A *-Continuous Kleene ω-Algebra for Real-Time Energy Problems

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Outline

Kleene Algebras

- idempotent semiring $S = (S, \vee, \cdot, \bot, 1)$
- with an operation $^* : S \to S$ which computes least fixed points:
- for all $x, y \in S$:
  - yx^* is the least fixed point of $z = zx \vee y$,
  - x^*y is the least fixed point of $z = xz \vee y$,

  with respect to the natural order $x \leq y$ iff $x \vee y = y$

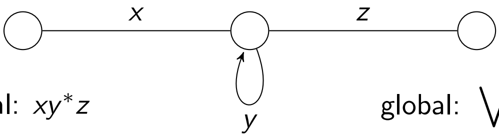
- Consequence (for $y = 1$):
  - x^* is the l.f.p. of $z = zx \vee 1$ and of $z = xz \vee 1$

*-Continuous Kleene Algebras

- Kleene algebra $S = (S, \vee, \cdot, ^*, \bot, 1)$
- in which all infinite suprema $\bigvee\{x^n \mid n \geq 0\}$ exist,
- and such that for all $x, y, z \in S$, $xy^*z = \bigvee_{n \geq 0} xy^nz$

- Consequence (for $x = z = 1$): $x^* = \bigvee_{n \geq 0} x^n$
- L.f.p. properties of * also follow

- Consequence: loop abstraction



local: xy^*z

global: $\bigvee_{n \geq 0} xy^nz$

Continuous Kleene Algebras

- Kleene algebra $S = (S, \vee, \cdot, ^*, \bot, 1)$
- in which all suprema $\bigvee X$, $X \subseteq S$ exist,
- and such that for all $X \subseteq S$, $y, z \in S$, $y(\bigvee X)z = \bigvee yXz$

- All continuous Kleene algebras are *-continuous, but not vice-versa
  - Example: regular languages over some Σ
- [Kozen 1990 (MFCS)]: not all Kleene algebras are *-continuous
  - Counterexample is necessarily infinite

Matrix Semirings

S semiring, $n \geq 1$

- $S^{n \times n}$: semiring of $n \times n$-matrices over S
- (with matrix addition and multiplication)
- If S is a *-continuous Kleene algebra, then so is $S^{n \times n}$
- with $M^*_{i,j} = \bigvee_{m \geq 0} \bigvee_{1 \leq k_1, \dots, k_m \leq n} M_{i,k_1} M_{k_1,k_2} \cdots M_{k_m,j}$
- and for $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$M^* = \begin{bmatrix} (a \vee bd^*c)^* & (a \vee bd^*c)^* bd^* \\ (d \vee ca^*b)^* ca^* & (d \vee ca^*b)^* \end{bmatrix}$$

(recursively)

Finite Runs in Weighted Automata

S *-continuous Kleene algebra, $n \geq 1$

- a weighted automaton over S (with n states): $A = (\alpha, M, \kappa)$
- $\alpha \in \{\bot, 1\}^n$ initial vector, $\kappa \in \{\bot, 1\}^n$ accepting vector, $M \in S^{n \times n}$ transition matrix
- finite behavior of A: $|A| = \alpha M^* \kappa$
- Theorem:

$$|A| = \bigvee \left\{ w_0 \cdots w_n \;\middle|\; s_i \xrightarrow{w_0} \cdots \xrightarrow{w_n} s_j \text{ accepting path in } S \right\}$$

$(s_i \xrightarrow{w_0} \cdots \xrightarrow{w_n} s_j$ accepting if $\alpha_i = \kappa_j = 1)$

Idempotent Semiring-Semimodule Pairs

- idempotent semiring $S = (S, \vee, \cdot, \bot, 1)$
- commutative idempotent monoid $V = (V, \vee, \bot)$
- left S-action $S \times V \to V$, $(s, v) \mapsto sv$
- such that for all $s, s' \in S$, $v \in V$:

$$(s \vee s')v = sv \vee s'v \qquad s(v \vee v') = sv \vee sv'$$
$$(ss')v = s(s'v) \qquad \bot s = \bot$$
$$s\bot = \bot \qquad 1v = v$$

