# Playing Games with Metrics
# Distances for Weighted Transition Systems

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68nqrt
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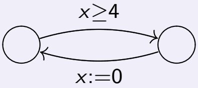
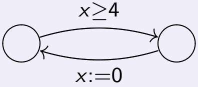
# Quantitative Analysis

## Quantitative *Models*



# Quantitative Quantitative Analysis

## Quantitative *Models*

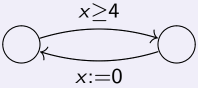



## Quantitative *Logics*

$$\mathrm{Pr}_{\leq .1}(\Diamond \mathit{error})$$

# Quantitative Quantitative Quantitative Analysis

## Quantitative *Models*

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## Quantitative *Verification*

$$[\![\varphi]\!](s) = 3.14$$
$$d(s, t) = 42$$

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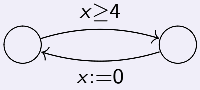


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## Quantitative *Verification*

$[\![\varphi]\!](s) = 3.14$

$d(s, t) = 42$

| **Boolean world** | **"Quantification"** |
|---|---|
| Trace equivalence $\equiv$ | Linear distance $d_L$ |
| Bisimilarity $\sim$ | Branching distance $d_B$ |
| $s \sim t$ implies $s \equiv t$ | $d_L(s, t) \leq d_B(s, t)$ |
| $s \models \varphi$ or $s \not\models \varphi$ | $[\![\varphi]\!](s)$ is a quantity |
| $s \sim t$ iff $\forall \varphi : s \models \varphi \Leftrightarrow t \models \varphi$ | $d_B(s, t) = \sup_\varphi d\big([\![\varphi]\!](s), [\![\varphi]\!](t)\big)$ |

# Quantitative Quantitative Quantitative Analysis

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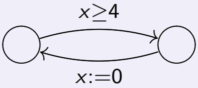


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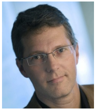
- Thrane, Fahrenberg, Larsen: *Quantitative analysis of weighted transition systems*. JLAP 79(7):689–703, 2010.
- Fahrenberg, Larsen, Thrane: *A quantitative characterization of weighted Kripke structures in temporal logic*. CAI 29, 2010.
- Fahrenberg, Thrane, Larsen: *Distances for weighted transition systems: Games and properties*. Submitted.



# Weighted Automata and Traces

**Definition**

A weighted automaton: states $S$, transitions $T \subseteq S \times \mathbb{K} \times S$

- $\mathbb{K}$: Set of weights.
- Standard example: $\mathbb{K} = L \times \mathbb{R}$. Discrete labels $L$, real weights $\mathbb{R}$.

**Definition**

A trace is an infinite sequence of weights; an element of $\mathbb{K}^\omega$.

- Notation: For $s \in S$ in a weighted automaton $(S, T)$, $\mathrm{Tr}(s)$ is the set of traces from $s$.

# Framework for Quantitative Analysis

## Trace distance

Assume given a hemimetric $d_T : \mathbb{K}^\omega \times \mathbb{K}^\omega \to [0, \infty]$.

That's it. We only assume some way to measure distance between traces.

- Think of the trace distance as application defined
- May or may not come from some metric on $\mathbb{K}$
- (This is very common *e.g.* in applications in real-time or hybrid systems)

(Hemimetric: not necessarily symmetric pseudometric:

- $d_T(x, x) = 0$ (indiscernibility of identicals)
- $d_T(x, y) + d_T(y, z) \geq d_T(x, z)$ (triangle inequality) )

# Examples of Trace Distances

- Let $\mathbb{K} = L \times \mathbb{R}$. Notation: Trace $\sigma = \big((\sigma_0^\ell, \sigma_0^w), (\sigma_1^\ell, \sigma_1^w), \dots\big)$.

