Infinite Runs in Priced Timed Automata: Discounting

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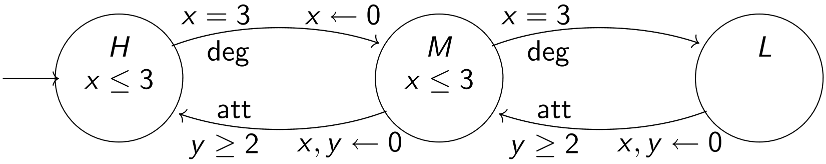
Motivation

- **The overall goal:** Find optimal infinite paths in priced timed automata
- (Priced timed automaton; PTA: timed automaton with weights ("prices") in locations and on the edges)
- Different versions:
  - Bouyer, Brinksma, Larsen, HSCC'04: *Staying alive as cheap as possible*: Given a PTA with *two* non-negative price functions, *cost* and *reward*, find infinite path with lowest ratio *accumulated cost/accumulated reward*
  - Bouyer, Fahrenberg, Larsen, Markey, Srba, FORMATS'08: *Infinite Runs in Weighted Timed Automata with Energy Constraints*: Given a PTA with positive or negative weights and possibly a pre-assigned threshold b, find infinite path for which accumulated cost ≥ 0 and possibly $\leq b$

Motivation

- This stuff: In the spirit of the HSCC'04 paper:
- One price function, all non-negative
- Interested in infinite paths with lowest accumulated cost
- But we apply discounting:
- Things which happen t time units in the future are taken into account only with a discount λ^t, for some fixed discounting factor λ
- (with Kim Larsen)
- INFINITY'08; QAPL'09

Priced timed automata



- Timed automaton:

  Finite automaton

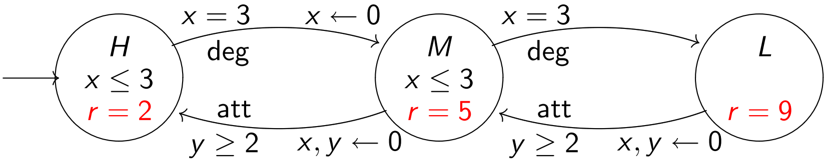
  $+$ finite set of clocks $C = \{x, y, \dots\}$

  $+$ location invariants

  $+$ edge constraints

  $+$ edge resets

Priced timed automata



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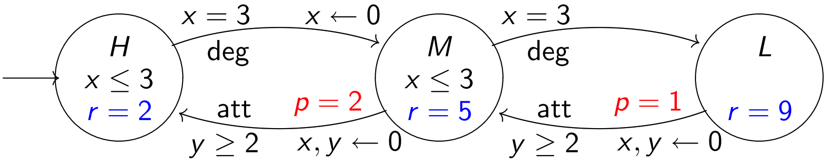
  + finite set of clocks $C = \{x, y, \ldots\}$
  + location invariants
  + edge constraints
  + edge resets
- Priced timed automaton:
  + price rates in locations (cost per time unit)

Priced timed automata

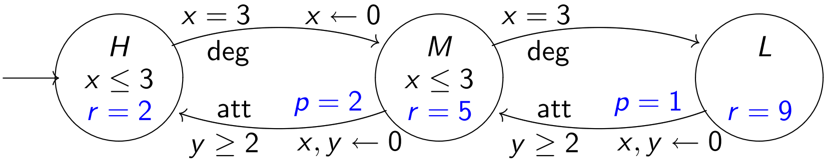


- Timed automaton:

  Finite automaton
  + finite set of clocks $C = \{x, y, \dots\}$
  + location invariants
  + edge constraints
  + edge resets

- Priced timed automaton:
  + price rates in locations (cost per time unit)
  + prices on transitions

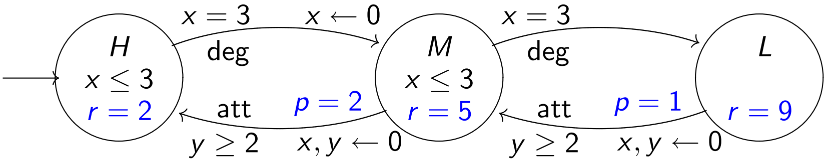
Priced timed automata



- An example (finite) run:

loc.	H	d	H	s	M	d	M	s	L	d	L	s	M	d	M	s	H
x	0		3		0		3		3		4		0		2		0
y	0		3		3		6		6		7		0		2		0
t	0		3		3		6		6		7		7		9		9

