Partial Higher-Dimensional Automata

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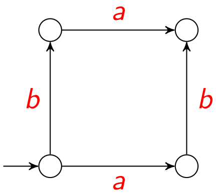
Motivation

$a|b$



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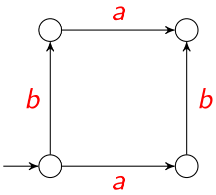
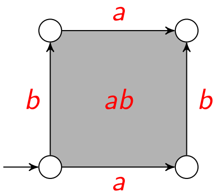


$a.b + b.a$



a and b are independent

Motivation

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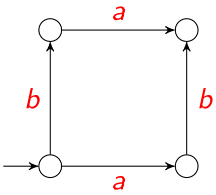
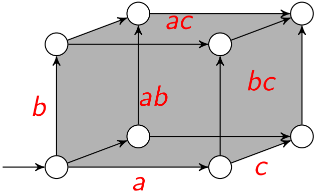
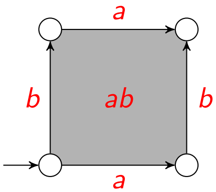
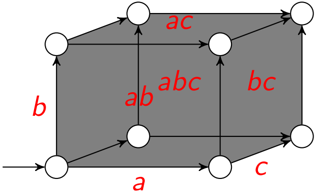


$a|b|c$



$a|b + a|c + b|c$

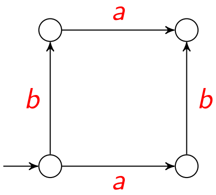


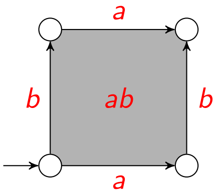


$\{a, b, c\}$ independent

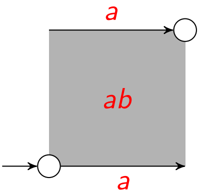
Motivation

$a|b$



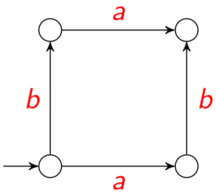


b “inside” a



Motivation

$a|b$



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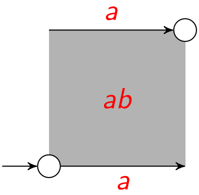
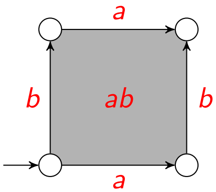
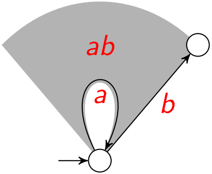




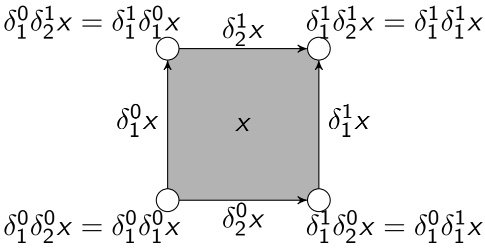
a looping; b priority



Higher-dimensional automata

A precubical set:

- a graded set $X = \{X_n\}_{n \in \mathbb{N}}$
- in each dimension n, $2n$ face maps $\delta_k^0, \delta_k^1 : X_n \to X_{n-1}$ ($k = 1, \dots, n$)
- the precubical identity: $\delta_k^\nu \delta_\ell^\mu = \delta_{\ell-1}^\mu \delta_k^\nu$ for all $k < \ell$



A higher-dimensional automaton: a pointed precubical set (precubical set with initial state)

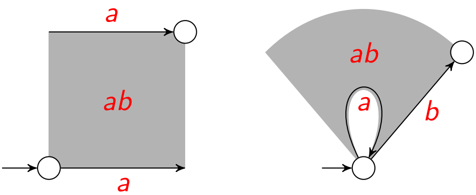
Higher-dimensional automata

HDA as a model for concurrency:

- points $x \in X_0$: states
- edges $a \in X_1$: transitions (labeled with events)
- n-squares $\alpha \in X_n$ ($n \geq 2$): independency relations (concurrently executing events)

van Glabbeek (TCS 2006): Up to history-preserving bisimilarity, HDA generalize "the main models of concurrency proposed in the literature"

Partial HDA



A partial precubical set (PPS):

