*-Continuous Kleene ω-Algebras for Energy Problems

Uli Fahrenberg

Inria Rennes, France

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Kleene Algebras

- idempotent semiring $S = (S, \vee, \cdot, \perp, 1)$
- with an operation $^* : S \to S$ which computes least fixed points:
- for all $x, y \in S$:
  - yx^* is the least fixed point of $z = zx \vee y$,
  - x^*y is the least fixed point of $z = xz \vee y$,

  with respect to the natural order $x \leq y$ iff $x \vee y = y$

- Consequence (for $y = 1$):
  - x^* is the l.f.p. of $z = zx \vee 1$ and of $z = xz \vee 1$

*-Continuous Kleene Algebras

- Kleene algebra $S = (S, \vee, \cdot, {}^*, \perp, 1)$
- in which all infinite suprema $\bigvee \{x^n \mid n \geq 0\}$ exist,
- and such that for all $x, y, z \in S$, $yx^*z = \bigvee_{n \geq 0} yx^nz$

- Consequence (for $x = z = 1$): $x^* = \bigvee_{n \geq 0} x^n$
- L.f.p. properties of * also follow

- Consequence: loop abstraction

Continuous Kleene Algebras

- Kleene algebra $S = (S, \vee, \cdot, ^*, \perp, 1)$
- in which all suprema $\bigvee X$, $X \subseteq S$ exist,
- and such that for all $X \subseteq S$, $y, z \in S$, $y(\bigvee X)z = \bigvee yXz$

- All continuous Kleene algebras are *-continuous, but not vice-versa
  - Example: regular languages over some Σ
- [Kozen 1990 (MFCS)]: not all Kleene algebras are *-continuous
  - Counterexample is necessarily infinite

Matrix Semirings

S semiring, $n \geq 1$

- $S^{n\times n}$: semiring of $n \times n$-matrices over S
- (with matrix addition and multiplication)
- If S is a *-continuous Kleene algebra, then so is $S^{n\times n}$
- with $M^*_{i,j} = \bigvee_{m\geq 0} \bigvee_{1\leq k_1,\dots,k_m\leq n} M_{i,k_1} M_{k_1,k_2} \cdots M_{k_m,j}$
- and for $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$M^* = \begin{bmatrix} (a \vee bd^*c)^* & (a \vee bd^*c)^* bd^* \\ (d \vee ca^*b)^* ca^* & (d \vee ca^*b)^* \end{bmatrix}$$

(recursively)

Finite Runs in Weighted Automata

S *-continuous Kleene algebra, $n \ge 1$

- a weighted automaton over S (with n states): $A = (\alpha, M, \kappa)$
- $\alpha \in \{\perp, 1\}^n$ initial vector, $\kappa \in \{\perp, 1\}^n$ accepting vector, $M \in S^{n \times n}$ transition matrix
- finite behavior of A: $|A| = \alpha M^* \kappa$
- Theorem:

$$|A| = \bigvee \left\{ w_0 \cdots w_n \;\middle|\; s_i \xrightarrow{w_0} \cdots \xrightarrow{w_n} s_j \text{ accepting path in } S \right\}$$

$(s_i \xrightarrow{w_0} \cdots \xrightarrow{w_n} s_j$ accepting if $\alpha_i = \kappa_j = 1)$

Idempotent Semiring-Semimodule Pairs

- idempotent semiring $S = (S, \vee, \cdot, \bot, 1)$
- commutative idempotent monoid $V = (V, \vee, \bot)$
- left S-action $S \times V \to V$, $(s, v) \mapsto sv$
- such that for all $s, s' \in S$, $v \in V$:

$$
\begin{aligned}
(s \vee s')v &= sv \vee s'v & \qquad s(v \vee v') &= sv \vee sv' \\
(ss')v &= s(s'v) & \bot s &= \bot \\
s\bot &= \bot & 1v &= v
\end{aligned}
$$

Continuous Kleene ω-Algebras

- idempotent semiring-semimodule pair (S, V)
- where S is a continuous Kleene algebra,
- V is a complete lattice,
- and the S-action on V preserves all suprema in either argument,
- with an infinite product $\prod : S^\omega \to V$ such that:
  - For all $x_0, x_1, \ldots \in S$, $\prod x_n = x_0 \prod x_{n+1}$.
  - Let $x_0, x_1, \ldots \in S$ and $0 = n_0 \le n_1 \le \cdots$ a sequence which increases without a bound. Let $y_k = x_{n_k} \cdots x_{n_{k+1}-1}$ for all $k \ge 0$. Then $\prod x_n = \prod y_k$.
  - For all $X_0, X_1, \ldots \subseteq S$, $\prod(\bigvee X_n) = \bigvee\{\prod x_n \mid x_n \in X_n, n \ge 0\}$.