## Point-wise trace distance

$$d_T^\bullet(\sigma,\tau) = \begin{cases} \sup_i \quad |\sigma_i^w - \tau_i^w| & \text{if } \sigma_i^\ell = \tau_i^\ell \text{ for all } i \\ \infty & \text{otherwise} \end{cases}$$

## Accumulating trace distance

$$d_T^+(\sigma,\tau) = \begin{cases} \sum_i \quad |\sigma_i^w - \tau_i^w| & \text{if } \sigma_i^\ell = \tau_i^\ell \text{ for all } i \\ \infty & \text{otherwise} \end{cases}$$

## Maximum-lead trace distance

$$d_T^\pm(\sigma,\tau) = \begin{cases} \sup_i \big|\sum_{j=0}^i \sigma_j^w - \sum_{j=0}^i \tau_j^w\big| & \text{if } \sigma_i^\ell = \tau_i^\ell \text{ for all } i \\ \infty & \text{otherwise} \end{cases}$$

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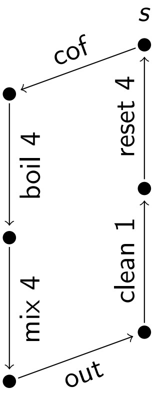
# Linear Distance

- (Recall: We assume given a hemimetric $d_T : \mathbb{K}^\omega \times \mathbb{K}^\omega \to [0, \infty]$ on traces.)
- Let $(S, T \subseteq S \times \mathbb{K} \times S)$ be a weighted automaton.
- Linear distance between states $s, t \in S$: use Hausdorff construction:

## Definition: Linear distance

$$d_L(s, t) = \sup_{\sigma \in \mathsf{Tr}(s)} \inf_{\tau \in \mathsf{Tr}(t)} d_T(\sigma, \tau)$$

# Example




Left: coffee machine

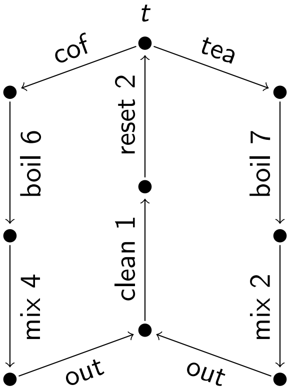
Right: coffee&tea

Labels are actions, numbers are energy use.
Discount factor $\lambda = .9$

Pointwise:
$d_L^\bullet(t,s) = \infty$, $d_L^\bullet(s,t) = 1.8$

Accumulated:
$d_L^+(t,s) = \infty$, $d_L^+(s,t) \approx 2.52$

Max-lead (no discounting):
$d_L^\pm(t,s) = \infty$, $d_L^\pm(s,t) = 2$



# Linear vs. Branching Distance: the Upshot

**Recall: Linear distance**

$$d_L(s,t) = \sup_{\sigma\in\mathrm{Tr}(s)} \inf_{\tau\in\mathrm{Tr}(t)} d_T(\sigma,\tau)$$

- This is inspired by trace inclusion
- and looks like it will be difficult to compute.
- (Indeed, for timed automata *e.g.*, $d_L$ is uncomputable.)

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- This is inspired by trace inclusion
- and looks like it will be difficult to compute.
- (Indeed, for timed automata *e.g.*, $d_L$ is uncomputable.)

- **Goal:** Find a corresponding branching distance $d_B$
- inspired by simulation
- which has $d_L \leq d_B$
- and may be easier to compute.

# Linear vs. Branching Distance: the Idea

## Recall: Linear distance

$$d_L(s,t) = \sup_{\sigma \in \mathrm{Tr}(s)} \inf_{\tau \in \mathrm{Tr}(t)} d_T(\sigma,\tau)$$

- This is a game!
- Player 1 chooses the worst trace $\sigma \in \mathrm{Tr}(s)$.
- Player 2 matches it with the best trace $\tau \in \mathrm{Tr}(t)$.
- $d_L(s,t) =$ value of the "1-blind weighted simulation game": Player 2 has perfect information, Player 1 is blind.

## Definition: Branching distance

$d_B(s,t) =$ value of the same game, but with perfect information

- Hence "$d_B(s,t) = \sup_{s \xrightarrow{\sigma_0} s_1} \inf_{t \xrightarrow{\tau_0} t_1} \sup_{s_1 \xrightarrow{\sigma_1} s_2} \inf_{t_1 \xrightarrow{\tau_1} t_2} \cdots \; d_T(\sigma,\tau)$".