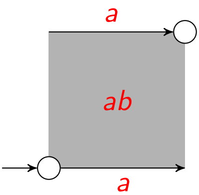
- a graded set $X = \{X_n\}_{n \in \mathbb{N}}$
- in each dimension n, partial face maps $\delta_k^0, \delta_k^1 : X_n \rightharpoonup X_{n-1}$ $(k = 1, \dots, n)$
- the precubical identity: $\delta_k^\nu \delta_\ell^\mu = \delta_{\ell-1}^\mu \delta_k^\nu$ for all $k < \ell$ whenever defined

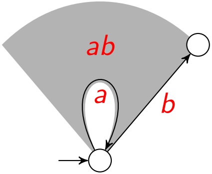
A partial higher-dimensional automaton: a pointed partial precubical set

A labeled PHDA over alphabet Σ:

- n-cubes labeled with elements of Σ^n
- compatible with boundaries

Partial HDA





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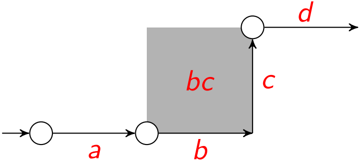
pointed comma category
$* \to \mathrm{PPS} \to !\Sigma$

Higher-dimensional paths

- a computation in a PHDA: a cube path: sequence $x_1, \dots, x_n$ of cubes connected by face maps, i.e. s.t. $x_i = \delta_k^0 x_{i+1}$ or $x_{i+1} = \delta_k^1 x_i$



- $x_i = \delta_k^0 x_{i+1}$: start of a new concurrent event
- $x_{i+1} = \delta_k^1 x_i$: end of a concurrent event
- a path object: a cube path with no extra relations
- $\mathrm{HDP} \hookrightarrow \mathrm{PHDA}$: subcategory of pointed path objects and path extensions (not full)

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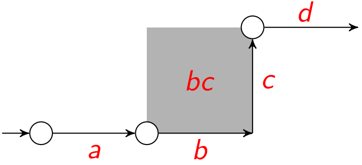

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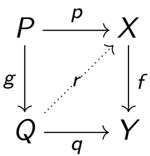
- a path object: a cube path with no extra relations
- $\mathrm{HDP} \hookrightarrow \mathrm{PHDA}$: subcategory of pointed path objects and path extensions (not full)
- (replacing cube paths with carrier sequences, i.e. $x_i = (\delta^0)^+ x_{i+1}$ or $x_{i+1} = (\delta^1)^+ x_i$, should not change much)

Open-maps bisimilarity

- PHDA morphism $f : X \to Y$ open if right-lifting w.r.t. HDP:



- PHDA X, Y bisimilar if span $X \leftarrow Z \to Y$ of open maps
- Theorem: PHDA X, Y bisimilar iff $\exists$ PHDA $R \subseteq X \times Y$ s.t. $\forall$ reachable $x \in X$, $y \in Y$ with $(x, y) \in R$:
  - $\forall x' = \delta_k^1 x : \exists y' = \delta_k^1 y : (x', y') \in R$
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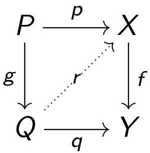

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  - $\forall x' = \delta_k^1 x : \exists y' = \delta_k^1 y : (x', y') \in R$
  - $\forall y' = \delta_k^1 y : \exists x' = \delta_k^1 x : (x', y') \in R$ (finish action)
  - $\forall x = \delta_k^0 x' : \exists y = \delta_k^0 y' : (x', y') \in R$
  - $\forall y = \delta_k^0 y' : \exists x = \delta_k^0 x' : (x', y') \in R$ (start action)

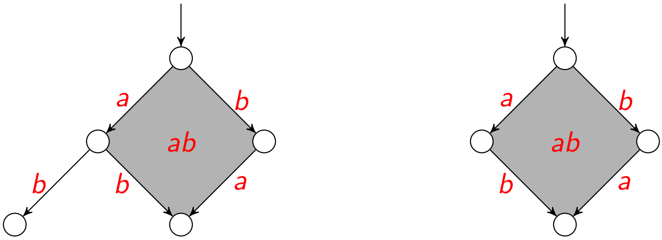
Wrong definition of open-maps bisimilarity using HDA