Matrix Semiring-Semimodule Pairs

(S, V) semiring-semimodule pair, $n \geq 1$

- $(S^{n \times n}, V^n)$ is again a semiring-semimodule pair
- (the action is matrix-vector product)
- If (S, V) is a continuous Kleene ω-algebra, then so is $(S^{n \times n}, V^n)$
- with $M_i^\omega = \bigvee_{1 \leq k_1, k_2, \ldots \leq n} M_{i,k_1} M_{k_1,k_2} \cdots$
- and for $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$M^\omega = \begin{bmatrix} (a \vee bd^*c)^\omega \vee (a \vee bd^*c)^* bd^\omega \\ (d \vee ca^*b)^\omega \vee (d \vee ca^*b)^* ca^\omega \end{bmatrix}$$

(recursively)

Infinite Runs in Weighted Automata

(S, V) continuous Kleene ω-algebra
(α, M, κ) weighted automaton over S

- Reorder $S = \{1, \dots, n\}$ so that $\kappa = (1, \dots, 1, \perp, \dots, \perp)$
  - i.e. the first $k \leq n$ states are accepting
- Büchi behavior of A: write $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, with $a \in S^{k \times k}$, then

$$\|A\| = \alpha \begin{bmatrix} (a + bd^* c)^\omega \\ d^* c (a + bd^* c)^\omega \end{bmatrix}$$

- Theorem:

$$\|A\| = \bigvee \Big\{ \prod w_n \;\Big|\; s_i \xrightarrow{w_0} \xrightarrow{w_1} \cdots \text{ Büchi path in } S \Big\}$$

($s_i \xrightarrow{w_0} \xrightarrow{w_1} \cdots$ Büchi path if $\alpha_i = 1$ and some s_j with $j \leq k$ is visited infinitely often)

Problem

continuous Kleene algebras	continuous Kleene ω-algebras
*-continuous Kleene algebras	???

Problem

continuous Kleene algebras	continuous Kleene ω-algebras
*-continuous Kleene algebras	*-continuous Kleene ω-algebras

[Esik, F., Legay 2015 (DLT)]

Generalized *-Continuous Kleene Algebras [EFL'15]

- semiring-semimodule pair (S, V)
- where S is a *-continuous Kleene algebra
- such that for all $x, y \in S$, $v \in V$, $xy^* v = \bigvee_{n \geq 0} xy^n v$

*-Continuous Kleene ω-Algebras [EFL'15]

- generalized *-continuous Kleene algebra (S, V)
- with an infinite product $\prod : S^\omega \to V$ such that:
  - For all $x_0, x_1, \ldots \in S$, $\prod x_n = x_0 \prod x_{n+1}$.
  - Let $x_0, x_1, \ldots \in S$ and $0 = n_0 \leq n_1 \leq \cdots$ a sequence which increases without a bound. Let $y_k = x_{n_k} \cdots x_{n_{k+1}-1}$ for all $k \geq 0$. Then $\prod x_n = \prod y_k$.
  - For all $x_0, x_1, \ldots, y, z \in S$, $\prod(x_n(y \vee z)) = \bigvee_{x_0', x_1', \ldots \in \{y,z\}} \prod x_n x_n'$.
  - For all $x, y_0, y_1, \ldots \in S$, $\prod x^* y_n = \bigvee_{k_0, k_1, \ldots \geq 0} \prod x^{k_n} y_n$.

