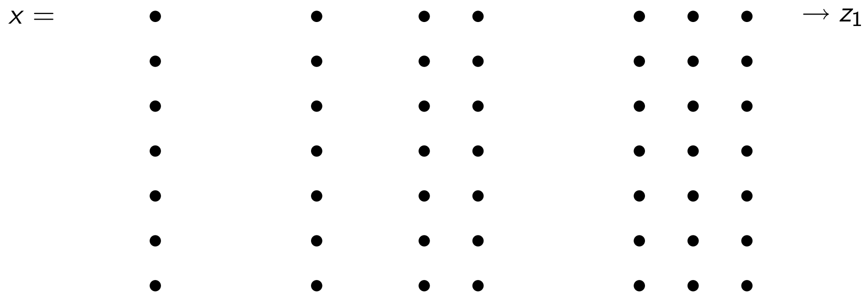

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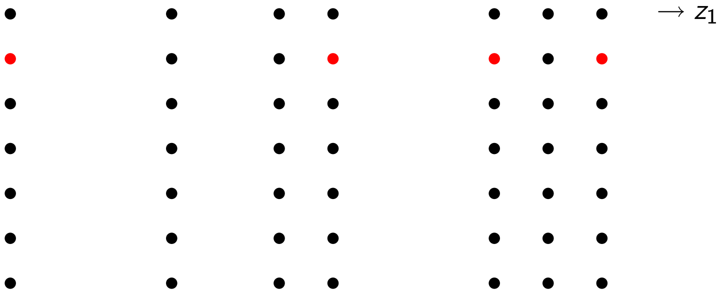

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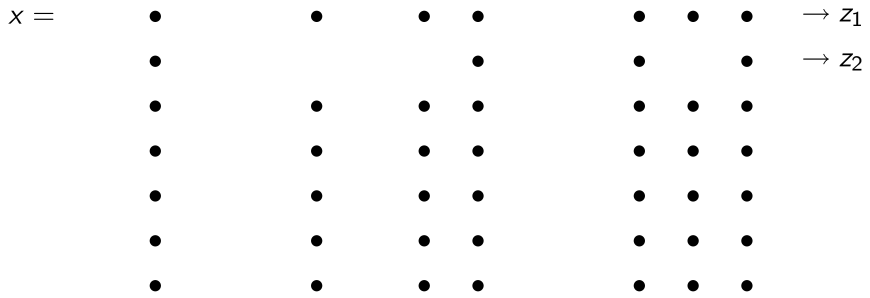

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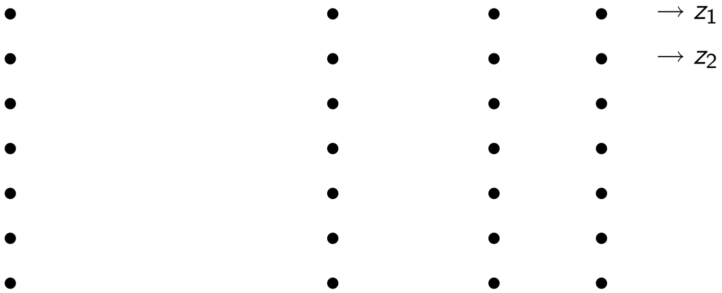

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$\Longrightarrow$ exists $z \in Z$ for which $f(z) = \inf_{y \in Z} f(y)$

Corollary: $Z \subseteq \mathbb{R}^\infty$ bounded and closed zone

- $f_1, f_2, \dots$ monotonous continuous functions $\text{pr}_i Z \to \mathbb{R}_{\geq 0}$
- $f(x_1, x_2, \dots) = \sum_{i=1}^\infty f_i(x_i) : Z \to [0, \infty]$ converges for some $x \in Z$

$\Longrightarrow$ exists corner point z of Z for which $f(z) = \inf_{y \in Z} f(y)$

So corner point abstraction *can* see the path with lowest price $\Longrightarrow$ done

Summary

- We show that, under certain mild assumptions, the discount-optimal infinite path problem is computable for priced timed automata
- We use a generalization of a well-known fact about monotonous functions defined on zones, to infinite zones
- (Other applications?)
- Computable yes, feasible NO
- But efficient, zone-based algorithm should exist
- Also, for λ close to 1, discount-optimal probably the same as limit-ratio optimal
- Games?