Partial Higher-Dimensional Automata

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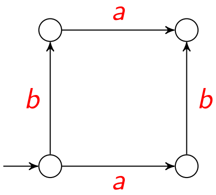
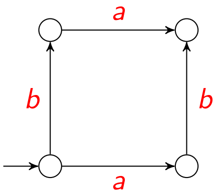
Motivation

$a|b$

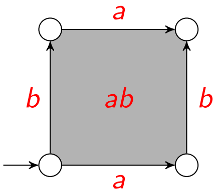


Motivation

$a|b$



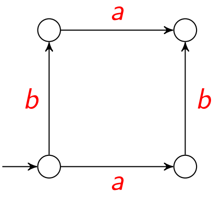
$a.b + b.a$



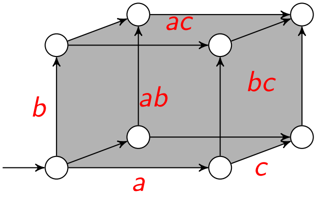
a and b are independent

Motivation

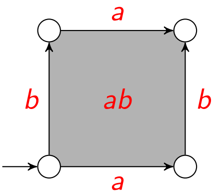
$a|b$

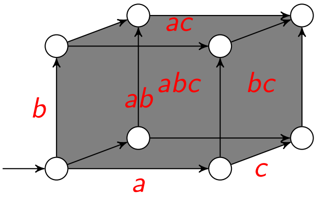


$a|b|c$



$a|b + a|c + b|c$

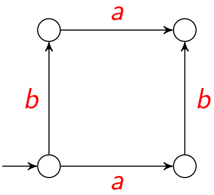


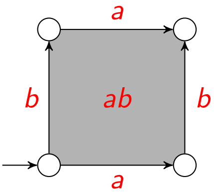


$\{a, b, c\}$ independent

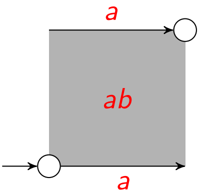
Motivation

$a|b$



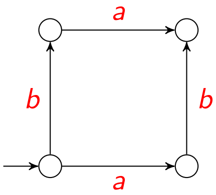


b “inside” a

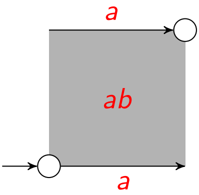


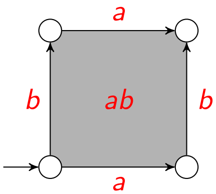
Motivation

$a|b$

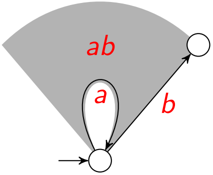


b "inside" a





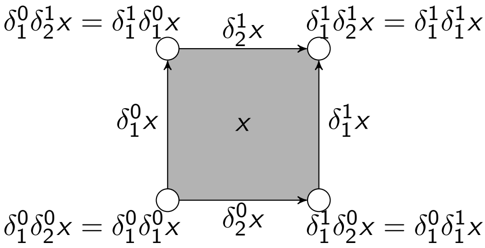
a looping; b priority



Higher-dimensional automata

A precubical set:

- a graded set $X = \{X_n\}_{n \in \mathbb{N}}$
- in each dimension n, $2n$ face maps $\delta_k^0, \delta_k^1 : X_n \to X_{n-1}$ $(k = 1, \dots, n)$
- the precubical identity: $\delta_k^\nu \delta_\ell^\mu = \delta_{\ell-1}^\mu \delta_k^\nu$ for all $k < \ell$



A higher-dimensional automaton: a pointed precubical set (precubical set with initial state)

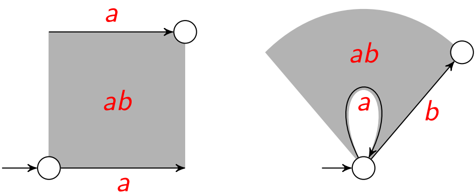
Higher-dimensional automata

HDA as a model for concurrency:

- points $x \in X_0$: states
- edges $a \in X_1$: transitions (labeled with events)
- n-squares $\alpha \in X_n$ ($n \geq 2$): independency relations (concurrently executing events)

van Glabbeek (TCS 2006): Up to history-preserving bisimilarity, HDA generalize "the main models of concurrency proposed in the literature"

Partial HDA



A partial precubical set (PPS):

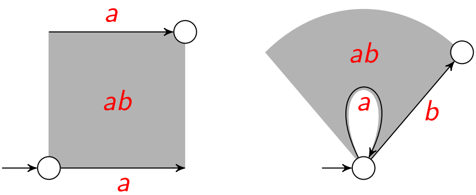
- a graded set $X = \{X_n\}_{n \in \mathbb{N}}$
- in each dimension n, partial face maps $\delta_k^0, \delta_k^1 : X_n \rightharpoonup X_{n-1}$ $(k = 1, \dots, n)$
- the precubical identity: $\delta_k^\nu \delta_\ell^\mu = \delta_{\ell-1}^\mu \delta_k^\nu$ for all $k < \ell$ whenever defined

