Energy Automata, Energy Functions, Kleene Algebra

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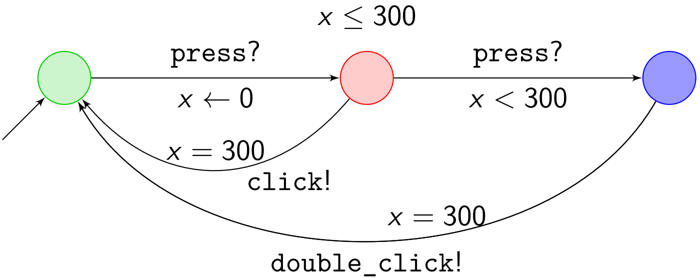
UNIVERSITÉ PARIS-SACLAY

Recall Timed Automata



Recall Timed Automata

Definition

The set $\Phi(C)$ of clock constraints ϕ over a finite set C is defined by the grammar

$$\phi ::= x \bowtie k \mid \phi_1 \wedge \phi_2 \qquad (x, y \in C, k \in \mathbb{Z}, \bowtie \in \{\leq, <, \geq, >\}).$$

Definition

A timed automaton is a tuple $(L, \ell_0, C, \Sigma, I, E)$ consisting of a finite set L of locations, an initial location $\ell_0 \in L$, a finite set C of clocks, a finite set Σ of actions, a location invariants mapping $I : L \to \Phi(C)$, and a set $E \subseteq L \times \Phi(C) \times \Sigma \times 2^C \times L$ of edges.

Recall Timed Automata

- Useful for modeling synchronous real-time systems
- Reachability, emptiness, LTL model checking PSPACE-complete
- Universality undecidable
- Fast on-the-fly algorithms, using zones, for reachability, liveness, and Timed CTL model checking
- UppAal
- Extensions to weighted timed automata, real-time games, etc.

- This work: Energy problems in timed automata

Energy Constraints

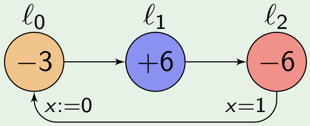
Energy is not only consumed, but can be regained.

$\leadsto$ "prices" can be negative;

$\leadsto$ the aim is to continuously satisfy cost constraints

$\leadsto$ in this paper, we focus on infinite runs.

Example



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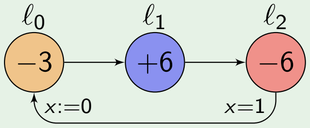

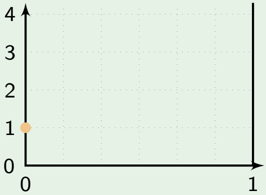

lower-bound problem

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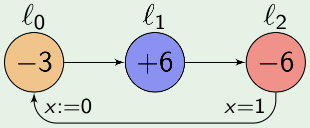

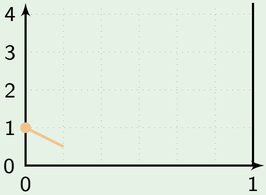

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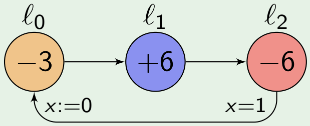

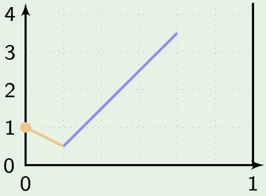

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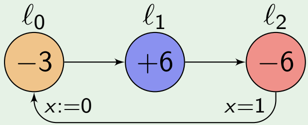

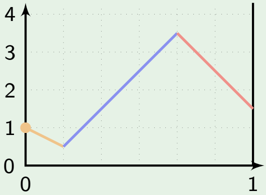

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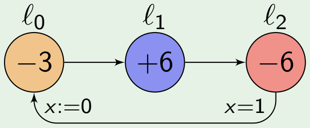

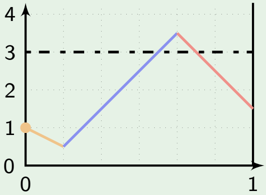

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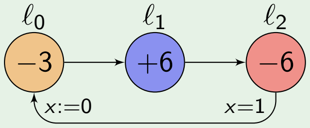

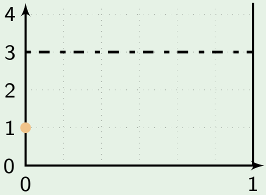

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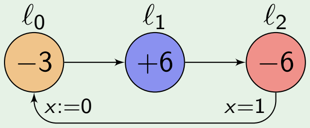

