Pomset Languages of Higher-Dimensional Automata

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Motivation

- The theory of regular languages is nice and beautiful. It's also fundamental for much of what we do.
- For non-interleaving models ("true concurrency"), no such theory
- It seems that this is mostly due to the choice of model: Petri nets are messy!
- Closest to what I want: [Fanchon-Morin 2002/2009] (abandoned after 2009)
- Here: pomset languages of higher-dimensional automata

Before we begin

Warning

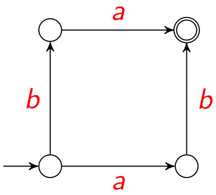
Much of this is work in progress.

Acknowledgement

Joint work with Christian Johansen (Oslo), Georg Struth (Sheffield), and Samuel Mimram

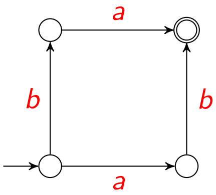
Higher-dimensional automata

$a|b$

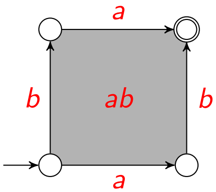


Higher-dimensional automata

$a|b$



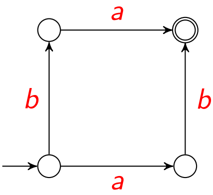
$a.b + b.a$



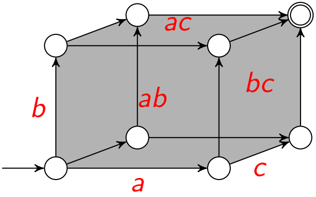
a and b are independent

Higher-dimensional automata

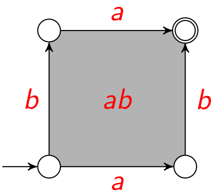
$a|b$

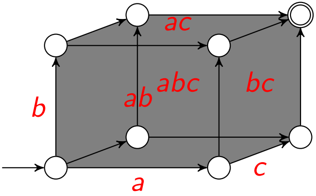


$a|b|c$



pairwise independent



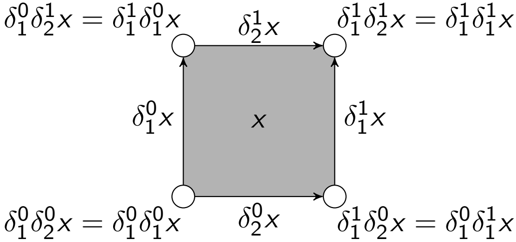


$\{a, b, c\}$ independent

Higher-dimensional automata

A precubical set:

- a graded set $X = \{X_n\}_{n\in\mathbb{N}}$
- in each dimension n, $2n$ face maps $\delta_k^0, \delta_k^1 : X_n \to X_{n-1}$ $(k = 1, \dots, n)$
- the precubical identity: $\delta_k^\nu \delta_\ell^\mu = \delta_{\ell-1}^\mu \delta_k^\nu$ for all $k < \ell$



Higher-dimensional automata

A (finite) higher-dimensional automaton (X, I, F, ℓ):

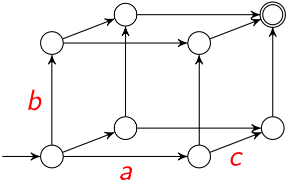
- a finite precubical set X
- with initial and final states $I, F \subseteq X_0$
- and labeling $\ell : X_1 \to \Sigma$
  - such that for all $x \in X_2$ and $i = 1, 2$, $\ell(\delta_i^0 x) = \ell(\delta_i^1 x)$
- [van Glabbeek-Pratt 1991]

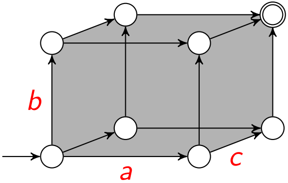
HDA as a model for concurrency:

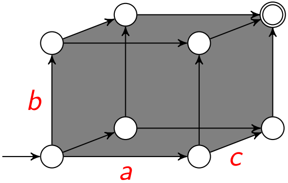
- points $x \in X_0$: states
- edges $a \in X_1$: transitions (labeled with events)
- n-squares $\alpha \in X_n$ $(n \geq 2)$: independency relations (concurrently executing events)

van Glabbeek 2006 (TCS): Up to history-preserving bisimilarity, HDA "generalize the main models of concurrency proposed in the literature"