Continuous Kleene ω-Algebras

- idempotent semiring-semimodule pair (S, V)
- where S is a continuous Kleene algebra,
- V is a complete lattice,
- and the S-action on V preserves all suprema in either argument,
- with an infinite product $\prod : S^\omega \to V$ such that:
  - For all $x_0, x_1, \ldots \in S$, $\prod x_n = x_0 \prod x_{n+1}$.
  - Let $x_0, x_1, \ldots \in S$ and $0 = n_0 \leq n_1 \leq \cdots$ a sequence which increases without a bound. Let $y_k = x_{n_k} \cdots x_{n_{k+1}-1}$ for all $k \geq 0$. Then $\prod x_n = \prod y_k$.
  - For all $X_0, X_1, \ldots \subseteq S$, $\prod(\bigvee X_n) = \bigvee\{\prod x_n \mid x_n \in X_n, n \geq 0\}$.

Matrix Semiring-Semimodule Pairs

(S, V) semiring-semimodule pair, $n \geq 1$

- $(S^{n\times n}, V^n)$ is again a semiring-semimodule pair
- (the action is matrix-vector product)
- If (S, V) is a continuous Kleene ω-algebra, then so is $(S^{n\times n}, V^n)$
- with $M_i^\omega = \bigvee_{1\leq k_1, k_2, \ldots \leq n} M_{i,k_1} M_{k_1,k_2} \cdots$
- and for $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$M^\omega = \begin{bmatrix} (a \vee bd^*c)^\omega \vee (a \vee bd^*c)^* bd^\omega \\ (d \vee ca^*b)^\omega \vee (d \vee ca^*b)^* ca^\omega \end{bmatrix}$$

(recursively)

Infinite Runs in Weighted Automata

(S, V) continuous Kleene ω-algebra
(α, M, κ) weighted automaton over S

- Reorder $S = \{1, \dots, n\}$ so that $\kappa = (1, \dots, 1, \bot, \dots, \bot)$
  - i.e. the first $k \le n$ states are accepting
- Büchi behavior of A: write $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, with $a \in S^{k \times k}$, then

$$\|A\| = \alpha \begin{bmatrix} (a + bd^* c)^\omega \\ d^* c(a + bd^* c)^\omega \end{bmatrix}$$

- Theorem:

$$\|A\| = \bigvee \{ \prod w_n \mid s_i \xrightarrow{w_0} \xrightarrow{w_1} \cdots \text{ Büchi path in } S \}$$

($s_i \xrightarrow{w_0} \xrightarrow{w_1} \cdots$ Büchi path if $\alpha_i = 1$ and some s_j with $j \le k$ is visited infinitely often)

Problem

continuous Kleene algebras	continuous Kleene ω-algebras
*-continuous Kleene algebras	???

Problem

continuous Kleene algebras	continuous Kleene ω-algebras
*-continuous Kleene algebras	*-continuous Kleene ω-algebras

[Ésik, F., Legay 2015 (DLT)]

Generalized *-Continuous Kleene Algebras [EFL'15]

- semiring-semimodule pair (S, V)
- where S is a *-continuous Kleene algebra
- such that for all $x, y \in S$, $v \in V$, $xy^*v = \bigvee_{n \geq 0} xy^n v$

*-Continuous Kleene ω-Algebras [EFL'15]

- generalized *-continuous Kleene algebra (S, V)
- with an infinite product $\prod : S^\omega \to V$ such that:
  - For all $x_0, x_1, \ldots \in S$, $\prod x_n = x_0 \prod x_{n+1}$.
  - Let $x_0, x_1, \ldots \in S$ and $0 = n_0 \leq n_1 \leq \cdots$ a sequence which increases without a bound. Let $y_k = x_{n_k} \cdots x_{n_{k+1}-1}$ for all $k \geq 0$. Then $\prod x_n = \prod y_k$.
  - For all $x_0, x_1, \ldots, y, z \in S$, $\prod (x_n(y \vee z)) = \bigvee_{x'_0, x'_1, \ldots \in \{y, z\}} \prod x_n x'_n$.
  - For all $x, y_0, y_1, \ldots \in S$, $\prod x^* y_n = \bigvee_{k_0, k_1, \ldots \geq 0} \prod x^{k_n} y_n$.

