Playing Games with Metrics
Distances for Weighted Transition Systems

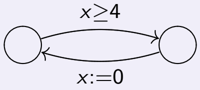
Uli Fahrenberg Claus Thrane Kim G. Larsen

IRISA Rennes, France / Aalborg University, Denmark

QAPL 2011

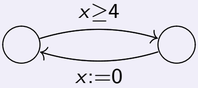
Quantitative Analysis

Quantitative *Models*



Quantitative Quantitative Analysis

Quantitative *Models*

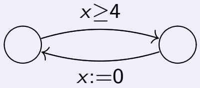


Quantitative *Logics*

$$\mathrm{Pr}_{\leq .1}(\Diamond \mathit{error})$$

Quantitative Quantitative Quantitative Analysis

Quantitative *Models*



Quantitative *Logics*

$$\Pr_{\leq .1}(\Diamond \mathit{error})$$

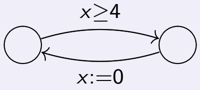
Quantitative *Verification*

$$[\![\varphi]\!](s) = 3.14$$

$$d(s, t) = 42$$

Quantitative Quantitative Quantitative Analysis

Quantitative *Models*



Quantitative *Logics*

$$\Pr_{\leq .1}(\Diamond \mathit{error})$$

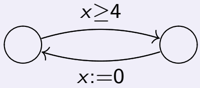
Quantitative *Verification*

$$[\![\varphi]\!](s) = 3.14$$
$$d(s,t) = 42$$

Boolean world	**"Quantification"**
Trace equivalence $\equiv$	Linear distance d_L
Bisimilarity $\sim$	Branching distance d_B
$s \sim t$ implies $s \equiv t$	$d_L(s,t) \leq d_B(s,t)$
$s \models \varphi$ or $s \not\models \varphi$	$[\![\varphi]\!](s)$ is a quantity
$s \sim t$ iff $\forall \varphi : s \models \varphi \Leftrightarrow t \models \varphi$	$d_B(s,t) = \sup_\varphi d\big([\![\varphi]\!](s), [\![\varphi]\!](t)\big)$

Quantitative Quantitative Quantitative Analysis

Quantitative *Models*



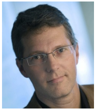
Quantitative *Logics*

$$\Pr_{\leq .1}(\Diamond \mathit{error})$$

Quantitative *Verification*

$$[\![\varphi]\!](s) = 3.14$$
$$d(s,t) = 42$$

- Thrane, Fahrenberg, Larsen: *Quantitative analysis of weighted transition systems*. JLAP 2010.
- Fahrenberg, Larsen, Thrane: *A quantitative characterization of weighted Kripke structures in temporal logic*. CAI 2010.
- Larsen, Fahrenberg, Thrane: *Metrics for weighted transition systems: Axiomatization and complexity*. TCS 2011.



The Framework

Idea:

- Qualitative and quantitative information should be orthogonal
- and both are inputs to the verification problem

Here:

- Qualitative information: labeled transition system
- Quantitative information: distance on traces

Definitions

- $\mathbb{K}$: set of labels
- $\mathbb{K}^\omega$: set of infinite traces in $\mathbb{K}$
- a labeled transition system: states S, transitions $T \subseteq S \times \mathbb{K} \times S$
- a trace distance: (extended) hemimetric $d_T : \mathbb{K}^\omega \times \mathbb{K}^\omega \to [0, \infty]$

Examples of Trace Distances

- Cantor distance: $d_T(\sigma, \tau) = 1 \,/\, (\text{length of longest common prefix})$
- Hamming distance: $d_T(\sigma, \tau) = \sum \delta(\sigma_j, \tau_j)$
- Levenshtein distance

Given a hemimetric $d : \mathbb{K} \times \mathbb{K} \to [0, \infty]$:

- point-wise distance: $d_T(\sigma, \tau) = \sup d(\sigma_j, \tau_j)$
- accumulating distance: $d_T(\sigma, \tau) = \sum d(\sigma_j, \tau_j)$

Given hemimetric $d : \mathbb{K} \times \mathbb{K} \to [0, \infty]$ and addition $+ : \mathbb{K} \times \mathbb{K} \to \mathbb{K}$:

- max-lead distance: $d_T(\sigma, \tau) = \sup d\big(\sum_0^n \sigma_j, \sum_0^n \tau_j \big)$

Useful for infinite traces:

- discounting: for $0 < \lambda < 1$, *e.g.* $d_T(\sigma, \tau) = \sum \lambda^j d(\sigma_j, \tau_j)$
- limit-average: *e.g.* $d_T(\sigma, \tau) = \liminf \frac{1}{n} \sum_0^n d(\sigma_j, \tau_j)$

Linear Distance

Let

- $(S, T \subseteq S \times \mathbb{K} \times S)$ be a labeled transition system,
- $d_T : \mathbb{K}^\omega \times \mathbb{K}^\omega \to [0, \infty]$ be a trace distance.