- HDA morphism $f : X \to Y$ open-s if right-lifting w.r.t. HDP-strict:

$$\begin{array}{ccc} P & \xrightarrow{p} & X \\ {\scriptstyle g}\downarrow & \overset{r}{\nearrow} & \downarrow{\scriptstyle f} \\ Q & \xrightarrow[q]{} & Y \end{array}$$

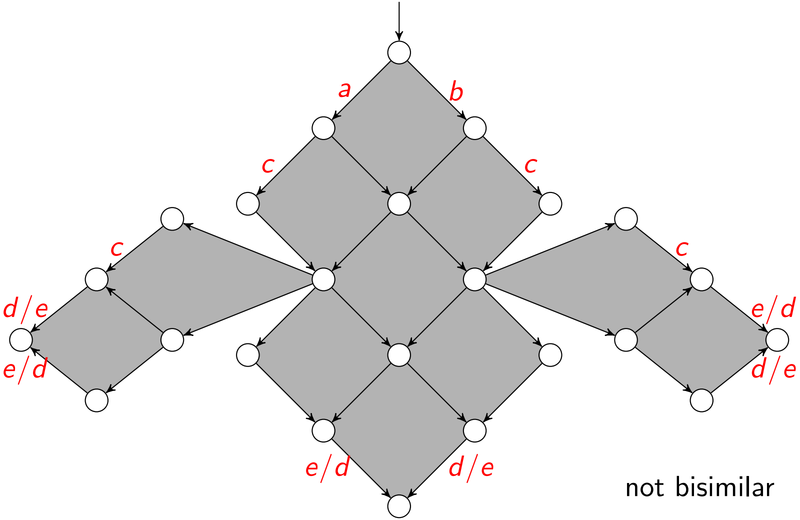
- HDA X, Y bisimilar-s if span $X \leftarrow Z \to Y$ of open-s maps
- Theorem: HDA X, Y bisimilar-s iff $\exists$ PHDA $R \subseteq X \times Y$ s.t. $\forall$ reachable $x \in X$, $y \in Y$ with $(x, y) \in R$:
  - $\forall k : (\delta^1_k x, \delta^1_k y) \in R$ (finish action)
  - $\forall x = \delta^0_k x' : \exists y = \delta^0_k y' : (x', y') \in R$ (start action)
  - $\forall y = \delta^0_k y' : \exists x = \delta^0_k x' : (x', y') \in R$ (start action)
  - $\forall k : (\delta^0_k x, \delta^0_k y) \in R$ (hereditarity with lower boundaries)

Open-maps bisimilarity



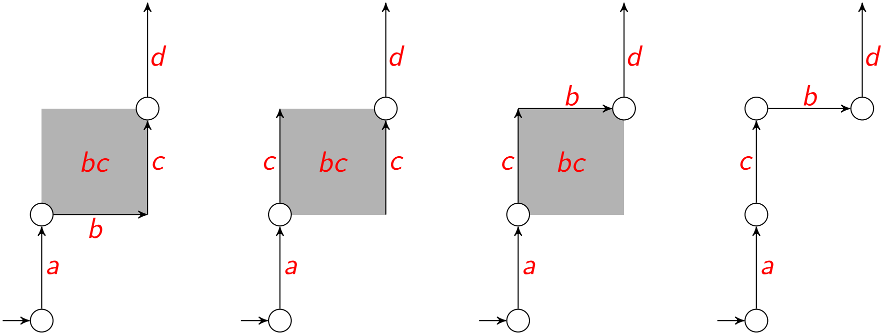
bisimilar

Open-maps bisimilarity



not bisimilar

Homotopy of computations



cube paths $x_1, \dots, x_n$, $y_1, \dots, y_n$ p-adjacent ($\overset{p}{\sim}$) if $x_i = y_i$ for $i \neq p$, and

- x_p and y_p are distinct lower faces of x_{p+1}, or
- x_p and y_p are distinct upper faces of x_{p-1}, or
- x_{p-1}, x_{p+1} are lower and upper faces of x_p, and y_p is an upper face of x_{p-1} and a lower face of x_{p+1}, or vice versa

homotopy $\sim$: reflexive, transitive closure of adjacency

Unfoldings

The unfolding of a PHDA:

