Pomset Languages of Higher-Dimensional Automata

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Motivation

- The theory of regular languages is nice and beautiful. It's also fundamental for much of what we do.
- For non-interleaving models ("true concurrency"), no such theory
- It seems that this is mostly due to the choice of model: Petri nets are messy!
- Closest to what I want: [Fanchon-Morin 2002/2009]
- Here: regular pomset languages of higher-dimensional automata

Before we begin

Warning

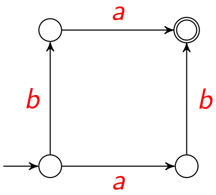
Much of this is work in progress.

Acknowledgement

I have started this work together with the late Zoltán Ésik when I visited him in Szeged in February 2016.

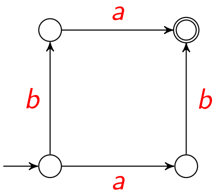
Higher-dimensional automata

$a|b$

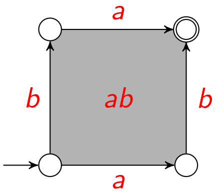


Higher-dimensional automata

$a|b$



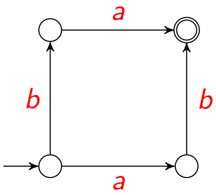
$a.b + b.a$



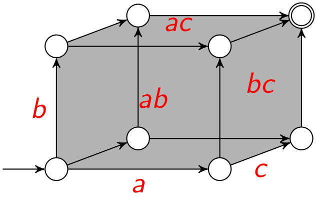
a and b are independent

Higher-dimensional automata

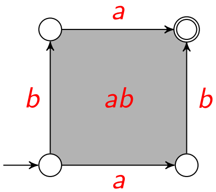
$a|b$

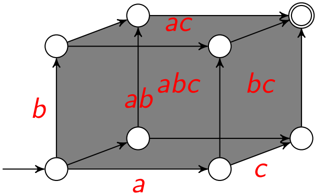


$a|b|c$



pairwise independent



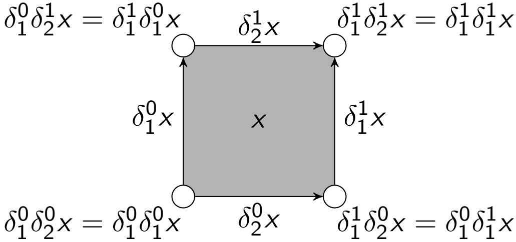


$\{a, b, c\}$ independent

Higher-dimensional automata

A precubical set:

- a graded set $X = \{X_n\}_{n\in\mathbb{N}}$
- in each dimension n, $2n$ face maps $\delta_k^0, \delta_k^1 : X_n \to X_{n-1}$ $(k = 1, \dots, n)$
- the precubical identity: $\delta_k^\nu \delta_\ell^\mu = \delta_{\ell-1}^\mu \delta_k^\nu$ for all $k < \ell$



Higher-dimensional automata

A (finite) higher-dimensional automaton (X, I, F, ℓ):

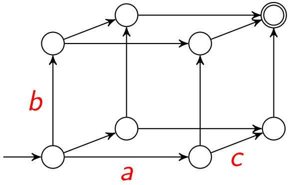
- a finite precubical set X
- with initial and final states $I, F \subseteq X_0$
- and labeling $\ell : X_1 \to \Sigma$
  - such that for all $x \in X_2$ and $i = 1, 2$, $\ell(\delta_i^0 x) = \ell(\delta_i^1 x)$
- [van Glabbeek-Pratt 1991]

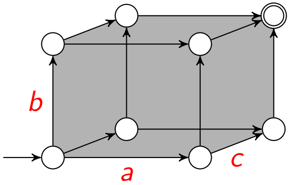
HDA as a model for concurrency:

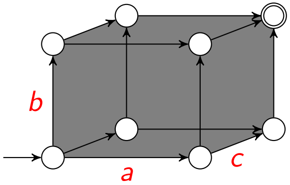
- points $x \in X_0$: states
- edges $a \in X_1$: transitions (labeled with events)
- n-squares $\alpha \in X_n$ $(n \geq 2)$: independency relations (concurrently executing events)

van Glabbeek 2006 (TCS): Up to history-preserving bisimilarity, HDA "generalize the main models of concurrency proposed in the literature"

Languages of HDA







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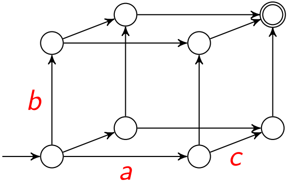
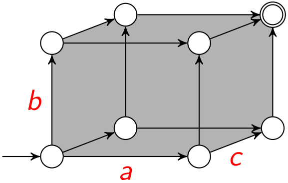
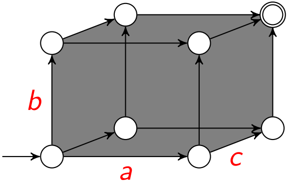
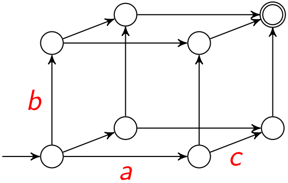






