Generalized Quantitative Analysis of Metric Transition Systems

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Elevator Statement

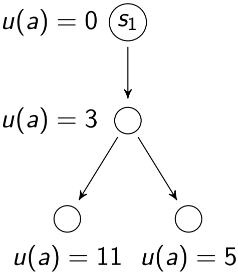
When formal models include quantities,
the standard Boolean relations
such as simulation, language inclusion, bisimulation, etc.
have little use.
They need to be replaced by distances.
There is, however, a lot of disagreement
how precisely to do this,
so a unifying metric theory
of quantitative analysis
is called for.

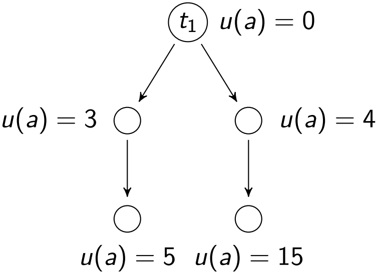
Metric Transition Systems

Quantitative model du jour:

- metric transition system: $(S, T, [\cdot])$, with $[\cdot] : S \to \mathcal{U}[\Sigma]$
- Σ: atomic propositions; $\mathcal{U}[\Sigma]$: set of valuations $u : \Sigma \to X$
- (X, d): (extended) hemimetric space
- (hemimetric: asymmetric pseudometric)
- essentially the setting from [Alfaro, Faella, Stoelinga: Linear and branching system metrics, IEEE Trans. Softw. Eng. 35(2):258–273, 2009]

Example





Distances

Propositional distance:

$$pd(u, v) = \sup_{a \in \Sigma} d(u(a), v(a))$$

State distance:

$$pd(s, t) = pd([s], [t])$$

- syntactic distance between states
- want: semantic distance between states' behaviors

Measuring distances between behaviors

- behavior = trace (finite or infinite)
- point-wise trace distance:

$$td(\sigma,\tau)=\begin{cases}\infty & \text{if } \mathrm{len}(\sigma)\neq\mathrm{len}(\tau),\\ \sup_i pd(\sigma_i,\tau_i) & \text{otherwise.}\end{cases}$$

- discounted accumulating trace distance ($\lambda \in]0,1[$):

$$td(\sigma,\tau)=\begin{cases}\infty & \text{if } \mathrm{len}(\sigma)\neq\mathrm{len}(\tau),\\ \sum_i \lambda^i pd(\sigma_i,\tau_i) & \text{otherwise.}\end{cases}$$

- limit-average trace distance:

$$td(\sigma,\tau)=\begin{cases}\infty & \text{if } \mathrm{len}(\sigma)\neq\mathrm{len}(\tau),\\ \liminf_j \frac{1}{j+1}\sum_{i=0}^{j} pd(\sigma_i,\tau_i) & \text{otherwise.}\end{cases}$$

- and a bunch of others, all with their own reasonable motivation

From behavioral distance to semantic state distance

[AFS09] consider only (discounted) point-wise distance:

- trace distance (recall): $td(\sigma, \tau) = \sup_i pd(\sigma_i, \tau_i)$
- linear distance:

$$ld(s, t) = \sup_{\sigma \in \mathrm{Tr}(s)} \inf_{\tau \in \mathrm{Tr}(t)} td(\sigma, \tau)$$

- generalizes trace inclusion; has symmetric cousin
- branching distance: least fixed point to

$$sd(s, t) = \sup_{s \to s'} \inf_{t \to t'} \max\{sd(s, t), sd(s', t')\}$$

- generalizes simulation; has symmetric cousin

How to generalize this to all the other useful distances?

General framework for system distances

Given: trace distance *td*. Want: linear & branching distances *ld*, *sd*

- for a set M, let $\mathbb{L}M = M \to \mathbb{R}_{\geq 0} \cup \{\infty\}$
  - complete lattice; $\alpha \sqsubseteq \beta$ iff $\forall x. \alpha(x) \leq \beta(x)$
  - addition $\alpha \oplus \beta = \lambda x. \alpha(x) + \beta(x)$ ("Girard quantale")

Definition

A *recursive specification* of a trace distance *td* consists of

- a set M and a lattice homomorphism $\mathrm{eval} : \mathbb{L}M \to \mathbb{R}_{\geq 0} \cup \{\infty\}$,
- a hemimetric $td^{\mathbb{L}} : \mathcal{U}[\Sigma]^\infty \times \mathcal{U}[\Sigma]^\infty \to \mathbb{L}M$ s.t. $td = \mathrm{eval} \circ td^{\mathbb{L}}$,
- and a distance iterator $F : \mathcal{U}[\Sigma] \times \mathcal{U}[\Sigma] \times \mathbb{L}M \to \mathbb{L}M$.

F must be monotone in the third coordinate and satisfy

$$td^{\mathbb{L}}(u.\sigma, v.\tau) = F(u, v, td^{\mathbb{L}}(\sigma, \tau))$$

Examples of recursive specifications

- point-wise: $M = \{*\}$

$$td(u.\sigma, v.\tau) = \max(pd(u, v), td(\sigma, \tau))$$

- discounted accumulating: $M = \{*\}$

$$td(u.\sigma, v.\tau) = pd(u, v) + \lambda td(\sigma, \tau)$$

- limit-average: $M = \mathbb{N}$

$$td^{\mathbb{L}}(u.\sigma, v.\tau)(j) = \frac{1}{j+1} pd(u, v) + \frac{j}{j+1} td(\sigma, \tau)$$
$$td(\sigma, \tau) = \liminf_j td^{\mathbb{L}}(\sigma, \tau)(j)$$

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$$td(\sigma, \tau) = \liminf_{j} td^{\mathbb{L}}(\sigma, \tau)(j)$$

All commonly used trace distances have recursive specifications.

From recursive specification to linear & branching distance

Given: trace distance td with recursive specification $td = \text{eval} \circ td^{\mathbb{L}}$, $td^{\mathbb{L}}(u.\sigma, v.\tau) = F(u, v, td^{\mathbb{L}}(\sigma, \tau))$

Definition

The linear distance from s to t is

$$ld(s, t) = \sup_{\sigma \in \text{Tr}(s)} \inf_{\tau \in \text{Tr}(t)} td(\sigma, \tau)$$

The branching distance from s to t is $sd = \text{eval} \circ sd^{\mathbb{L}}$, with $sd^{\mathbb{L}}$ the least fixed point to

$$sd^{\mathbb{L}}(s, t) = \sup_{s \to s'} \inf_{t \to t'} F([s], [t], sd^{\mathbb{L}}(s', t'))$$

Conclusion

- From a recursive specification of a trace distance, we get definitions of corresponding linear and branching distances
- These are generalizations of trace inclusion and simulation
- Theorem: always $ld(s, t) \leq sd(s, t)$
- This generalizes a number of approaches in the litterature
- Similarly one can get: trace equivalence distance, bisimulation distance, nested simulation distance, ready trace distance, etc.
- A quantitative linear-time–branching-time spectrum!

- Next step: Transfer this to probabilistic automata and relate to prior work in this area