- unfolding up to homotopy, AKA universal covering
- unfolding of PHDA X is $\tilde{X}$, set of homotopy classes of cube paths in X
- with suitable face maps:
  - $\tilde{\delta}^1_k[x_1, \dots, x_m] = [x_1, \dots, x_m, \delta^1_k x_m]$ if $\delta^1_k x_m$ exists; otherwise undefined
  - $\tilde{\delta}^0_k[x_1, \dots, x_m] = \big\{(y_1, \dots, y_p) \mid y_p = \delta^0_k x_m, (y_1, \dots, y_p, x_m) \sim (x_1, \dots, x_m)\big\}$ provided this set is non-empty; else undefined
- and a projection $\pi_X : \tilde{X} \to X$

Unfoldings

Properties:

- unfoldings are (partial) higher-dimensional trees
- if X is a higher-dimensional tree, then $\pi_X : \tilde{X} \to X$ is an isomorphism
- projections $\pi_X : \tilde{X} \to X$ are open maps
- hence: PHDA X, Y are bisimilar iff $\tilde{X}$ and $\tilde{Y}$ are bisimilar

History-preserving bisimilarity

Let $* \xrightarrow{i} X \xrightarrow{\lambda} !\Sigma$, $* \xrightarrow{j} Y \xrightarrow{\mu} !\Sigma$ be labeled PHDA.

Theorem

X and Y are bisimilar iff $\exists$ relation R between pointed cube paths in X and Y for which $((i),(j)) \in R$, and such that for all $(\rho,\sigma) \in R$,

- $\lambda(\rho) \sim \mu(\sigma)$,
- $\forall \rho \rightsquigarrow \rho' : \exists \sigma \rightsquigarrow \sigma' : (\rho',\sigma') \in R$,
- $\forall \sigma \rightsquigarrow \sigma' : \exists \rho \rightsquigarrow \rho' : (\rho',\sigma') \in R$,
- $\forall \rho \sim \rho' : \exists \sigma \sim \sigma' : (\rho',\sigma') \in R$,
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Definition

X and Y are history-preserving bisimilar iff $\exists$ relation R between pointed cube paths in X and Y for which $((i),(j)) \in R$, and such that $\forall (\rho,\sigma) \in R$,

- $\lambda(\rho) = \mu(\sigma)$,
- $\forall \rho \rightsquigarrow \rho' : \exists \sigma \rightsquigarrow \sigma' : (\rho',\sigma') \in R$,
- $\forall \sigma \rightsquigarrow \sigma' : \exists \rho \rightsquigarrow \rho' : (\rho',\sigma') \in R$,
- $\forall \rho \overset{p}{\sim} \rho' : \exists \sigma \overset{p}{\sim} \sigma' : (\rho',\sigma') \in R$,
- $\forall \sigma \overset{p}{\sim} \sigma' : \exists \rho \overset{p}{\sim} \rho' : (\rho',\sigma') \in R$.

Conclusion

- Our bisimilarity is strictly weaker than history-preserving bisimilarity, but not weaker than split bisimilarity.
- Its relation with ST-bisimilarity is unclear.

- But contrary to the others, our bisimilarity has a simple precubical definition (no paths!)
- and a simple game characterization,
- hence it is decidable in polynomial time (for finite PHDA).

Some next steps

- Coalgebraic characterization?
- Implementation?
- How is the geometry of PHDA?
  - Do Lisbeth's (et.al.) results on carrier sequences still hold?
  - Homotopy vs. dihomotopy
  - What is the geometric realization of a PHDA?

Natural homology of PHDA [DGGL15]

- concatenation of cube paths: if $\alpha = (x_1, \dots, x_n)$ and $\beta = (y_1, \dots, y_p)$ with $x_n = y_1$, then $\alpha * \beta = (x_1, \dots, x_n, y_2, \dots, y_p)$
- identities: $1_x := (x)$
- thus: small category $\mathrm{TrC}(X)$: objects cubes, morphisms cube paths

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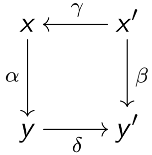
- identities: $1_x := (x)$
- thus: small category $\text{TrC}(X)$: objects cubes, morphisms cube paths
- let $Fc_X := F\text{TrC}(X)$: the category of factorizations of $\text{TrC}(X)$:
  - objects cube paths
  - morphisms commutative diagrams of extensions (γ, δ):



