The Quantitative Linear-Time–Branching-Time Spectrum

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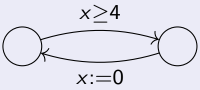
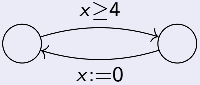
Quantitative Analysis

Quantitative *Models*



Quantitative Quantitative Analysis

Quantitative *Models*

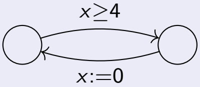


Quantitative *Logics*

$\mathsf{Pr}_{\leq .1}(\Diamond \mathit{error})$

Quantitative Quantitative Quantitative Analysis

Quantitative *Models*



Quantitative *Logics*

$$\Pr_{\leq .1}(\Diamond \mathit{error})$$

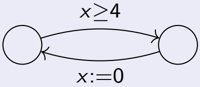
Quantitative *Verification*

$$[\![\phi]\!](s) = 3.14$$

$$d(s, t) = 42$$

Quantitative Quantitative Quantitative Analysis

Quantitative *Models*



Quantitative *Logics*

$\Pr_{\leq .1}(\Diamond \mathit{error})$

Quantitative *Verification*

$[\![\phi]\!](s) = 3.14$

$d(s, t) = 42$

Boolean world	**"Quantification"**
Trace equivalence $\equiv$	Linear distances d_L
Bisimilarity $\sim$	Branching distances d_B
$s \sim t$ implies $s \equiv t$	$d_L(s, t) \leq d_B(s, t)$
$s \models \phi$ or $s \not\models \phi$	$[\![\phi]\!](s)$ is a quantity
$s \sim t$ iff $\forall \phi : s \models \phi \Leftrightarrow t \models \phi$	$d_B(s, t) = \sup_\phi d\big([\![\phi]\!](s), [\![\phi]\!](t)\big)$

Quantitative Quantitative Quantitative Analysis

Problem: For processes with quantities, lots of different ways to measure distance

- point-wise $D(\sigma, \tau) = \sup_i |\sigma_i - \tau_i|$
- accumulating $D(\sigma, \tau) = \sum_i |\sigma_i - \tau_i|$
- limit-average $D(\sigma, \tau) = \limsup_N \frac{1}{N} \sum_{i=0}^N |\sigma_i - \tau_i|$
- discounted $D(\sigma, \tau) = \sum_i \lambda^i |\sigma_i - \tau_i|$
- maximum-lead $D(\sigma, \tau) = \sup_N |\sum_{i=0}^N \sigma_i - \sum_{i=0}^N \tau_i|$
- Cantor $D(\sigma, \tau) = 1/(1 + \inf\{j \mid \sigma_j \neq \tau_j\})$
- discrete $D(\sigma, \tau) = 0$ if $\sigma = \tau$; ∞ otherwise
- etc.

Upshot

- For an application, it is easiest to define distance between system traces (executions)
- Use games to convert these *linear* distances to *branching* distances
- Use other games to compute branching distances
- Extends also to specifications

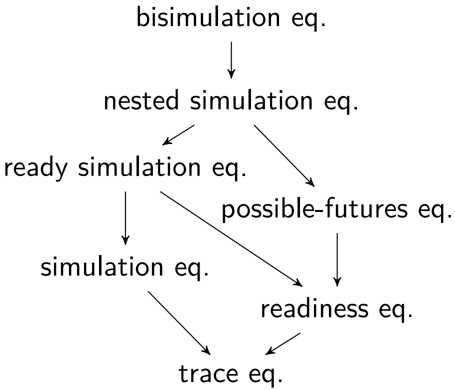
Nice people

- Kim G. Larsen, Claus R. Thrane (DK)
- Axel Legay, Louis-Marie Traonouez (FR)
- Karin Quaas (DE)
- Jan Křetínský, Nikola Beneš (CZ)

