What is Known about Weighted Games?

Uli Fahrenberg Axel Legay Karin Quaas

Inria Rennes, France

University of Leipzig, Germany

WATA 2016

What is Known about Weighted Games?

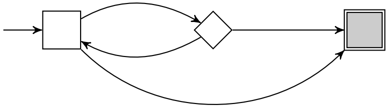
Uli Fahrenberg Axel Legay Karin Quaas

Inria Rennes, France

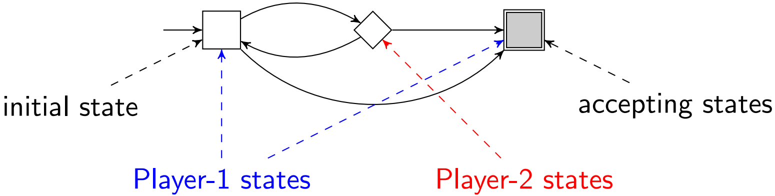
University of Leipzig, Germany

WATA 2016

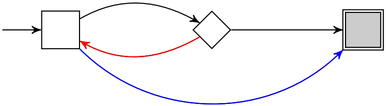
A reachability game



A reachability game

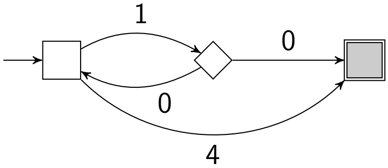


A reachability game

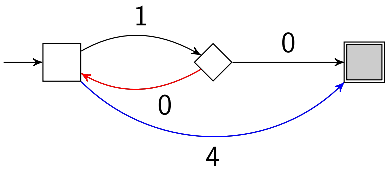


a Player-1 strategy
a Player-2 strategy

A minimum reachability game



A minimum reachability game

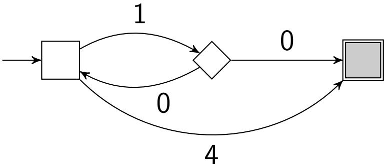


a Player-1 strategy
a Player-2 strategy

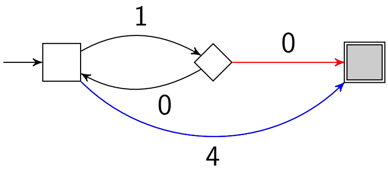
value of game:

- minimum over all Player-1 strategies
  - *of* maximum over all Player-2 strategies
    - *of* weight of induced path (or ∞ if not accepting)

A maximum reachability game



A maximum reachability game



a Player-1 strategy
a Player-2 strategy

value of game:

- maximum over all Player-1 strategies
  - *of* minimum over all Player-2 strategies
    - *of* weight of induced path (or $-\infty$ if not accepting)

Games, strategies and outcomes

- let K be a set (*for now*)
- a game structure: (S_1, S_2, s^0, T, F)
  - S_1 Player-1 states, S_2 Player-2 states (disjoint)
  - $s^0 \in S = S_1 \sqcup S_2$ initial, $F \subseteq S$ accepting
  - $T \subseteq S \times K \times S$ transitions
- non-blocking: $\forall s \in S : \exists (s, x, s') \in T$
- a finite Player-i path: $s^0 \xrightarrow{x_1} \cdots \xrightarrow{x_{n-1}} s_n \in S_i$
  - s.t. $s^0 \xrightarrow{x_1} s_2, \dots, s_{n-1} \xrightarrow{x_{n-1}} s_n \in T$
- a Player-i strategy: $\theta : \mathrm{fPa}_i \to T$
  - s.t. $\forall \pi = s^0 \to \cdots \to s_n \in \mathrm{fPa}_i : \theta(\pi) = (s_n, \cdot, \cdot)$ is an extension of π
- a pair of strategies $(\theta_1, \theta_2) \in \Theta_1 \times \Theta_2$ induces a unique infinite path: its outcome $\pi(\theta_1, \theta_2)$
  - start in s^0 and use θ_1 and θ_2 to extend indefinitely