Natural homology of PHDA [DGGL15]

Let $X \in \mathrm{PPS}$ and $n \geq 1$. $\vec{H}_n(X) : Fc_X \to \mathrm{Ab}$ is the functor given by:

- on objects: $\alpha = (x_1, \dots, x_n)$ maps to $H_{n-1}(|X|; x_1^{\mathsf{ctr}}, x_n^{\mathsf{ctr}})$
- on morphisms: $(\gamma, \delta) : \alpha \to \beta$ maps to Ab-homomorphism $H_{n-1}(\gamma, \delta)$ given by $H_{n-1}(\gamma, \delta)([\pi]) = [\gamma * \pi * \delta]$
- (x^{ctr}: the center point of the geometric realization $|x| \subseteq |X|$; Ab: category of Abelian groups)

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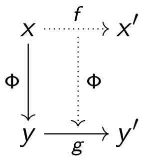
- (x^{ctr}: the center point of the geometric realization $|x| \subseteq |X|$; Ab: category of Abelian groups)
- (I hope I got this right?)
- (What do you do with the initial state? This info seems to get lost?)
- (This needs work on the geometry of partial precubical sets!)

Open maps in natural homology [DGGL15]

Let $X, Y \in \mathrm{Cat}$ and $F : X \to \mathrm{Ab}$, $G : Y \to \mathrm{Ab}$ functors.

- a morphism $F \to G$: a functor $\Phi : X \to Y$ and a natural transformation $\sigma : F \stackrel{\cdot}{\to} G \circ \Phi$
  - ([DGGL15] require σ to be a natural iso. Why?)
- $(\Phi, \sigma) : F \to G$ an open map iff
  - Φ surjective on objects
  - σ a natural iso
  - $\forall x \in X : \forall g : \Phi(x) \to y' \in Y : \exists f : x \to x' \in X : \Phi(f) = g$



Open maps in PHDA vs. in natural homology

Let $f : X \to Y \in \mathrm{PHDA}$ and $n \geq 1$.

- f induces a functor $f : Fc_X \to Fc_Y$: for $\alpha = (x_1, \dots, x_n)$ cube path, $f(\alpha) := (f(x_1), \dots, f(x_n))$
- Conjecture: f induces a morphism $(f, \sigma) : \vec{H}_n(X) \to \vec{H}_n(Y)$, where $\sigma : \vec{H}_n(X) \dot{\to} \vec{H}_n(Y) \circ f$ is as follows: for $\alpha = (x_1, \dots, x_n)$, $\sigma_\alpha : H_{n-1}(|X|; x_1^{\mathrm{ctr}}, x_n^{\mathrm{ctr}}) \to H_{n-1}(|Y|; f(x_1)^{\mathrm{ctr}}, f(x_n)^{\mathrm{ctr}})$ is the homomorphism $\sigma_\alpha([\pi]) = [|f| \circ \pi]$
- Wish: If f is an open map and Y is completely reachable, then (f, σ) is an open map.

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  - f surjective on objects: because Y completely reachable ✓
  - σ a natural iso: seems plausible ☺
  - f lifts extensions: probably not!

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  - σ a natural iso: seems plausible ☺
  - f lifts extensions: probably not!
  - f lifts future extensions $(1, \delta)$, but not past extensions $(\gamma, 1)$

"Conclusion"

- PHDA morphisms induce morphisms in natural homology
- but PHDA open maps do not induce open maps in natural homology?
- fix PHDA open maps? $f : X \to Y$ open iff $\forall$ reachable $x_1 \in X$:
  - $\forall y_2 \in Y$ with $f(x_1) = \delta_k^0 y_2$, $\exists x_2 \in X : x_1 = \delta_k^0 x_2, y_2 = f(x_2)$
  - $\forall y_2 \in Y$ with $y_2 = \delta_k^1 f(x_1)$, $\exists x_2 \in X : x_2 = \delta_k^1 x_1, y_2 = f(x_2)$

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- but what is the path category? what is the relation to (h)hp-bisimilarity? is this useful?
- fix natural homology??