From Linear to Branching Distances
The Role of Recursion

Uli Fahrenberg Axel Legay

IRISA / Inria, Rennes, France

Dagstuhl 2014

Motivation

$$d(A, B) = \sup_{w \in \Sigma^*} d(\llbracket A \rrbracket(w), \llbracket B \rrbracket(w))$$

Motivation

$$d(A, B) = \sup_{w \in \Sigma^*} d(\llbracket A \rrbracket(w), \llbracket B \rrbracket(w))$$

$$d(A, B) = \sup_{u \in \llbracket A \rrbracket} \inf_{v \in \llbracket B \rrbracket} d(u, v)$$

Motivation

$$d(A, B) = \sup_{w \in \Sigma^*} d(\llbracket A \rrbracket(w), \llbracket B \rrbracket(w))$$

$$d(A, B) = \sup_{u \in \llbracket A \rrbracket} \inf_{v \in \llbracket B \rrbracket} d(u, v)$$

$$d(s, t) = \sup_{s \xrightarrow{a} s'} \inf_{t \xrightarrow{b} t'} d(a, b) + d(s', t') \quad \text{(l.f.p.)}$$

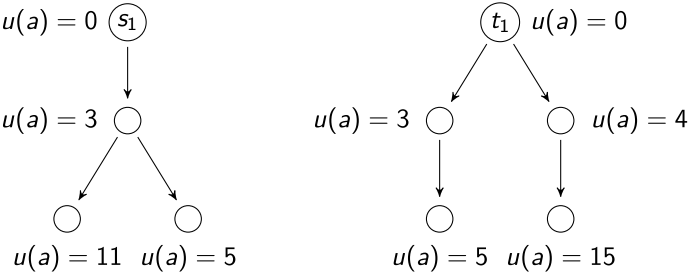
1. Metric Transition Systems
2. Distances
3. General framework for system distances
4. Conclusion

Metric Transition Systems

Quantitative model du jour:

- metric transition system: $(S, T, [\cdot])$, with $[\cdot] : S \to \mathcal{U}[\Sigma]$
- Σ: atomic propositions; $\mathcal{U}[\Sigma]$: set of valuations $u : \Sigma \to X$
- (X, d): (extended) hemimetric space
- (hemimetric: asymmetric pseudometric)
- essentially the setting from [Alfaro, Faella, Stoelinga: Linear and branching system metrics, IEEE Trans. Softw. Eng. 35(2):258–273, 2009]

Example



Distances

Propositional distance:

$$pd(u, v) = \sup_{a \in \Sigma} d(u(a), v(a))$$

State distance:

$$pd(s, t) = pd([s], [t])$$

- syntactic distance between states
- want: semantic distance between states' behaviors

Measuring distances between behaviors

- behavior = trace (finite or infinite)
- point-wise trace distance:

$$td(\sigma, \tau) = \begin{cases} \infty & \text{if } \mathsf{len}(\sigma) \neq \mathsf{len}(\tau), \\ \sup_i pd(\sigma_i, \tau_i) & \text{otherwise.} \end{cases}$$

- discounted accumulating trace distance ($\lambda \in]0, 1[$):

$$td(\sigma, \tau) = \begin{cases} \infty & \text{if } \mathsf{len}(\sigma) \neq \mathsf{len}(\tau), \\ \sum_i \lambda^i pd(\sigma_i, \tau_i) & \text{otherwise.} \end{cases}$$

- limit-average trace distance:

$$td(\sigma, \tau) = \begin{cases} \infty & \text{if } \mathsf{len}(\sigma) \neq \mathsf{len}(\tau), \\ \liminf_j \frac{1}{j+1} \sum_{i=0}^{j} pd(\sigma_i, \tau_i) & \text{otherwise.} \end{cases}$$

- and a bunch of others, all with their own reasonable motivation

From behavioral distance to semantic state distance

[AFS09] consider only (discounted) point-wise distance:

- trace distance (recall): $td(\sigma, \tau) = \sup_i pd(\sigma_i, \tau_i)$
- linear distance:

$$ld(s,t) = \sup_{\sigma \in \mathrm{Tr}(s)} \inf_{\tau \in \mathrm{Tr}(t)} td(\sigma, \tau)$$

- generalizes trace inclusion; has symmetric cousin
- branching distance: least fixed point to

$$sd(s,t) = \sup_{s \to s'} \inf_{t \to t'} \max\{sd(s,t), sd(s',t')\}$$

- generalizes simulation; has symmetric cousin

How to generalize this to all the other useful distances?