A partial higher-dimensional automaton: a pointed partial precubical set

A labeled PHDA over alphabet Σ:

- n-cubes labeled with elements of Σ^n
- compatible with boundaries

Partial HDA



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A partial higher-dimensional automaton: a pointed partial precubical set

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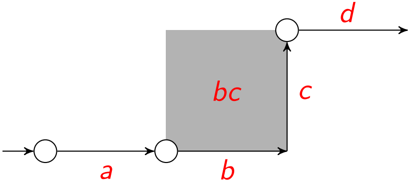
pointed comma category
$* \to \text{PPS} \to !\Sigma$

Higher-dimensional paths

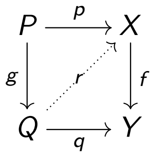
- a computation in a PHDA: a cube path: sequence $x_1, \dots, x_n$ of cubes connected by face maps, i.e. s.t. $x_i = \delta_k^0 x_{i+1}$ or $x_{i+1} = \delta_k^1 x_i$



- $x_i = \delta_k^0 x_{i+1}$: start of a new concurrent event
- $x_{i+1} = \delta_k^1 x_i$: end of a concurrent event
- a path object: a cube path with no extra relations
- HDP $\hookrightarrow$ HDA: subcategory of pointed path objects and path extensions (not full)

Open-maps bisimilarity

- PHDA morphism $f : X \to Y$ open if right-lifting w.r.t. HDP:



- PHDA X, Y bisimilar if span $X \leftarrow Z \to Y$ of open maps

- Theorem: PHDA X, Y bisimilar iff $\exists$ PHDA $R \subseteq X \times Y$ s.t. $\forall$ reachable $x \in X$, $y \in Y$ with $(x, y) \in R$:
  - $\forall x' = \delta_k^1 x : \exists y' = \delta_k^1 y : (x', y') \in R$
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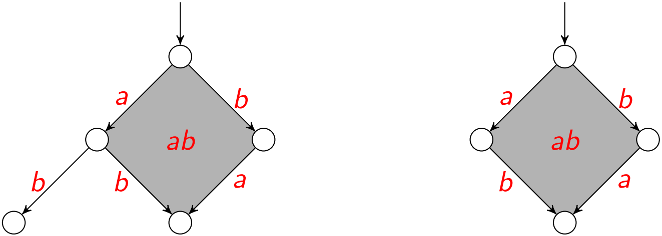
Open-maps bisimilarity

- PHDA morphism $f : X \to Y$ open if right-lifting w.r.t. HDP:

$$\begin{array}{ccc} P & \xrightarrow{p} & X \\ {\scriptstyle g}\downarrow & \overset{r}{\nearrow} & \downarrow{\scriptstyle f} \\ Q & \xrightarrow[q]{} & Y \end{array}$$

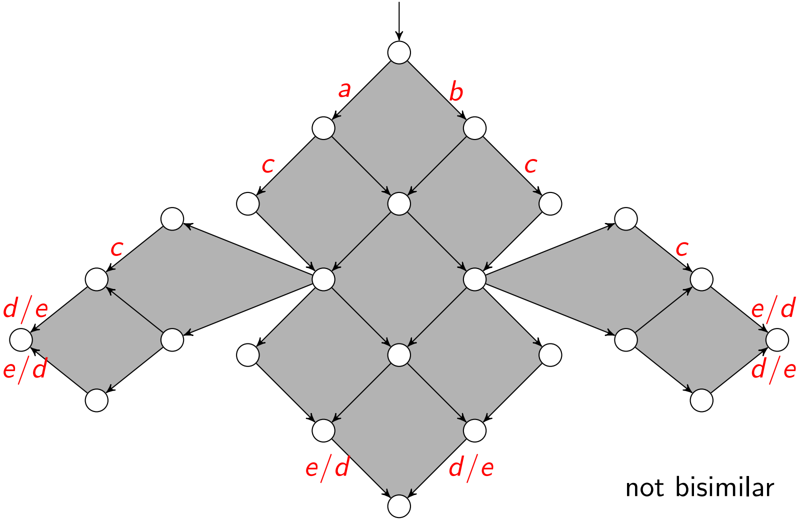
- PHDA X, Y bisimilar if span $X \leftarrow Z \to Y$ of open maps
- Theorem: PHDA X, Y bisimilar iff $\exists$ PHDA $R \subseteq X \times Y$ s.t. $\forall$ reachable $x \in X$, $y \in Y$ with $(x, y) \in R$:
  - $\forall x' = \delta^1_k x : \exists y' = \delta^1_k y : (x', y') \in R$
  - $\forall y' = \delta^1_k y : \exists x' = \delta^1_k x : (x', y') \in R$ (finish action)
  - $\forall x = \delta^0_k x' : \exists y = \delta^0_k y' : (x', y') \in R$
  - $\forall y = \delta^0_k y' : \exists x = \delta^0_k x' : (x', y') \in R$ (start action)