Matrix Semiring-Semimodule Pairs, Revisited [EFL'15]

(S, V) *-continuous Kleene ω-algebra, $n \geq 1$

- $(S^{n\times n}, V^n)$ is a generalized *-continuous Kleene algebra
- with an operation $^\omega : S^{n\times n} \to V^n$ given by
  $M_i^\omega = \bigvee_{1\leq k_1, k_2, \ldots \leq n} M_{i,k_1} M_{k_1,k_2} \cdots$
  - (not a general infinite product)
- and for $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$M^\omega = \begin{bmatrix} (a \vee bd^*c)^\omega \vee (a \vee bd^*c)^* bd^\omega \\ (d \vee ca^*b)^\omega \vee (d \vee ca^*b)^* ca^\omega \end{bmatrix}$$

(recursively)

References

- Ésik, F., Legay, Quaas, ATVA 2013: Energy automata
- Ésik, F., Legay, DLT 2015: *-continuous Kleene ω-algebras
- Ésik, F., Legay, FICS 2015: *-continuous Kleene ω-algebras for energy automata
- Cachera, F., Legay, FSTTCS 2015: *-continuous Kleene ω-algebras for real-time energy problems

Outline

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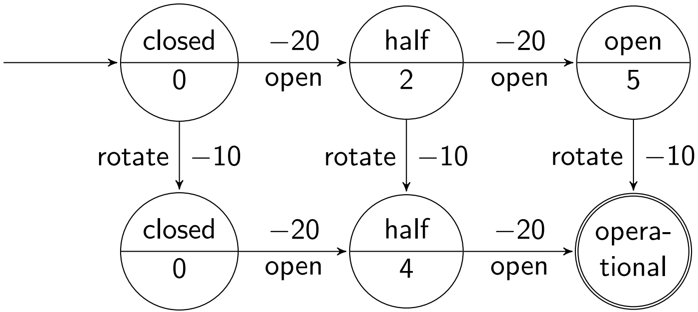
Example

Example



- given initial energy and time budget
- spend time in states to regain energy
- lose energy on transitions
- decide reachability / Büchi acceptance

Real-Time Energy Automata

A Real-Time Energy Automaton (S, s_0, F, T, r):

- finite set S of states,
- initial state $s_0 \in S$,
- accepting states $F \subseteq S$,
- transitions $T \subseteq S \times \mathbb{R}_{\le 0} \times \mathbb{R}_{\ge 0} \times S$
  - $s \xrightarrow[b]{p} s'$; p price, b bound

Semantics:

- configurations $(s, x, t) \in C = S \times \mathbb{R}_{\ge 0} \times \mathbb{R}_{\ge 0}$
  - x energy value; t available time
- $(s, x, t) \rightsquigarrow (s', x', t')$ iff $d = t - t' \ge 0$ and $\exists (s, p, b, s') \in T$ such that
  - $x + d\, r(s) \ge b$ and
  - $x' = x + d\, r(s) + p$

Problems

Let $A = (S, s_0, F, T, r)$ be a computable real-time energy automaton and $x_0, t, y \in [0, \infty]$ computable numbers.

State reachability: Does there exist a finite run $(s_0, x_0, t) \rightsquigarrow \cdots \rightsquigarrow (s, x, t')$ in A with $s \in F$?

Coverability: Does there exist a finite run $(s_0, x_0, t) \rightsquigarrow \cdots \rightsquigarrow (s, x, t')$ in A with $s \in F$ and $x \geq y$?

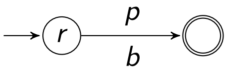
Büchi acceptance: Does there exist $s \in F$ and an infinite run $(s_0, x_0, t) \rightsquigarrow (s_1, x_1, t_1) \rightsquigarrow \cdots$ in A in which $s_n = s$ for infinitely many $n \geq 0$?