Definition: Linear distance from s to t

$$d_L(s,t) = \sup_{\sigma \in \mathrm{Tr}(s)} \inf_{\tau \in \mathrm{Tr}(t)} d_T(\sigma, \tau)$$

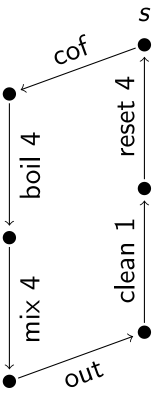
- $\mathrm{Tr}(s)$: set of infinite traces from s
- This is the Hausdorff construction

Lemma

If (S, T) is finitely branching, then

$$d(s,t) \le \varepsilon \iff \forall \sigma \in \mathrm{Tr}(s)\, \exists \tau \in \mathrm{Tr}(t) : d_T(\sigma, \tau) \le \varepsilon.$$

Example



Left: coffee machine

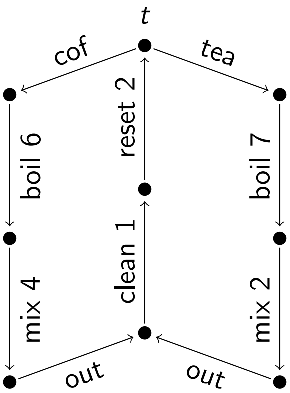
Right: coffee&tea

Labels are actions, numbers are energy use.
Discount factor $\lambda = .9$

Pointwise:
$d_L^\bullet(t,s) = \infty$, $d_L^\bullet(s,t) = 1.8$

Accumulated:
$d_L^+(t,s) = \infty$, $d_L^+(s,t) \approx 2.52$

Max-lead (no discounting):
$d_L^\pm(t,s) = \infty$, $d_L^\pm(s,t) = 2$



Linear vs. Branching Distance

Recall: Linear distance

$$d_L(s,t) = \sup_{\sigma \in \mathrm{Tr}(s)} \inf_{\tau \in \mathrm{Tr}(t)} d_T(\sigma,\tau)$$

- This is a game!
- Player 1 chooses the worst trace $\sigma \in \mathrm{Tr}(s)$.
- Player 2 matches it with the best trace $\tau \in \mathrm{Tr}(t)$.
- $d_L(s,t)$ = value of the "1-blind weighted simulation game": Player 2 has perfect information, Player 1 is blind.

Definition: Branching distance

$d_B(s,t)$ = value of the same game, but with perfect information

- Hence "$d_B(s,t) = \sup_{s \xrightarrow{\sigma_0} s_1} \inf_{t \xrightarrow{\tau_0} t_1} \sup_{s_1 \xrightarrow{\sigma_1} s_2} \inf_{t_1 \xrightarrow{\tau_1} t_2} \cdots \; d_T(\sigma,\tau)$".

Linear vs. Branching Distance

Precise definition of how this works:

- Imagine a game of two players taking turns to build two paths:
- A strategy from s, t: $\theta : \mathrm{fPa}(s) \times \mathrm{fPa}(t) \to T$
  - for Player 1: $\mathrm{start}(\theta(\pi_1, \pi_2)) = \mathrm{end}(\pi_1)$
  - for Player 2: $\mathrm{start}(\theta(\pi_1, \pi_2)) = \mathrm{end}(\pi_2)$
- A round of the game under strategies θ_1, θ_2:
  $\mathrm{Round}_{(\theta_1, \theta_2)}(\pi_1, \pi_2) = \big(\pi_1 \cdot \theta_1(\pi_1, \pi_2), \pi_2 \cdot \theta_2(\pi_1 \cdot \theta_1(\pi_1, \pi_2), \pi_2)\big)$
- The limit of the game under strategies θ_1, θ_2:
  $\mathrm{limit} = \lim_{j \to \infty} \mathrm{Round}^j_{(\theta_1, \theta_2)}(s_0, t_0)$ (a pair of infinite paths)
- The utility of the strategies θ_1, θ_2: $u(\theta_1, \theta_2) = d_T\big(\mathrm{tr}(\mathrm{limit})\big)$
- The value of the game: $v(s, t) = \sup_{\theta_1} \inf_{\theta_2} u(\theta_1, \theta_2)$