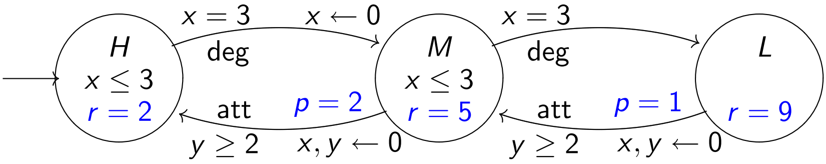
Priced timed automata



- An example (finite) run:

loc.	H	d	H	s	M	d	M	s	L	d	L	s	M	d
x	0		3		0		3		3		4		0	
y	0		3		3		6		6		7		0	
t	0		3		3		6		6		7		7	
		$2\int_0^3 \lambda^t dt$		0		$5\int_3^6 \lambda^t dt$		0		$9\int_6^7 \lambda^t dt$		$1\lambda^7$		$5\int_7^9 \lambda^t$

Priced timed automata



- An example (finite) run:

loc.	H	d	H	s	M	d	M	s	L	d	L	s	M	d	M	s	H
x	0		3		0		3		3		4		0		2		0
y	0		3		3		6		6		7		0		2		0
t	0		3		3		6		6		7		7		9		9
$\lambda = .9$		5.14		0		9.38		0		4.54		.48		4.31		.77	

- Total discounted price of run: 24.62
- Total discounted price of infinite loop: 40.2

Discounted price

Formally: Let A be a priced timed transition system and $\lambda \in]0,1]$.

- Discounted price of *finite* alternating path $\pi = s_0 \xrightarrow{t_0} s'_0 \to s_1 \to \cdots \xrightarrow{t_{n-1}} s'_{n-1} \to s_n$:

$$P(\pi) = \sum_{i=0}^{n-1} \Big(\int_{T_{i-1}}^{T_i} \lambda^t r(s_i^t) dt + \lambda^{T_i} p(s'_i \to s_{i+1}) \Big)$$

with $T_i = \sum_{j=0}^{i} t_j$.

- Discounted price of *infinite* alternating path $\pi = s_0 \xrightarrow{t_0} s'_0 \to s_1 \to \cdots$: limit

$$P(\pi) = \lim_{n \to \infty} P\big(s_0 \to s'_0 \to \cdots \to s'_{n-1} \to s_n\big)$$

provided that it exists. (!)

Problem and solution

Problem:

- Given: priced timed transition system A, state $s \in A$
- Find: an infinite path from s with lowest discounted price
- (or one that comes arbitrarily close)

Solution:

- For priced timed automata,
- which are *bounded*
- and *time-divergent*,
- and *rational* λ,
- starting in the *initial state*,
- our problem is computable

Proof structure

Given priced timed automaton A:

1. Use corner point abstraction to construct *finite weighted graph* $\mathrm{cp}(A)$ (refinement of *region graph*; Bouyer, Brinksma, Larsen, HSCC'04)
2. Find infinite path $\tilde{\pi}$ with lowest discounted price in $\mathrm{cp}(A)$ using linear programming
   (*discounted payoff games*; Andersson, MSc. thesis, Uppsala '06)
3. Find cheapest path lying over $\tilde{\pi}$ in A

This needs a soundness and completeness result:

- Given an infinite path $\tilde{\pi}$ in $\mathrm{cp}(A)$ for which $P(\tilde{\pi})$ converges, then for all $\varepsilon > 0$ there exists an infinite path $\pi \in \mathrm{cp}^{-1}(\tilde{\pi})$ for which $\big|P(\pi) - P(\tilde{\pi})\big| < \varepsilon$.
- Given an infinite path π in A, there exists an infinite path $\tilde{\pi} \in \mathrm{cp}(\pi)$ for which $P(\tilde{\pi}) \leq P(\pi)$.

Proof details

Given an infinite path π in A, there exists an infinite path $\tilde{\pi} \in \mathrm{cp}(\pi)$ for which $P(\tilde{\pi}) \leq P(\pi)$.