Open-maps bisimilarity



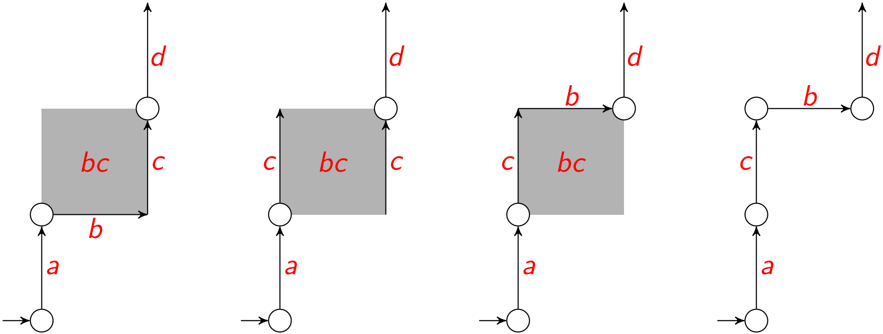
bisimilar

Open-maps bisimilarity



not bisimilar

Homotopy of computations



cube paths $x_1, \dots, x_n$, $y_1, \dots, y_n$ p-adjacent ($\overset{p}{\sim}$) if $x_i = y_i$ for $i \neq p$, and

- x_p and y_p are distinct lower faces of x_{p+1}, or
- x_p and y_p are distinct upper faces of x_{p-1}, or
- x_{p-1}, x_{p+1} are lower and upper faces of x_p, and y_p is an upper face of x_{p-1} and a lower face of x_{p+1}, or vice versa

homotopy $\sim$: reflexive, transitive closure of adjacency

Unfoldings

The unfolding of a PHDA:

- unfolding up to homotopy, AKA universal covering
- unfolding of PHDA X is $\tilde{X}$, set of homotopy classes of cube paths in X
- with suitable face maps:
  - $\tilde{\delta}_k^1[x_1, \dots, x_m] = [x_1, \dots, x_m, \delta_k^1 x_m]$ if $\delta_k^1 x_m$ exists; otherwise undefined
  - $\tilde{\delta}_k^0[x_1, \dots, x_m] = \big\{(y_1, \dots, y_p) \mid y_p = \delta_k^0 x_m, (y_1, \dots, y_p, x_m) \sim (x_1, \dots, x_m)\big\}$ provided this set is non-empty; else undefined
- and a projection $\pi_X : \tilde{X} \to X$

Unfoldings

Properties:

- unfoldings are (partial) higher-dimensional trees
- if X is a higher-dimensional tree, then $\pi_X : \tilde{X} \to X$ is an isomorphism
- projections $\pi_X : \tilde{X} \to X$ are open maps
- hence: PHDA X, Y are bisimilar iff $\tilde{X}$ and $\tilde{Y}$ are bisimilar

History-preserving bisimilarity

Let $* \xrightarrow{i} X \xrightarrow{\lambda} !\Sigma$, $* \xrightarrow{j} Y \xrightarrow{\mu} !\Sigma$ be labeled PHDA.

Theorem

X and Y are bisimilar iff $\exists$ relation R between pointed cube paths in X and Y for which $((i),(j)) \in R$, and such that for all $(\rho, \sigma) \in R$,

- $\lambda(\rho) \sim \mu(\sigma)$,
- $\forall \rho \rightsquigarrow \rho' : \exists \sigma \rightsquigarrow \sigma' : (\rho', \sigma') \in R$,
- $\forall \sigma \rightsquigarrow \sigma' : \exists \rho \rightsquigarrow \rho' : (\rho', \sigma') \in R$,
- $\forall \rho \sim \rho' : \exists \sigma \sim \sigma' : (\rho', \sigma') \in R$,
- $\forall \sigma \sim \sigma' : \exists \rho \sim \rho' : (\rho', \sigma') \in R$.

Definition

X and Y are history-preserving bisimilar iff $\exists$ relation R between pointed cube paths in X and Y for which $((i),(j)) \in R$, and such that $\forall (\rho, \sigma) \in R$,

- $\lambda(\rho) = \mu(\sigma)$,
- $\forall \rho \rightsquigarrow \rho' : \exists \sigma \rightsquigarrow \sigma' : (\rho', \sigma') \in R$,
- $\forall \sigma \rightsquigarrow \sigma' : \exists \rho \rightsquigarrow \rho' : (\rho', \sigma') \in R$,
- $\forall \rho \overset{p}{\sim} \rho' : \exists \sigma \overset{p}{\sim} \sigma' : (\rho', \sigma') \in R$,
- $\forall \sigma \overset{p}{\sim} \sigma' : \exists \rho \overset{p}{\sim} \rho' : (\rho', \sigma') \in R$.

Conclusion

- Our bisimilarity is strictly weaker than history-preserving bisimilarity, but not weaker than split bisimilarity.
- Its relation with ST-bisimilarity is unclear.

- But contrary to the others, our bisimilarity has a simple precubical definition (no paths!)
- and a simple game characterization,
- hence it is decidable in polynomial time (for finite PHDA).

- Coalgebraic characterization?
- Implementation?