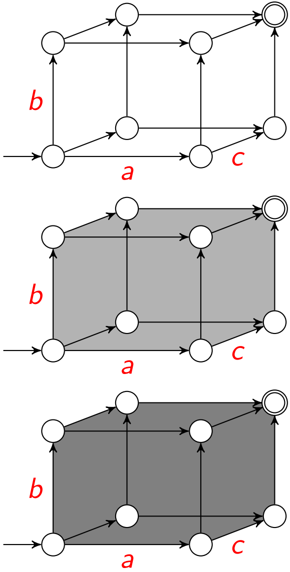
Languages of HDA





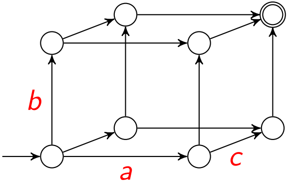


Languages of HDA



$$L_1 = \{abc, acb, bac, bca, cab, cba\}$$

Languages of HDA



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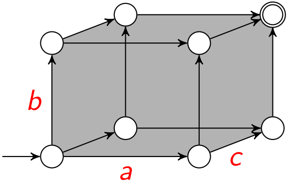
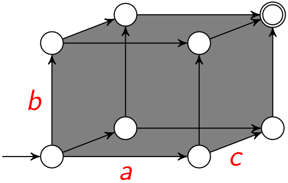
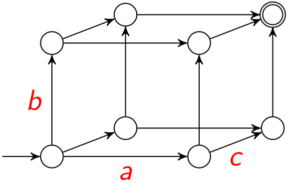




$$L_3 = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \dots \right\}$$

Languages of HDA



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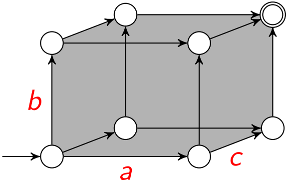
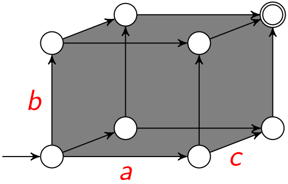


$$L_2 = \left\{ \begin{pmatrix} a & \\ b \to c \end{pmatrix}, \begin{pmatrix} a & \\ c \to b \end{pmatrix}, \begin{pmatrix} b & \\ a \to c \end{pmatrix}, \begin{pmatrix} b & \\ c \to a \end{pmatrix}, \begin{pmatrix} c & \\ a \to b \end{pmatrix}, \begin{pmatrix} c & \\ b \to a \end{pmatrix}, \dots \right\}$$



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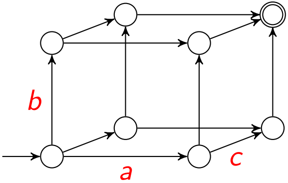
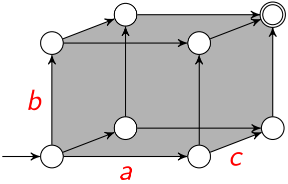


$$L_1 = \{abc, acb, bac, bca, cab, cba\}$$



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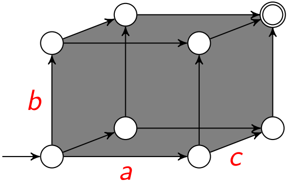
sets of pomsets



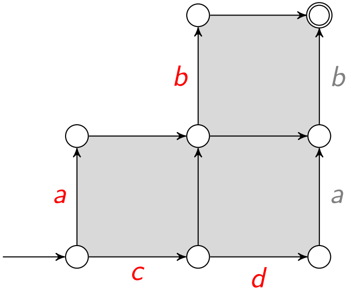
$$L_3 = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\} \cup L_2$$

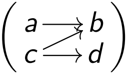
Pomsets

A (finite) pomset ("partially ordered multiset") $(P, \leq, \ell)$:

- a finite partially ordered set $(P, \leq)$
- with labeling $\ell : P \to \Sigma$
- (AKA labeled partial order)
- [Lamport 1978]

Example



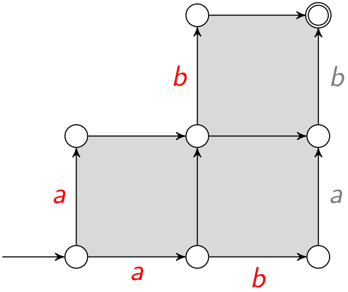


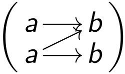
- (not series-parallel!)
- gluing product of pomsets:

$$\begin{pmatrix} a\\ c\end{pmatrix} \underset{\smile}{(a)} \begin{pmatrix} a\\ d\end{pmatrix} \underset{\smile}{(d)} \begin{pmatrix} b\\ d\end{pmatrix} = \begin{pmatrix} a & & \\ c & \longrightarrow & d\end{pmatrix} \underset{\smile}{(d)} \begin{pmatrix} b\\ d\end{pmatrix} = \begin{pmatrix} a & \longrightarrow & b \\ c & \nearrow\longrightarrow & d\end{pmatrix}$$