# Linear vs. Branching Distance: the Dirty Details

Precise definition of how this works:

- Given: Weighted automaton $(S, T \subseteq S \times \mathbb{K} \times S)$, states $s, t \in S$
- (Imagine a game of two players taking turns to build two paths:)
- A strategy from $s$, $t$: $\theta : \mathrm{fPa}(s) \times \mathrm{fPa}(t) \to T$
  - for Player 1: $\mathrm{start}(\theta(\pi_1, \pi_2)) = \mathrm{end}(\pi_1)$
  - for Player 2: $\mathrm{start}(\theta(\pi_1, \pi_2)) = \mathrm{end}(\pi_2)$
- A round of the game under strategies $\theta_1$, $\theta_2$:
  $\mathrm{Round}_{(\theta_1,\theta_2)}(\pi_1, \pi_2) = \big(\pi_1 \cdot \theta_1(\pi_1, \pi_2), \pi_2 \cdot \theta_2(\pi_1 \cdot \theta_1(\pi_1, \pi_2), \pi_2)\big)$
- The limit of the game under strategies $\theta_1$, $\theta_2$:
  $\mathrm{limit} = \lim_{j\to\infty} \mathrm{Round}^j_{(\theta_1,\theta_2)}(s_0, t_0)$ (a pair of infinite paths)
- The utility of the strategies $\theta_1$, $\theta_2$: $u(\theta_1, \theta_2) = d_T\big(\mathrm{tr}(\mathrm{limit})\big)$
- The value of the game: $v(s, t) = \sup_{\theta_1} \inf_{\theta_2} u(\theta_1, \theta_2)$

# Perfect vs. Imperfect Information

- $\Theta_1(s,t)$, $\Theta_2(s,t)$: sets of all Player-1 resp. Player-2 strategies $\text{fPa}(s) \times \text{fPa}(t) \to T$
- Games with imperfect information: Restrict available strategies to proper subsets of $\Theta_1$ or $\Theta_2$
- Special case: blind Player-1 strategies $\tilde{\Theta}_1 = T^{\text{fPa}(s)}$
- Do not depend on Player-2 choices: Player 1 cannot "see" what Player 2 is doing

- Branching distance: $d_B(s,t) = \sup_{\theta_1 \in \Theta_1(s,t)} \inf_{\theta_2 \in \Theta_2(s,t)} u(\theta_1, \theta_2)$
- Linear distance: $d_L(s,t) = \sup_{\theta_1 \in \tilde{\Theta}_1(s,t)} \inf_{\theta_2 \in \Theta_2(s,t)} u(\theta_1, \theta_2)$

# Properties

**Proposition**

- $d_L$ is a hemimetric.
- If the simulation game is *determined*, $d_B$ is a hemimetric.

- Need determinacy to show triangle inequality
- (But have no counterexample)

**Theorem**

*For all* $s, t \in S$, $d_L(s,t) \le d_B(s,t)$.

**Proof:**

For $d_B$, Player 1 (the *sup player*) has more strategies to choose from!

# Properties

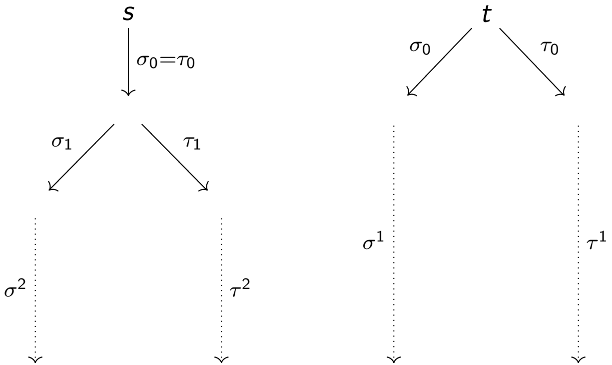
**Theorem**

*There exists a weighted automaton on which $d_L$ and $d_B$ are topologically inequivalent.*

- Unless for all traces $\sigma$, $\tau$ : $\sigma_0 = \tau_0$ implies $d_T(\sigma, \tau) = 0$.
- (*i.e.* $d_T$ measures only on *first* trace element; not very useful!)

# Proof

Let $\sigma, \tau \in \mathbb{K}^\omega$ such that $\sigma_0 = \tau_0$, $d_T(\sigma, \tau) > 0$, and $d_T(\tau, \sigma) > 0$.