The reachability game

- the reachability objective:

$$\mathcal{R} = \{\text{infinite paths } \pi = s^0 \to s_2 \to \cdots \mid \exists n : s_n \in F\}$$

- Player 1 wins the reachability game if

$$\exists\, \theta_1 \in \Theta_1 : \forall\, \theta_2 \in \Theta_2 : \pi(\theta_1, \theta_2) \in \mathcal{R}$$

The weighted reachability game

- let $K = (K, \leq, \cdot, \mathbf{1})$ be a unital quantale
  - $(K, \leq)$ a complete lattice
  - $(K, \cdot, \mathbf{1})$ a monoid
  - $x(\bigvee Y)z = \bigvee(xYz)$ and $x(\bigwedge Y)z = \bigwedge(xYz)$ for all $x, z \in K$, $Y \subseteq K$.
- the reachability weight of an infinite path $\pi = s_1 \xrightarrow{x_1} s_2 \xrightarrow{x_2} \cdots$:

$$w_{\mathcal{R}}(\pi) = \bigvee \big\{ x_1 \cdots x_{n-1} \mid s_n \in F \big\}$$

  - the $\bigvee$ of the weights of all finite accepting prefixes of π
- the value of the game:

$$v_{\mathcal{R}} = \bigvee_{\theta_1 \in \Theta_1} \bigwedge_{\theta_2 \in \Theta_2} w_{\mathcal{R}}(\pi(\theta_1, \theta_2))$$

Controllable predecessors for the reachability game

- controllable predecessors of F:

$$U_0 = \{s_1 \in S_1 \mid \exists s_1 \to \cdots \to s_n : s_1, \dots, s_{n-1} \in S_1, s_n \in F\}$$

- uncontrollable predecessors of U_0:

$$V_1 = \{s_1 \in S_2 \mid \forall s_1 \to \cdots \to s_n : s_1, \dots, s_{n-1} \in S_2 \implies s_n \in U_0\}$$

- controllable predecessors of V_1:

$$U_1 = \{s_1 \in S_1 \mid \exists s_1 \to \cdots \to s_n : s_1, \dots, s_{n-1} \in S_1, s_n \in V_1\}$$

- etc.

Theorem

Player 1 wins the reachability game iff $s^0 \in \bigcup_{n\geq 0} U_n$

Controllable predecessors for weighted reachability

- let $U_0 : S_1 \to K$ defined by

$$U_0(s_1) = \bigvee \big\{ x_1 \cdots x_{n-1} \mid s_1 \xrightarrow{x_1} \cdots \xrightarrow{x_{n-1}} s_n, s_1, \dots, s_{n-1} \in S_1, s_n \in F \big\}$$

- let $V_1 : S_2 \to K$ defined by

$$V_1(s_1) = \bigwedge \big\{ x_1 \cdots x_{n-1} U_0(s_n) \mid s_1 \xrightarrow{x_1} \cdots \xrightarrow{x_{n-1}} s_n, s_1, \dots, s_{n-1} \in S_2, s_n \in S_1 \big\}$$

- Let $U_1 : S_1 \to K$ defined by

$$U_1(s_1) = \bigvee \big\{ x_1 \cdots x_{n-1} V_1(s_n) \mid s_1 \xrightarrow{x_1} \cdots \xrightarrow{x_{n-1}} s_n, s_1, \dots, s_{n-1} \in S_1, s_n \in S_2 \big\}.$$

Conjecture

$v_{\mathcal{R}} = \bigvee_{n \ge 0} U_n(s^0)$

What do I know about weighted games?

- integer- or real-weighted reachability games
- mean-payoff games
- discounted games
- energy games
- timed games, real-weighted timed games
- Nash equilibria
- secure equilibria
- subgame perfect equilibria
- no attempt at general theory so far?

You know nothing U.F.

- integer- or real-weighted reachability games
- mean-payoff games
- discounted games
- energy games
- timed games, real-weighted timed games
- Nash equilibria
- secure equilibria
- subgame perfect equilibria
- no attempt at general theory so far?
- Winter is coming!