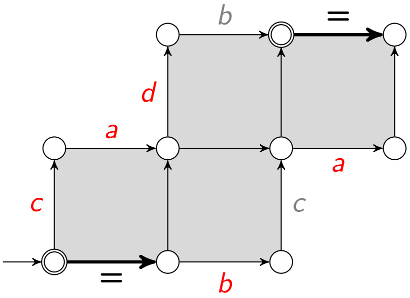
- (new ternary operation; more later)

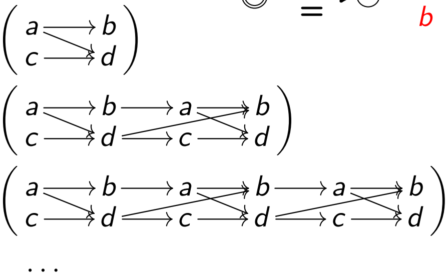
Auto-concurrency





A loop





Are all pomsets generated by HDA?

No, only (labeled) interval orders

- Poset $(P, \leq)$ is an interval order iff it does not contain $(\rightrightarrows)$
  - (iff it is "2+2-free")
- iff it has an interval representation:
  - a set $I = \{[l_i, r_i]\}$ of real intervals
  - with order $[l_i, r_i] \preceq [l_j, r_j]$ iff $r_i \leq l_j$
  - and an order isomorphism $(P, \leq) \leftrightarrow (I, \preceq)$
- [Fishburn 1970]

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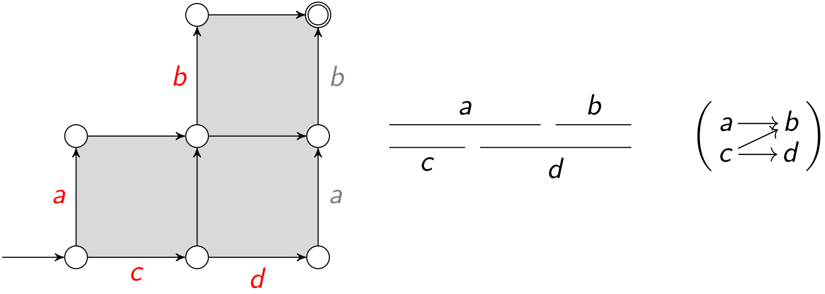
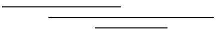
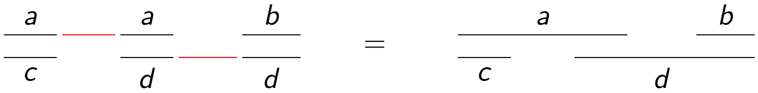


Interval orders vs ST-traces

- An ST-trace: $a^+ \, b^+ \, a^+ \, a^- \, a^- \, b^-$ [van Glabbeek 1990]
- as intervals: 
- Lemma: ST-traces up to the equivalence generated by $a^+b^+ \sim b^+a^+$ and $a^-b^- \sim b^-a^-$ are in bijection with interval orders.

Gluing product of interval orders

$$\begin{pmatrix}a\\c\end{pmatrix}\underset{\smile}{(a)}\begin{pmatrix}a\\d\end{pmatrix}\underset{\smile}{(d)}\begin{pmatrix}b\\d\end{pmatrix}=\begin{pmatrix}a & \\ c\longrightarrow d\end{pmatrix}\underset{\smile}{(d)}\begin{pmatrix}b\\d\end{pmatrix}=\begin{pmatrix}a\longrightarrow b\\ c\longrightarrow d\end{pmatrix}$$



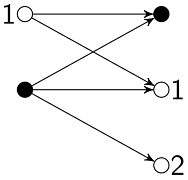
Intuitively clear: but need to make this precise!

Posets with interfaces

Definition

A poset with interfaces (i-poset) is a cospan $s : [n] \to P \leftarrow [m] : t$ of monomorphisms s, t into a poset P such that $s[n]$ is minimal and $t[m]$ is maximal in P.