We have $\mathrm{Tr}(s) = \mathrm{Tr}(t)$, hence $d_L(s, t) = 0$. On the other hand, $d_B(s, t) = \min\big(d_T(\sigma, \tau), d_T(\tau, \sigma)\big) > 0$. That's it.

# Fixed-Point Characterization

- Back to trace distance examples:

$$d_T^\bullet(\sigma,\tau) = \sup_i |\sigma_i^w - \tau_i^w| = \max\left(|\sigma_0^w - \tau_0^w|, d_T^\bullet(\sigma^1,\tau^1)\right)$$

Similarly:

$$d_T^+(\sigma,\tau) = |\sigma_0^w - \tau_0^w| + d_T^+(\sigma^1,\tau^1)$$

**Theorem**

If $d_T(\sigma,\tau) = f\left(\sigma_0,\tau_0,d_T(\sigma^1,\tau^1)\right)$ for some function $f : \mathbb{K}\times\mathbb{K}\times[0,\infty] \to [0,\infty]$ which is monotone in the third coordinate and all $\sigma,\tau\in\mathbb{K}^\omega$, then $d_B$ is the least fixed point to the set of equations

$$h(s,t) = \sup_{s \xrightarrow{x} s'} \inf_{t \xrightarrow{y} t'} f\left(x,y,h(s',t')\right)$$

# Fixed-Point Characterization

## Theorem (again)

If $d_T(\sigma, \tau) = f\big(\sigma_0, \tau_0, d_T(\sigma^1, \tau^1)\big)$ for some function $f : \mathbb{K} \times \mathbb{K} \times [0, \infty] \to [0, \infty]$ which is monotone in the third coordinate and all $\sigma, \tau \in \mathbb{K}^\omega$, then $d_B$ is the least fixed point to the set of equations

$$h(s, t) = \sup_{s \xrightarrow{x} s'} \inf_{t \xrightarrow{y} t'} f\big(x, y, h(s', t')\big)$$

- So if trace distance has a simple recursive characterization, then so does branching distance.
- Applies to $d_T^\bullet$ and $d_T^+$, but not to $d_T^\pm$.
- Have extension to "recursive characterization with memory" which applies to $d_T^\pm$ (and other interesting distances, *e.g.* lim-avg).

# Conclusion

- For most applications, trace distances are easy to think of
- We show how to go from any trace distance to a linear *(easy)* and branching *(difficult)* distance
- (Using games with quantitative objectives)

- Our definition of linear and branching distance may not be very operational
- (*E.g.*, linear distance is *uncomputable* for some models, and so may branching distance)

- But we claim that our definition is (or should be) the canonical one
- (And we show that for a number of interesting examples, we
  - get an operational definition (using fixed points)
  - and recover previously considered distances)

# Mathematical Wish List

- Relate equivalence of trace distances to equivalence of linear distances. Like this:

**Theorem**

*If trace distances $d_T^1$ and $d_T^2$ are Lipschitz equivalent, then the corresponding linear distances $d_L^1$ and $d_L^2$ are topologically equivalent.*

- Relate equivalence of trace distances to equivalence of branching distances.
- Classify trace distances (up to equivalence).

# Other games

- Recall: $d_B(s,t)$ — value of weighted simulation game
- $d_L(s,t) = \sup_{\sigma \in \mathrm{Tr}(s)} \inf_{\tau \in \mathrm{Tr}(t)} d_T(\sigma,\tau)$ — value of 1-blind game
- The 2-blind game: $\inf_{\tau \in \mathrm{Tr}(t)} \sup_{\sigma \in \mathrm{Tr}(s)} d_T(\sigma,\tau)$
- (Oh: what about minimax theorems here?)
- The weighted bisimulation game: At each turn, give Player 1 the choice whether to prolong the path from $s$ or from $t$ $\Longrightarrow$ bisimulation distance!
- The weighted similarity game: Player 1 gets to choose which path to build before first turn only $\Longrightarrow$ similarity distance
- Player 1 gets to choose before first turn and is blinded $\Longrightarrow$ language equivalence distance
- *etc.*