$$L_1 = \{abc, acb, bac, bca, cab, cba\}$$

Languages of HDA



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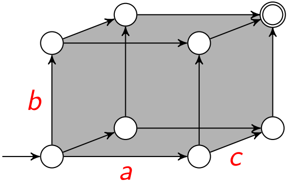
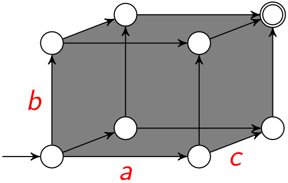
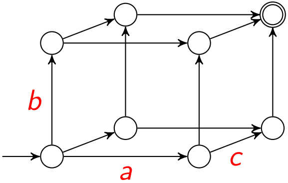




$$L_3 = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \dots \right\}$$

Languages of HDA



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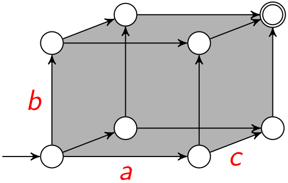
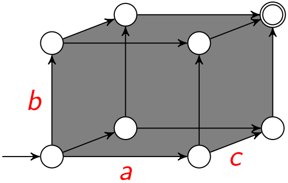


$$L_2 = \left\{ \begin{pmatrix} a \\ b \to c \end{pmatrix}, \begin{pmatrix} a \\ c \to b \end{pmatrix}, \begin{pmatrix} b \\ a \to c \end{pmatrix}, \begin{pmatrix} b \\ c \to a \end{pmatrix}, \begin{pmatrix} c \\ a \to b \end{pmatrix}, \begin{pmatrix} c \\ b \to a \end{pmatrix}, \dots \right\}$$



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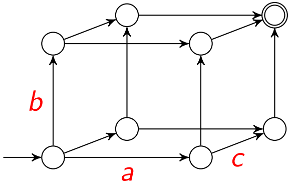
Languages of HDA



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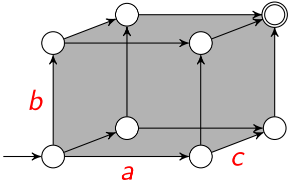

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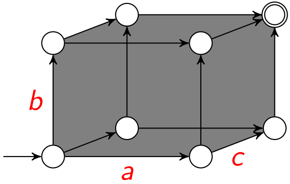
sets of pomsets



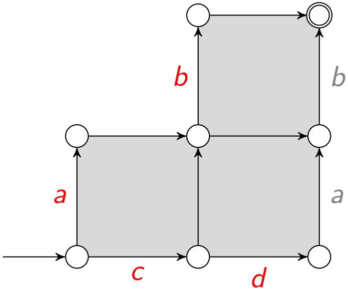
$$L_3 = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\} \cup L_2$$

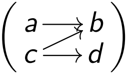
Pomsets

A (finite) pomset ("partially ordered multiset") $(P, \leq, \ell)$:

- a finite partially ordered set $(P, \leq)$
- with labeling $\ell : P \to \Sigma$
- (AKA labeled partial order)
- [Lamport 1978]

Example



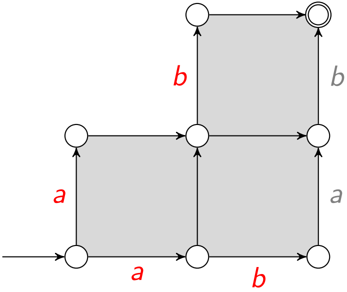


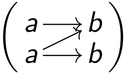
- (not series-parallel!)
- gluing product of pomsets:

$$\begin{pmatrix} a \\ c \end{pmatrix} \overset{(\,a\,)}{\smile} \begin{pmatrix} a \\ d \end{pmatrix} \overset{(\,d\,)}{\smile} \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} a & & \\ c & \longrightarrow & d \end{pmatrix} \overset{(\,d\,)}{\smile} \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} a & \longrightarrow & b \\ c & \nearrow\!\!\longrightarrow & d \end{pmatrix}$$

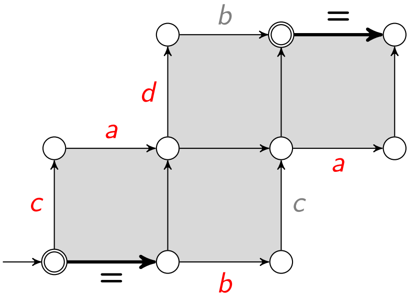
- (new ternary operation which generates all pomsets)

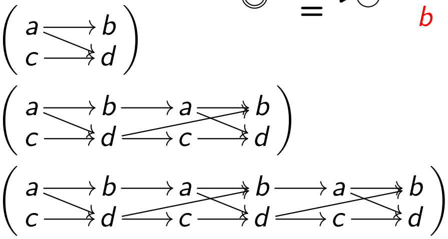
Auto-concurrency





A loop





. . .

Are all pomsets generated by HDA?