Write $\pi = (q_0, \nu_0) \to (q_0, \nu_0 + t_0) \xrightarrow[p_0]{} (q_1, \nu_1) \to (q_1, \nu_1 + t_1) \xrightarrow[p_1]{} \cdots$

Then
$$P(\pi) = \sum_{i=0}^{\infty} \Big(\int_{T_{i-1}}^{T_i} \lambda^t r(q_i) dt + \lambda^{T_i} p_i \Big)$$
$$= \sum_{i=0}^{\infty} \Big(\frac{1}{\ln \lambda} r(q_i) \big(\lambda^{T_i} - \lambda^{T_{i-1}} \big) + p_i \lambda^{T_i} \Big)$$

– a function in variables $T_0, T_1, \dots$.

$\Longrightarrow$ Optimization problem: Minimize $P(\pi) = f(T_0, T_1, \dots)$ under the constraint that $T_0, T_1, \dots$ lie in a specific *zone* defined by π above.

Proof details, 2.

$$P(\pi) = f(T_0, T_1, \dots) = \sum_{i=0}^{\infty} \Big(\frac{1}{\ln \lambda} r(q_i) \big(\lambda^{T_i} - \lambda^{T_{i-1}} \big) + p_i \lambda^{T_i} \Big)$$

Task: Minimize $f(T_0, T_1, \dots)$ under the constraint that $(T_0, T_1, \dots) \in Z$ for a given (bounded, closed) zone Z.

- (can be shown that) f is (weakly) monotonic
- easy to see: monotonic functions over *finite-dimensional* closed zones attain their minimum in a *corner point*

$\Longrightarrow$ for *finite paths*, the corner point abstraction can "see" the path with lowest price (because it goes through corners) $\Longrightarrow$ done

- Need: Generalization of the above to infinite-dimensional zones
- not easy, because infinite-dimensional zones are not compact
- difficult part: show that infimum is attained somewhere

Proof details, 3.

Theorem: $Z \subseteq \mathbb{R}^\infty$ bounded and closed (in the supremum metric)

- $f_1, f_2, \dots$ continuous functions $\mathrm{pr}_i\, Z \to \mathbb{R}_{\geq 0}$ (*non-negative* values!)
- $f(x_1, x_2, \dots) = \sum_{i=1}^\infty f_i(x_i) : Z \to [0, \infty]$ converges for some $x \in Z$

$\Longrightarrow$ exists $z \in Z$ for which $f(z) = \inf_{y \in Z} f(y)$

Proof: Let $x : \mathbb{N} \to Z$ be a sequence for which $\lim f(x_i) = \inf_{y \in Z} f(y)$

Standard argument: Z is compact $\Longrightarrow$ x contains converging subsequence $x' \Longrightarrow$ let $z = \lim x' \Longrightarrow$ done.

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But Z is not compact.

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$x = \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \quad x_7 \quad x_8 \quad x_9$

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$$x = \begin{matrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{matrix}$$

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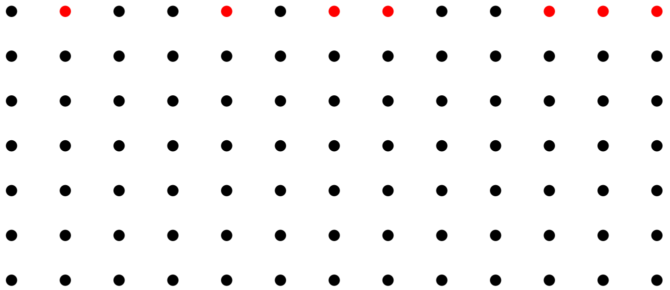

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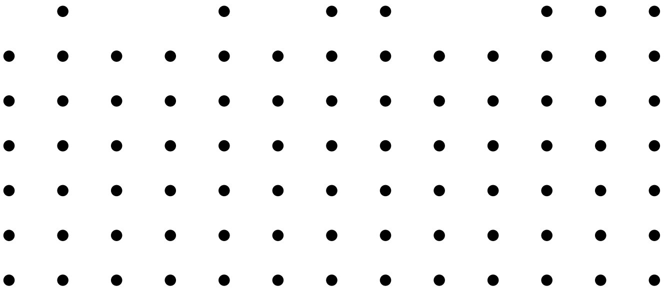