- $[n]$: discrete poset $\{1, \dots, n\}$ with $i \leq j$ iff $i = j$; $[0] = \emptyset$



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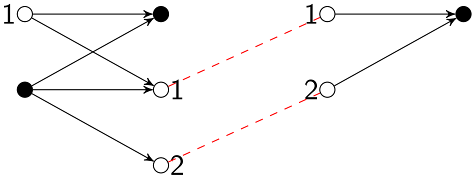


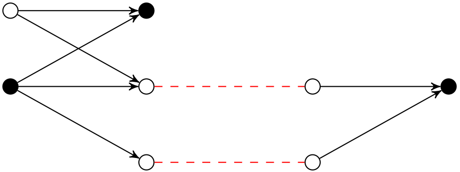
- interfaces for composition!

Gluing product

Definition

The concatenation / gluing product of i-posets
$s_P : [n] \to (P, \leq_P) \leftarrow [m] : t_P$ and $s_Q : [m] \to (Q, \leq_Q) \leftarrow [k] : t_Q$:

- $P \triangleright Q = s : [n] \to (R, \leq) \leftarrow [k] : t$
- $R = (P \sqcup Q)_{/ t_P(i) = s_Q(i)}$
- $\leq = \big(\leq_P \cup \leq_Q \cup (P \setminus t_P[m]) \times (Q \setminus s_Q[m])\big)^*$



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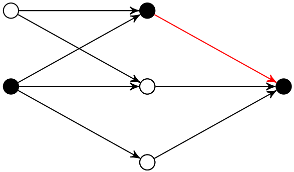
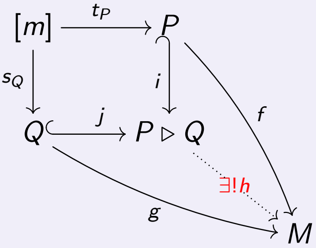


An ordered pushout

Lemma

Let $s_P : [n] \to P \leftarrow [m] : t_P$ and $s_Q : [m] \to Q \leftarrow [k] : t_Q$ be i-posets and $f : P \to M$, $g : Q \to M$ morphisms into a poset $(M, \leq_M)$ such that the diagram



(1)

commutes and such that for all $x \in P \setminus t_P[m]$, $y \in Q \setminus s_Q[m]$, $f(x) \leq_M g(y)$. Then there exists a unique poset morphism $h : P \triangleright Q \to M$ in (1).

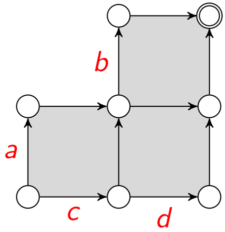
A small symmetric monoidal category

- a small category: objects $n \in \mathbb{N}$, morphisms i-posets $(s, P, t) : n \to m$, composition $\triangleright$, identities $(\mathrm{id}, [n], \mathrm{id}) : n \to n$
- parallel product / disjoint union of posets:
  $[n_1] \to P_1 \leftarrow [m_1], \quad [n_2] \to P_2 \leftarrow [m_2] \ \mapsto$
  $$[n_1 + n_2] \overset{\cong}{\to} [n_1] \sqcup [n_2] \to P_1 \sqcup P_2 \leftarrow [m_1] \sqcup [m_2] \overset{\cong}{\leftarrow} [m_1 + m_2]$$
- symmetries: isomorphisms $n \to n$

$\Longrightarrow$ small symmetric monoidal category

Weak pomset languages

- Languages of HDA are subsumption-closed: if $P \in L$, then $Q \in L$ for any *refinement* $Q \preceq P$
  - $P \succeq Q$ iff $\exists$ bijective morphism $P \to Q$



- L weak if $L = {\downarrow}L := \{Q \mid \exists P \in L : P \succeq Q\}$
- Let $\mathcal{W}$: the class of all weak sets of i-pomsets
- operations on $\mathcal{W}$:
  - $L \cup M$
  - $L \otimes M = {\downarrow}\{P \otimes Q \mid P \in L, Q \in M\}$
  - $L \triangleright M = {\downarrow}\{P \triangleright Q \mid P \in L, Q \in M, P \text{ and } Q \text{ composable}\}$

Proposition

- $(\mathcal{W}, \cup, \otimes, \emptyset, \mathbb{1}_\otimes)$ and $(\mathcal{W}, \cup, \triangleright, \emptyset, \mathbb{1}_\triangleright)$ idempotent semirings
- units $\mathbb{1}_\otimes = \{\text{id}_0\}$, $\mathbb{1}_\triangleright = \{\text{id}_n \mid n \in \mathbb{N}\}$
- interchange: $(A \otimes B) \triangleright (C \otimes D) \subseteq (A \triangleright C) \otimes (B \triangleright D)$