Matrix Semiring-Semimodule Pairs, Revisited [EFL'15]

(S, V) *-continuous Kleene ω-algebra, $n \geq 1$

- $(S^{n\times n}, V^n)$ is a generalized *-continuous Kleene algebra
- with an operation $^\omega : S^{n\times n} \to V^n$ given by
  $$M_i^\omega = \bigvee_{1\leq k_1, k_2, \ldots \leq n} M_{i,k_1} M_{k_1,k_2} \cdots$$
  - (not a general infinite product)
- and for $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,
  $$M^\omega = \begin{bmatrix} (a \vee bd^*c)^\omega \vee (a \vee bd^*c)^* bd^\omega \\ (d \vee ca^*b)^\omega \vee (d \vee ca^*b)^* ca^\omega \end{bmatrix}$$

  (recursively)

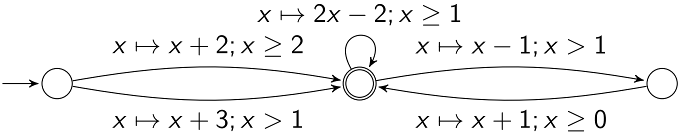
Energy Automata

Energy function:

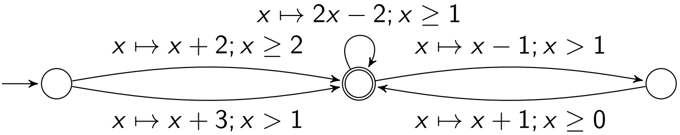
- partial function $f : \mathbb{R}_{\geq 0} \hookrightarrow \mathbb{R}_{\geq 0}$
- which is defined on some closed interval $[l_f, \infty[$ or on some open interval $]l_f, \infty[$,
- and such that for all $x \leq y$ for which f is defined,

$$f(y) - f(x) \geq y - x$$

Energy automaton: finite automaton labeled with energy functions



Energy Automata, Semantically



- Start with initial energy x_0 and update at transitions according to label function
- If label function undefined on input, transition is disabled

Reachability: Given x_0, does there exist an accepting (finite) run with initial energy x_0?

Büchi: Given x_0, does there exist a Büchi (infinite) run with initial energy x_0?

Energy Automata, Algebraically

- Let $L = [0, \infty]_\perp$: extended nonnegative real numbers plus bottom
  - (a complete lattice)
- Extended energy function: function $f : L \to L$
- with $f(\perp) = \perp$, and $f(\infty) = \infty$ unless $f(x) = \perp$ for all $x \in L$,
- and $f(y) - f(x) \geq y - x$ for all $x \leq y$.
- Set $\mathcal{E}$ of such functions is an idempotent semiring with operations $\vee$ (pointwise max) and $\circ$ (composition)
- in fact, a *-continuous Kleene algebra
  - $f^*(x) = x$ if $f(x) \leq x$; $f^*(x) = \infty$ if $f(x) > x$
  - not a continuous Kleene algebra

$\Longrightarrow$ There exists an accepting (finite) run from initial energy x_0 iff $|A|(x_0) \neq \perp$

Energy Automata, Algebraically, 2.

- Let $\mathbf{2} = \{\mathbf{ff}, \mathbf{tt}\}$: the Boolean lattice
- Let $\mathcal{V}$ be the set of monotone and $\top$-continuous functions $L \to \mathbf{2}$
  - $f : L \to \mathbf{2}$ $\top$-continuous if $f(x) \equiv \mathbf{ff}$ or for all $X \subseteq L$ with $\bigvee X = \infty$, also $\bigvee f(X) = \mathbf{tt}$.
- $(\mathcal{E}, \mathcal{V})$ is an idempotent semiring-semimodule pair
- Define $\prod : \mathcal{E}^\omega \to \mathcal{V}$ by

$$(\prod f_n)(x) = \mathbf{tt} \text{ iff } \forall n \geq 0 : f_n(f_{n-1}(\cdots(x)\cdots)) \neq \bot$$

- Lemma: $\prod f_n$ is indeed $\top$-continuous for all $f_0, f_1, \ldots \in \mathcal{E}$
- Theorem: $(\mathcal{V}, \mathcal{E})$ is a *-continuous Kleene ω-algebra
  - not a continuous Kleene ω-algebra

$\Longrightarrow$ There exists a Büchi run from initial energy x_0 iff $\|A\|(x_0) \neq \mathbf{ff}$

Conclusion

- *-continuous Kleene ω-algebras: a useful generalization of continuous Kleene ω-algebras
  - (like *-continuous Kleene algebras are a useful generalization of continuous Kleene algebras)
- can be used to solve general one-dimensional energy problems

Future / ongoing work:

- application to real-time energy problems (with D. Cachera; submitted)
- application to VASS? (with M. Droste & K. Quaas)