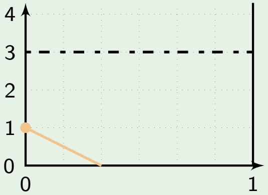

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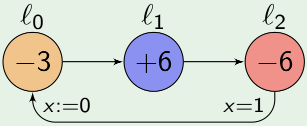

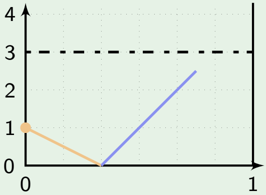

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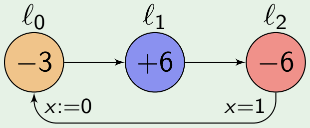

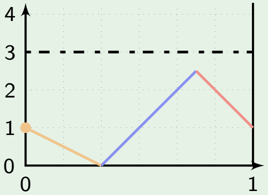

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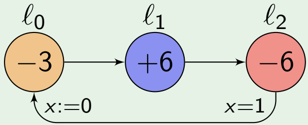

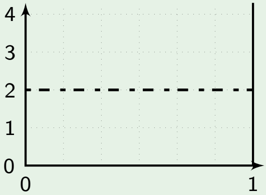

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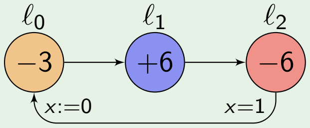

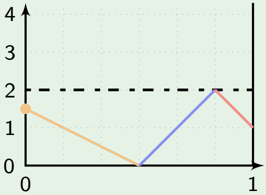

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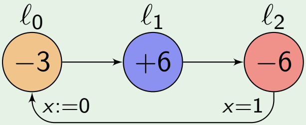

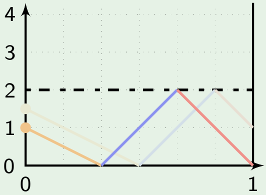

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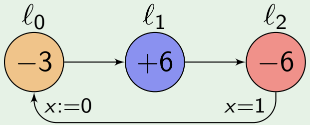

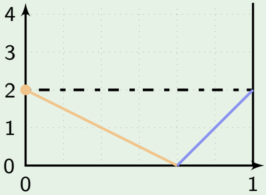

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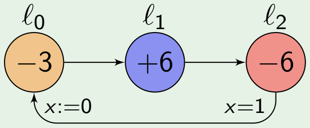

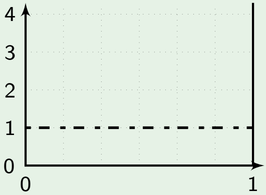

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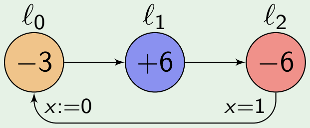
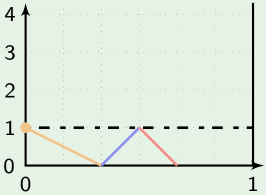




lower-upper-bound problem

Problems

Definition:

- $\gamma = s_0 \xrightarrow{p_1} s_1 \xrightarrow{p_2} \cdots \xrightarrow{p_n} s_n$ a finite path in a weighted transition system
- $c \in \mathbb{R}_{\geq 0}$ initial credit
- $b \in \mathbb{R}_{\geq 0}$ (possible) upper bound
- accumulated cost of γ with initial credit c: $c + p_1 + p_2 + \dots$

Problems:

- lower bound: Find infinite run γ for which $c + p_1 + \ldots + p_n \geq 0$ for all finite prefixes
- interval bound: Find infinite γ for which $c + p_1 + \ldots + p_n \in [0, b]$ for all finite prefixes

Results

- First paper: FORMATS 2008 (170 citations for now)
- Lots of work since then, by lots of people
- Last paper for now: FM 2018 (Best Paper Award)
- My acknowledgements: Kim G. Larsen, Patricia Bouyer, Nicolas Markey[1], Jiří Srba, Zoltán Ésik†, Karin Quaas, David Cachera, Axel Legay, Pierre-Alain Reynier, Claus Thrane

- lower-bound problem decidable for 1-clock WTA
  undecidable for 4-clock WTA
- interval problem undecidable for 2-clock WTA

- Applications in scheduling
  - of batch plants
  - of satellites

[1] also for some of the slides. . .

GOMSPACE: Scheduling of Nanosatellites Using UppAal

The Lower-Bound Problem for 1-Clock WTA

Theorem

For 1-clock WTA without weights on transitions, the lower-bound problem is solvable in polynomial time.