$\to z_1$

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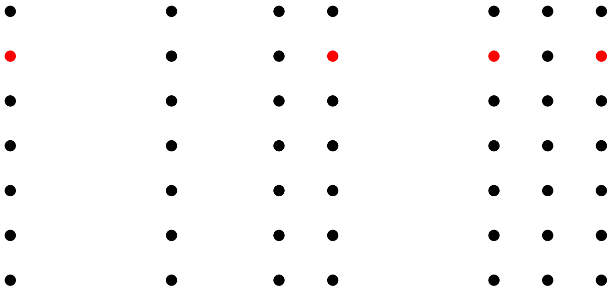

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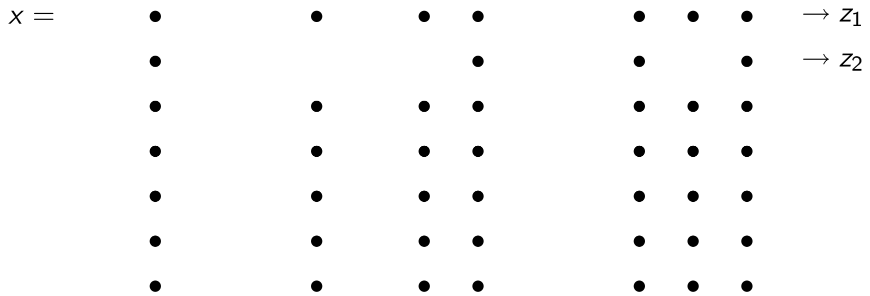

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$\Longrightarrow$ exists $z \in Z$ for which $f(z) = \inf_{y \in Z} f(y)$

Corollary: $Z \subseteq \mathbb{R}^\infty$ bounded and closed zone

- $f_1, f_2, \dots$ monotonous continuous functions $\mathrm{pr}_i Z \to \mathbb{R}_{\geq 0}$
- $f(x_1, x_2, \dots) = \sum_{i=1}^\infty f_i(x_i) : Z \to [0, \infty]$ converges for some $x \in Z$

$\Longrightarrow$ exists corner point z of Z for which $f(z) = \inf_{y \in Z} f(y)$

So corner point abstraction *can* see the path with lowest price $\Longrightarrow$ done

Summary

- The discount-optimal infinite path problem is computable for priced timed automata
- (with arbitrarily many clocks)
- (under certain restrictions; only “real” restriction: time divergence)
- (time-divergence assumption similar to reward-divergence assumption in HSCC’04 paper; can it be avoided?)
- Uses generalization of a well-known fact about monotonous functions defined on zones, to infinite zones
- (Other applications?)

- Computable yes, feasible NO
- For zone-based iterative computations: need some fixed-point property / additivity

Additivity

- For $\longrightarrow$ a switch and π a path out of the target of the switch:

$$P(\longrightarrow \circ \pi) = p(\longrightarrow) + P(\pi)$$

For $\xrightarrow{t}$ a delay and π a path out of the end state of the delay:

$$P(\xrightarrow{t} \circ \pi) = p(\xrightarrow{t}) + \lambda^t P(\pi)$$

- Similar property not satisfied for other formalisms (*e.g.* cost / reward, or step-wise discounting)
- Question: What do we know about the cost function if we require

$$P(\xrightarrow{t} \circ \pi) = p(\xrightarrow{t}) + g(t) P(\pi)$$

- (Natural property also because of additivity of delays!)
- Answer: Everything!

Additivity

Theorem: If P is an accumulated price function in a PTA which satisfies

$$P(s \xrightarrow{t_1} s' \xrightarrow{t_2} s'') = P(s \xrightarrow{t_1} s') + g(t_1)\, P(s' \xrightarrow{t_2} s'')$$

for all states and delays, then

$$g(t) = \lambda^t \quad \text{and} \quad P(s \xrightarrow{t} s') = \alpha(s) \int_0^t \lambda^t dt$$

for some $\lambda \in \mathbb{R}_{\geq 0}$ and $\alpha : S \to \mathbb{R}$.

Reason: The functional equation

$$f(t_1 + t_2) = f(t_1) + g(t_1)\, f(t_2)$$

has essentially only one solution.

Proof: Just do like Cauchy in 1821 !

Conclusion

- If you want to discount in real-time formalisms, use our kind of discounting
- (unless you have your own good reasons not to; but you'll lose additivity)
- Then you'll get computability of optimal infinite paths
- And if you give us a little more time, maybe you'll even have an efficient zone-based algorithm