General framework for system distances

Given: trace distance *td*. Want: linear & branching distances *ld*, *sd*

- for a set M, let $\mathbb{L}M = M \to \mathbb{R}_{\geq 0} \cup \{\infty\}$
  - complete lattice; $\alpha \sqsubseteq \beta$ iff $\forall x. \alpha(x) \leq \beta(x)$
  - addition $\alpha \oplus \beta = \lambda x. \alpha(x) + \beta(x)$ ("Girard quantale")

Definition

A *recursive specification* of a trace distance *td* consists of

- a set M and a lattice homomorphism $\text{eval} : \mathbb{L}M \to \mathbb{R}_{\geq 0} \cup \{\infty\}$,
- a hemimetric $td^{\mathbb{L}} : \mathcal{U}[\Sigma]^\infty \times \mathcal{U}[\Sigma]^\infty \to \mathbb{L}M$ s.t. $td = \text{eval} \circ td^{\mathbb{L}}$,
- and a distance iterator $F : \mathcal{U}[\Sigma] \times \mathcal{U}[\Sigma] \times \mathbb{L}M \to \mathbb{L}M$.

F must be monotone in the third coordinate and satisfy

$$td^{\mathbb{L}}(u.\sigma, v.\tau) = F(u, v, td^{\mathbb{L}}(\sigma, \tau))$$

Examples of recursive specifications

- point-wise: $M = \{*\}$

$$td(u.\sigma, v.\tau) = \max(pd(u, v), td(\sigma, \tau))$$

- discounted accumulating: $M = \{*\}$

$$td(u.\sigma, v.\tau) = pd(u, v) + \lambda td(\sigma, \tau)$$

- limit-average: $M = \mathbb{N}$

$$td^{\mathbb{L}}(u.\sigma, v.\tau)(j) = \frac{1}{j+1} pd(u, v) + \frac{j}{j+1} td(\sigma, \tau)$$

$$td(\sigma, \tau) = \liminf_{j} td^{\mathbb{L}}(\sigma, \tau)(j)$$

Examples of recursive specifications

- point-wise: $M = \{*\}$

$$td(u.\sigma, v.\tau) = \max(pd(u, v), td(\sigma, \tau))$$

- discounted accumulating: $M = \{*\}$

$$td(u.\sigma, v.\tau) = pd(u, v) + \lambda td(\sigma, \tau)$$

- limit-average: $M = \mathbb{N}$

$$td^{\mathbb{L}}(u.\sigma, v.\tau)(j) = \frac{1}{j+1} pd(u, v) + \frac{j}{j+1} td(\sigma, \tau)$$
$$td(\sigma, \tau) = \liminf_{j} td^{\mathbb{L}}(\sigma, \tau)(j)$$

All commonly used trace distances have recursive specifications.

From recursive specification to linear & branching distance

Given: trace distance td with recursive specification $td = \text{eval} \circ td^{\mathbb{L}}$, $td^{\mathbb{L}}(u.\sigma, v.\tau) = F(u, v, td^{\mathbb{L}}(\sigma, \tau))$

Definition

The linear distance from s to t is

$$ld(s, t) = \sup_{\sigma \in \text{Tr}(s)} \inf_{\tau \in \text{Tr}(t)} td(\sigma, \tau)$$

The branching distance from s to t is $sd = \text{eval} \circ sd^{\mathbb{L}}$, with $sd^{\mathbb{L}}$ the least fixed point to

$$sd^{\mathbb{L}}(s, t) = \sup_{s \to s'} \inf_{t \to t'} F([s], [t], sd^{\mathbb{L}}(s', t'))$$

Conclusion

- From a recursive specification of a trace distance, we get definitions of corresponding linear and branching distances
- These are generalizations of trace inclusion and simulation
- Theorem: always $ld(s,t) \leq sd(s,t)$
- This generalizes a number of approaches in the litterature
- Similarly one can get: trace equivalence distance, bisimulation distance, nested simulation distance, ready trace distance, etc.
- A quantitative linear-time–branching-time spectrum!