Proof Idea

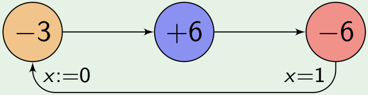
- can assume that delays within a region are elapsed in the most profitable location
- hence can use corner-point abstraction

Corner-point abstraction

Idea

Delays within a region are elapsed in the most profitable location.

Example



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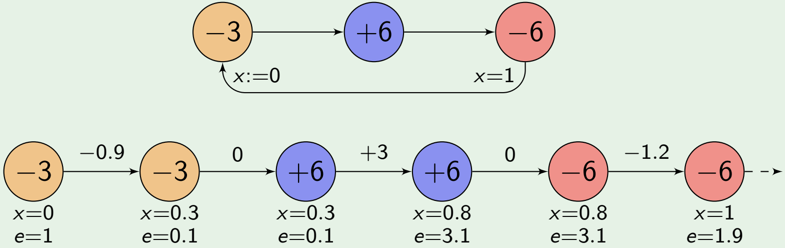

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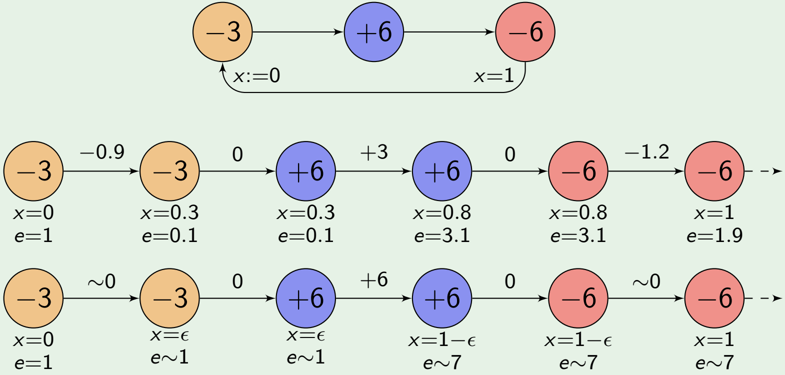

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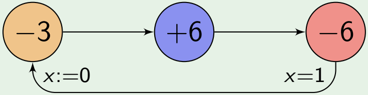

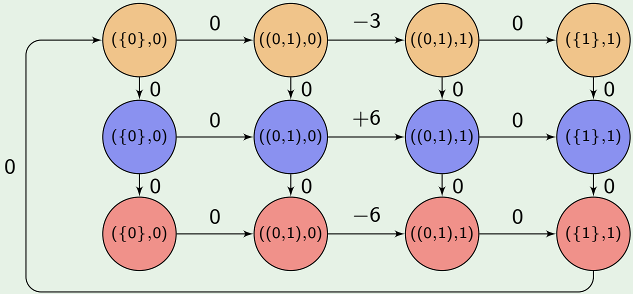

Corner-point abstraction

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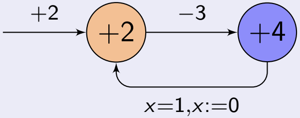
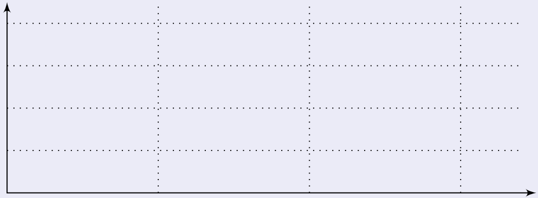
Remark

The corner-point abstraction is not correct if discrete transitions are weighted:





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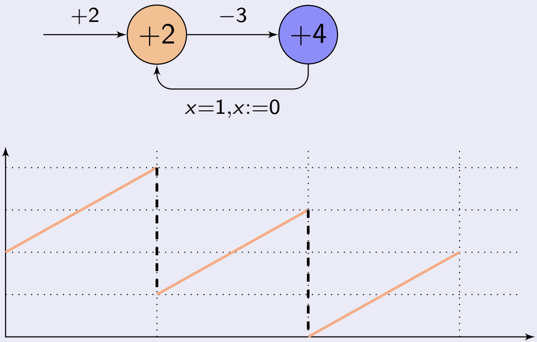

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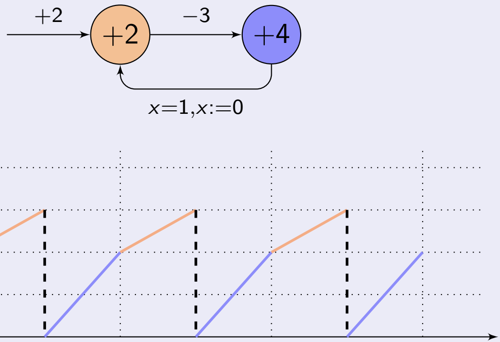

The Lower-Bound Problem for 1-Clock WTA

Theorem (FORMATS 2008)

For 1-clock WTA without weights on transitions, the lower-bound problem is solvable in polynomial time.