Perfect vs. Imperfect Information

- Θ_1, Θ_2: sets of all strategies $\text{fPa}(s) \times \text{fPa}(t) \to T$
- Games with imperfect information: Restrict available strategies to proper subsets of Θ_1 or Θ_2
- Special case: blind Player-1 strategies $\tilde{\Theta}_1 = T^{\text{fPa}(s)}$
- Do not depend on Player-2 choices: Player 1 cannot "see" what Player 2 is doing

- Branching distance: $d_B(s,t) = \sup_{\theta_1 \in \Theta_1} \inf_{\theta_2 \in \Theta_2} u(\theta_1, \theta_2)$
- Linear distance: $d_L(s,t) = \sup_{\theta_1 \in \tilde{\Theta}_1} \inf_{\theta_2 \in \Theta_2} u(\theta_1, \theta_2)$

Properties

Proposition

- d_L is a hemimetric
- if the game is determined, then d_B is a hemimetric

Theorem

For all $s, t \in S$, $d_L(s,t) \leq d_B(s,t)$.

Proof:

For d_B, Player 1 (the *sup player*) has more strategies to choose from!

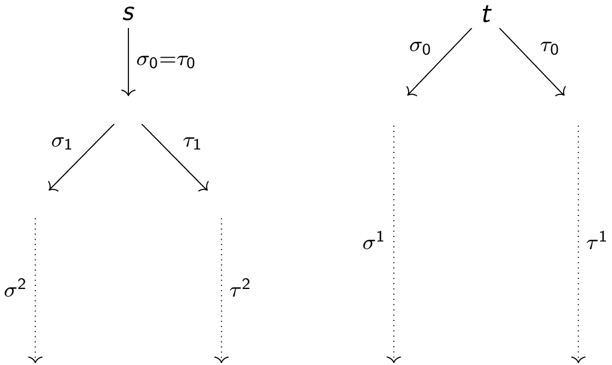
Theorem

There exists a weighted automaton on which d_L *and* d_B *are topologically inequivalent.*

- Unless for all traces $\sigma, \tau : \sigma_0 = \tau_0$ implies $d_T(\sigma, \tau) = 0$

Proof

Let $\sigma, \tau \in \mathbb{K}^\omega$ such that $\sigma_0 = \tau_0$, $d_T(\sigma, \tau) > 0$, and $d_T(\tau, \sigma) > 0$.



We have $\mathrm{Tr}(s) = \mathrm{Tr}(t)$, hence $d_L(s,t) = 0$. On the other hand, $d_B(s,t) = \min\big(d_T(\sigma,\tau), d_T(\tau,\sigma)\big) > 0$. That's it.

Fixed-Point Characterization

Theorem

If $d_T(\sigma, \tau) = F\big(\sigma_0, \tau_0, d_T(\sigma^1, \tau^1)\big)$ for some "iterator" function $F : \mathbb{K} \times \mathbb{K} \times [0, \infty] \to [0, \infty]$ which is monotone in the third coordinate and all $\sigma, \tau \in \mathbb{K}^\omega$, then d_B is the least fixed point to the set of equations

$$h(s, t) = \sup_{s \xrightarrow{x} s'} \inf_{t \xrightarrow{y} t'} F\big(x, y, h(s', t')\big)$$

- So if trace distance has a simple recursive characterization, then so does branching distance
- Applies to $d_T^\bullet$ and d_T^+, but not to $d_T^\pm$
- Have extension to "recursive characterization with memory" which applies to all examples given previously

Conclusion

Given:

- an arbitrary labeled transition system
- an arbitrary trace distance

we construct

- a linear system distance
- a branching system distance
- (corresponding to trace inclusion and simulation)

This generalizes a number of previous approaches.

Next paper: a host of other system distances

- Coming to a conf near you real soon.

FORMATS 2011

9th International Conference on Formal Modeling and Analysis of Timed Systems

Mathematical Wish List

- Relate equivalence of trace distances to equivalence of linear distances. Like this:

Theorem

If trace distances d_T^1 and d_T^2 are Lipschitz equivalent, then the corresponding linear distances d_L^1 and d_L^2 are topologically equivalent.

- Relate equivalence of trace distances to equivalence of branching distances
- Classify trace distances (up to equivalence)