No, only (labeled) interval orders

- Poset $(P, \leq)$ is an interval order iff it does not contain ($\rightrightarrows$)
  - (iff it is "2+2-free")
- iff it has an interval representation:
  - a set $I = \{[l_i, r_i]\}$ of real intervals
  - with order $[l_i, r_i] \preceq [l_j, r_j]$ iff $r_i \leq l_j$
  - and an order isomorphism $(P, \leq) \leftrightarrow (I, \preceq)$
- [Fishburn 1970]

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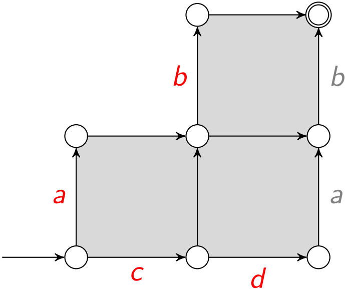
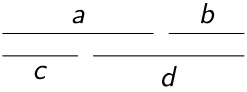
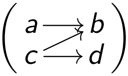
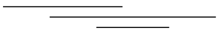




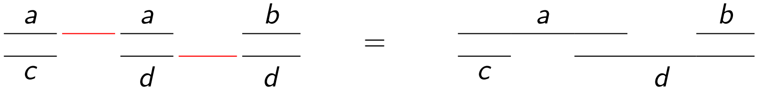


Interval orders vs ST-traces

- An ST-trace: $a^+ \, b^+ \, a^+ \, a^- \, a^- \, b^-$ [van Glabbeek 1990]
- as intervals: 
- Lemma: ST-traces up to the equivalence generated by $a^+ b^+ \sim b^+ a^+$ and $a^- b^- \sim b^- a^-$ are in bijection with interval orders.

Gluing product of interval orders

$$\begin{pmatrix} a \\ c \end{pmatrix} \big(\underset{\smile}{a}\big) \begin{pmatrix} a \\ d \end{pmatrix} \big(\underset{\smile}{d}\big) \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} a & \\ c \longrightarrow d \end{pmatrix} \big(\underset{\smile}{d}\big) \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} a \longrightarrow b \\ c \nearrow \!\!\!\longrightarrow d \end{pmatrix}$$



Properties

- For $P = (P, \leq, \ell)$ and $P' = (P, \leq', \ell)$ pomsets with the same underlying set, write $P \succeq P'$ if $\forall x, y \in P : x \leq y \Longrightarrow x \leq' y$
  - the subsumption order [Gischer 1988]
  - P has fewer dependencies than P'
- Let $\mathcal{I}$ be the set of labeled interval orders.
- For $L \subseteq \mathcal{I}$, let $\downarrow L = \{Q \in \mathcal{I} \mid \exists P \in L : P \succeq Q\}$
- Say that $L \subseteq \mathcal{I}$ is subsumption-closed if $\downarrow L = L$
- Theorem: For any HDA X, $L(X) \subseteq \mathcal{I}$ is subsumption closed.

Higher-dimensional regular and rational languages

The following works only without auto-concurrency (for now):

- Let HReg $\subseteq 2^{\mathcal{I}}$ denote the class of languages of HDA.
- Theorem: HReg is closed under $\cup$.
- For $L_1, L_2 \subseteq \mathcal{I}$ and R a multiset, define the gluing product $L_1 \overset{R}{\smile} L_2 = {\downarrow}\{P \overset{R}{\smile} Q \mid P \in L_1, Q \in L_2\}$.
- Theorem: HReg is closed under gluing product.
- For $L \subseteq \mathcal{I}$ and R a multiset, define the gluing star $L \overset{R}{\smile}{}^{*} = \{\epsilon\} \cup L \cup L \overset{R}{\smile} L \cup L \overset{R}{\smile} L \overset{R}{\smile} L \cup \cdots$.
- Theorem: HReg is closed under gluing star.
- Let HRat $\subseteq 2^{\mathcal{I}}$ be the class generated by $\emptyset$, $\{\epsilon\}$ and ${\downarrow}\{R\}$ for all multisets R, closed under $\cup$, gluing product, and gluing star.
- Conjecture: HRat $=$ HReg.

Other results

- Algebraic characterization:
  - Let $\mathcal{T}$ denote the set of multisets over Σ.
  - Conjecture: HReg is the free $\mathcal{T}$-indexed Kleene algebra and the free $\mathcal{T}$-indexed *-continuous Kleene algebra.
- Theorem: $L \in \text{HReg} \Longrightarrow L \cap \Sigma^*$ regular.
  - but not the other way: take

$$L = \downarrow\left\{ \begin{pmatrix} a \\ b \end{pmatrix}^n \cdot (ab + ba)^n \;\middle|\; n \geq 0 \right\}$$

- Conjecture: HReg is closed under complement.
  - via new notion of deterministic and complete HDA
  - here forbidding auto-concurrency seems to be necessary!
- Conjecture: If HDA X and Y are ST-bisimilar, then $L(X) = L(Y)$.

Ongoing and future work

- Prove conjectures!
- Parallel composition of HDA: probably $L(X\|Y) = L(X) \otimes L(Y)$
  - $\otimes$ parallel product of pomsets [Gischer 1988]
  - Definition: $L_1 \otimes L_2 = \{P \otimes Q \mid P \in L_1, Q \in L_2\} \cap \mathcal{I}$
- Weighted HDA?
- Real-time HDA?