- corner-point abstraction

Theorem (HSCC 2010)

For 1-clock WTA with weights on transitions, the lower-bound problem is solvable in double-exponential time.

- completely different method, introducing energy functions

The Interval Problem for 1-Clock WTA

(Work in Progress)

Definition

An interval timed automaton $A = (L, E, I, F, r)$ consists of a finite set L of locations, a finite set $E \subseteq L \times \mathbb{Q}^3 \times L$ of transitions, subsets $I, F \subseteq L$ of initial and accepting locations, and weight rates $r : L \to \mathbb{Q}$.

- transitions $l \xrightarrow[{[a,b]}]{p} l'$: $[a, b]$ interval bound; p price
- spend some time in location l; take transition if $x \in [a, b]$; add p to x
- runs have initial energy and initial time budget
- can only spend time budget: no resets
  - almost a 1-clock WTA, but not quite

Interval Time Relations

- A basic interval timed automaton

$$l \xrightarrow[{[a,b]}]{p} l'$$

defines a relation

$$R = \{(x, t, x') \mid a \le x + r(l)\, t \le b, x' = x + r(l)\, t + p\}$$

- These can be composed:

$$l \xrightarrow[{[a,b]}]{p} l' \xrightarrow[{[c,d]}]{q} l''$$

corresponds to

$$R_1 \triangleright R_2 = \{(x_0, t_1{+}t_2, x_2) \mid \exists x_1 : (x_0, t_1, x_1) \in R_1, (x_1, t_2, x_2) \in R_2\}\,.$$

Theorem

With operations $\cup$ and $\triangleright$, relations as above form an idempotent semiring.

The Algebraic Approach to Energy Problems, I

Let $\mathcal{Q} = \mathbb{Q}^\infty \times \mathbb{Q}^\infty_{\ge 0} \times \mathbb{Q}^\infty$: the set of interval timed relations

- together with operations $\cup$ (addition) and $\rhd$ (multiplication)
  - $R_1 \rhd R_2 = \{(x_0, t_1 + t_2, x_2) \mid \exists x_1 : (x_0, t_1, x_1) \in R_1, (x_1, t_2, x_2) \in R_2\}$

$\mathcal{Q}$ forms an idempotent semiring:

- $\cup$ is associative & commutative, with unit $\emptyset$
- $\rhd$ is associative, with unit $\mathbb{1} = \{(x, 0, x) \mid x \in \mathbb{Q}^\infty\}$
- $\rhd$ distributes over $\cup$; $x \rhd \emptyset = \emptyset \rhd x = \emptyset$ for all x
- $x \cup x = x$ for all x

$\mathcal{Q}$ forms a continuous Kleene algebra:

- for all $Y \subseteq \mathcal{Q}$ and $x, z \in \mathcal{Q}$, $\bigcup Y$ exists and

$$x \rhd \big(\bigcup Y\big) \rhd z = \bigcup x \rhd Y \rhd z$$

The Algebraic Approach to Energy Problems, II

Let $n \geq 1$. $\mathcal{Q}^{n \times n}$: the semiring of $n \times n$ matrices over $\mathcal{Q}$

- with matrix addition $\cup$ and matrix multiplication $\rhd$

$\mathcal{Q}^{n \times n}$ is again a continuous Kleene algebra

- with $M^*_{i,j} = \bigcup_{m \geq 0} \bigcup_{1 \leq k_1, \dots, k_m \leq n} M_{i,k_1} M_{k_1,k_2} \cdots M_{k_m,j}$
- and for $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, can compute

$$M^* = \begin{bmatrix} (a \cup bd^*c)^* & (a \cup bd^*c)^* bd^* \\ (d \cup ca^*b)^* ca^* & (d \cup ca^*b)^* \end{bmatrix}$$

(recursively; "generalized Floyd-Warshal")

The Algebraic Approach to Energy Problems, III

$A = (\alpha, M, \kappa)$ an interval timed automaton

- $\alpha \in \{\emptyset, \mathbb{1}\}^n$ initial vector, $\kappa \in \{\emptyset, \mathbb{1}\}^n$ accepting vector, $M \in \mathcal{Q}^{n \times n}$ transition matrix
- finite path $s_i \xrightarrow{w_0} \cdots \xrightarrow{w_n} s_j$ accepting if $\alpha_i = \kappa_j = \mathbb{1}$
- finite behavior of A:

$$|A| = \bigvee \left\{ w_0 \cdots w_n \mid s_i \xrightarrow{w_0} \cdots \xrightarrow{w_n} s_j \text{ accepting finite path} \right\}$$