Real-Time Energy Functions

Idea: a real-time energy automaton computes a function

$$(x, t) \mapsto y$$
$$(\text{input energy}, \text{available time}) \mapsto \text{output energy}$$

Atomic functions:

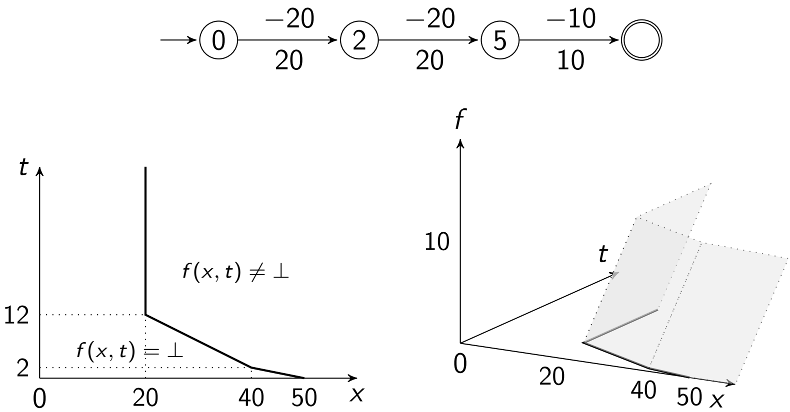


$$f(x, t) = \begin{cases} x + r\,t + p & \text{if } x + r\,t \geq b\,, \\ \bot & \text{otherwise} \end{cases}$$

Composition:

$$f \triangleright g(x, t) = \bigvee_{t_1 + t_2 = t} g(f(x, t_1), t_2)$$

Example



Operations on Real-Time Energy Functions

Composition: $$f \rhd g(x,t) = \bigvee_{t_1+t_2=t} g(f(x,t_1),t_2)$$

Maximum: $$f \vee g(x,t) = \max(f(x,t), g(x,t))$$

Star: $$f^*(x,t) = \bigvee_{n\geq 0} f^n(x,t)$$

Definition

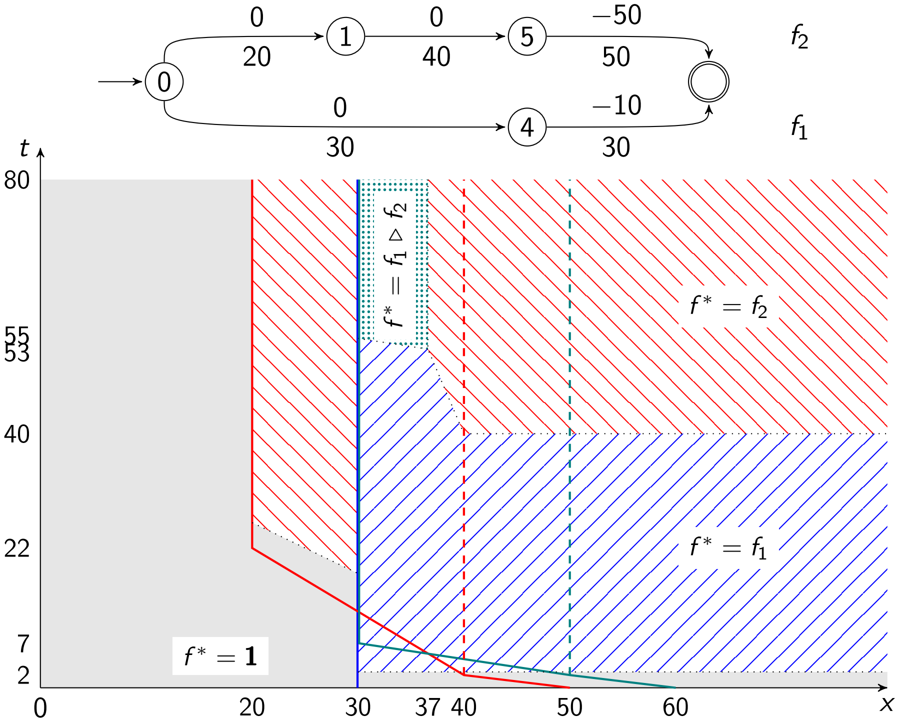
$\mathcal{E}$: set of functions generated by atomic functions under $\vee$ and $\rhd$.

Lemma

For every $f \in \mathcal{E}$ there exists $N \geq 0$ so that $f^* = \bigvee_{n=0}^N f^n$.

Corollary

*$\mathcal{E}$ is locally closed, hence a *-continuous Kleene algebra.*



0
20
0
40
−50
50
0
30
−10
30
f_2
f_1
t
80
55
53
40
22
7
2
0
20
30
37
40
50
60
x
$f^* = f_1 \rhd f_2$
$f^* = f_2$
$f^* = f_1$
$f^* = \mathbf{1}$

Reachability & Coverability

Let $A = (\alpha, M, \kappa)$ be a computable real-time energy automaton.

- (Recall: $\alpha \in \{\bot, 1\}^n$ initial vector, $\kappa \in \{\bot, 1\}^n$ accepting vector, $M \in S^{n \times n}$ transition matrix)

Compute $|A| = \alpha M^* \kappa$.

Let $x_0, t, y \in [0, \infty]$ computable numbers.

Theorem

There exists a finite run $(s_0, x_0, t) \rightsquigarrow \cdots \rightsquigarrow (s, x, t')$ in A with $s \in F$ iff $|A|(x_0, t) > \bot$.

Theorem

There exists a finite run $(s_0, x_0, t) \rightsquigarrow \cdots \rightsquigarrow (s, x, t')$ in A with $s \in F$ and $x \geq y$ iff $|A|(x_0, t) \geq y$.

Büchi Acceptance

- $\mathbb{B} = \{\mathbf{ff}, \mathbf{tt}\}$: the Boolean lattice, $\mathbf{ff} < \mathbf{tt}$
- $\mathcal{V}$: set of monotonic functions $[0, \infty] \times [0, \infty] \to \mathbb{B}$
- infinite product $\mathcal{E}^\omega \to \mathcal{V}$: $\prod_{n \geq 0} f_n(x, t) = \mathbf{tt}$ iff $\exists\, t_0, t_1, \ldots \in [0, \infty] : \sum_{n=0}^{\infty} t_n = t$ and $\forall\, n \geq 0$, $f_n(t_n) \circ \cdots \circ f_0(t_0)(x) \neq \bot$
- $\mathcal{U}$: subset of $\mathcal{V}$ generated by infinite products of $\mathcal{E}$-functions: a left $\mathcal{E}$-semimodule

Theorem

*$(\mathcal{E}, \mathcal{U})$ forms a *-continuous Kleene ω-algebra.*

Büchi Acceptance

Let $A = (\alpha, M, \kappa)$ be a computable real-time energy automaton.

Write $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, with $a \in S^{k\times k}$, and compute

$$\|A\| = \alpha \begin{bmatrix} (a + bd^* c)^\omega \\ d^* c(a + bd^* c)^\omega \end{bmatrix}$$

Let $x_0, t, y \in [0, \infty]$ computable numbers.

Theorem

There exists $s \in F$ and an infinite run $(s_0, x_0, t) \rightsquigarrow (s_1, x_1, t_1) \rightsquigarrow \cdots$ in A in which $s_n = s$ for infinitely many $n \geq 0$ iff $\|A\|(x_0, t) = \mathbf{tt}$.

- Functions in $\mathcal{E}$ are computable piecewise linear, hence $|A|$ and $\|A\|$ are computable
- (probably in EXPTIME)

Istead of a Conclusion

Uli Fahrenberg, PhD in maths 2005, 64 publications,
looking for a job

- higher-dimensional automata
- weighted timed automata
- energy problems
- quantitative verification
- interface and specification theories
- algebraic methods
- teaching
- project administration