Theorem: $|A| = \alpha M^* \kappa$

The Algebraic Approach to Energy Problems, IV

Let $\mathcal{V} = \mathbb{Q}^\infty \times \mathbb{Q}^\infty_{\ge 0}$: interval timed relations without output

- for infinite runs
- with operation $\cup$ and unit $\emptyset$, $\mathcal{V}$ forms a commutative idempotent monoid

left $\mathcal{Q}$-action $\mathcal{Q} \times \mathcal{V} \to \mathcal{V}$: $(R, U) \mapsto R \rhd U$

- $(\mathcal{Q}, \mathcal{V})$ semiring-semimodule pair

infinite product $\mathcal{Q}^\omega \to \mathcal{V}$: for $R_0, R_1, \ldots \in \mathcal{Q}$, define

$$\prod R_n = \Big\{(x, t) \;\Big|\; \exists x_0, x_1, \ldots \in \mathbb{Q}^\infty, t_1, t_2, \ldots \in \mathbb{Q}^\infty_{\ge 0} : \sum_{n=0}^{\infty} t_n = t, \forall n \ge 0 : (x_n, t_{n+1}, x_{n+1}) \in R_n\Big\}$$

- $(\mathcal{Q}, \mathcal{V})$ continuous Kleene ω-algebra

The Algebraic Approach to Energy Problems, V

$(\mathcal{Q}, \mathcal{V})$ continuous Kleene ω-algebra:

- $\mathcal{Q}$ continuous Kleene algebra; $\mathcal{V}$ complete lattice
- $\mathcal{Q}$-action on $\mathcal{V}$ preserves all suprema: $x \triangleright \left(\bigcup Y\right) \triangleright u = \bigcup x \triangleright Y \triangleright u$
- and three axioms for the infinite product:
  - For all $x_0, x_1, \dots \in \mathcal{Q}$, $\prod x_n = x_0 \prod x_{n+1}$.
  - Let $x_0, x_1, \dots \in \mathcal{Q}$ and $0 = n_0 \le n_1 \le \cdots$ a sequence which increases without a bound. Let $y_k = x_{n_k} \cdots x_{n_{k+1}-1}$ for all $k \ge 0$. Then $\prod x_n = \prod y_k$.
  - For all $X_0, X_1, \dots \subseteq \mathcal{Q}$, $\prod(\bigvee X_n) = \bigvee\{\prod x_n \mid x_n \in X_n, n \ge 0\}$.

The Algebraic Approach to Energy Problems, VI

$(\mathcal{Q}^{n\times n}, \mathcal{V}^n)$ is again a continuous Kleene ω-algebra

- with $M_i^\omega = \bigcup_{1\le k_1,k_2,\ldots\le n} M_{i,k_1} M_{k_1,k_2} \cdots$
- and for $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, can compute

$$M^\omega = \begin{bmatrix} (a \cup bd^* c)^\omega \cup (a \cup bd^* c)^* bd^\omega \\ (d \cup ca^* b)^\omega \cup (d \cup ca^* b)^* ca^\omega \end{bmatrix}$$

(recursively)

The Algebraic Approach to Energy Problems, VII

$A = (\alpha, M, \kappa)$ an interval timed automaton

- infinite path $s_i \xrightarrow{w_0}\xrightarrow{w_1} \cdots$ accepting if $\alpha_i = \mathbb{1}$ and some s_j with $\kappa_j = \mathbb{1}$ is visited infinitely often
- Büchi behavior of A:

$$\|A\| = \bigvee \{ \prod w_n \mid s_i \xrightarrow{w_0}\xrightarrow{w_1} \cdots \text{accepting infinite path} \}$$

- Re-order states so that $\kappa = (\mathbb{1}, \dots, \mathbb{1}, \emptyset, \dots, \emptyset)$
- i.e. the first $k \le n$ states are accepting

Theorem: with $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for $a \in S^{k \times k}$, $\|A\| = \alpha \begin{bmatrix} (a + bd^* c)^\omega \\ d^* c (a + bd^* c)^\omega \end{bmatrix}$

Conclusion

- Formal methods for solving energy problems
- Applications in scheduling
- Continuous Kleene ω-algebras: obscure algebraic theory with real-world applications!

The work on the interval problem presented here is only half complete: we've found a nice algebraic setting; but we've said nothing about actual computations

- See our FM 2018 paper for actual computations in a restricted setting ("segmented energy timed automata")
- Rest is future work

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