- [UF, C.R.Thrane. K.G.Larsen: *Quantitative analysis of weighted transition systems*. JLAP 2010]
- [UF, A.Legay: *The Quantitative Linear-Time–Branching-Time Spectrum*. TCS 2014]
- [UF, J.Křetínský, A.Legay, L.-M.Traonouez: *Compositionality for quantitative specifications*. SoCo 2018]
- [N.Beneš, UF, J.Křetínský, A.Legay, L.-M.Traonouez: *Logical vs. Behavioural Specifications*. I&C 2020]
- [UF, A.Legay, K.Quaas: *Computing Branching Distances Using Quantitative Games*. *Proc. ICTAC* 2019]

The Linear-Time–Branching-Time Spectrum

van Glabbeek, 2001 (excerpt):



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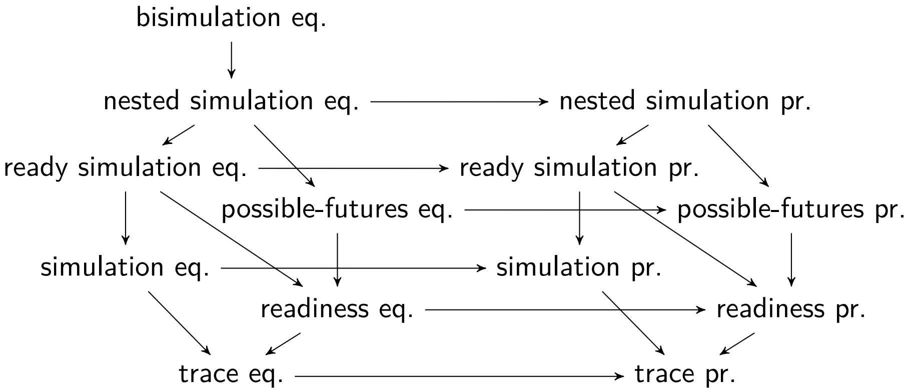

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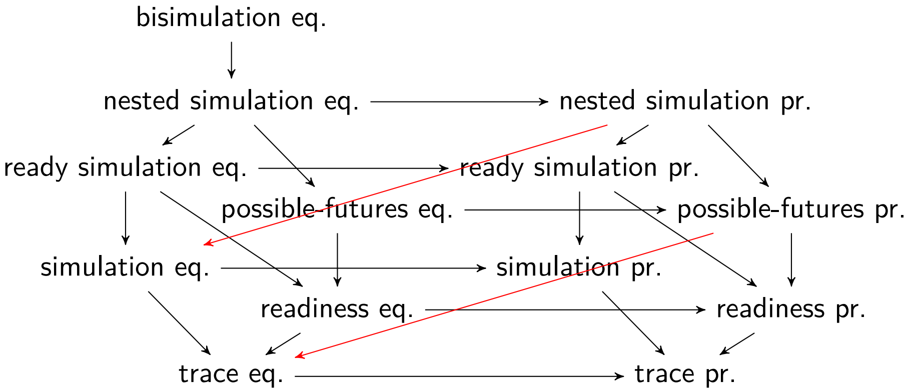
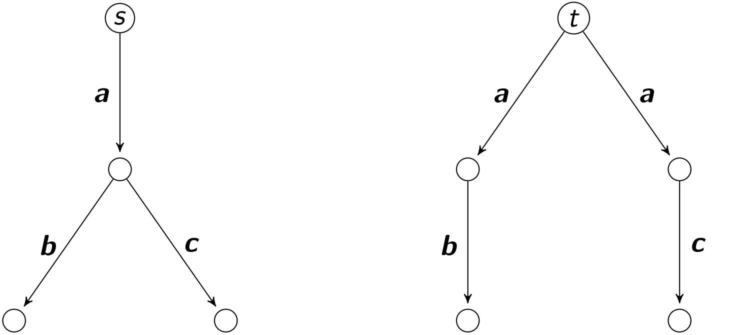


The Simulation Game



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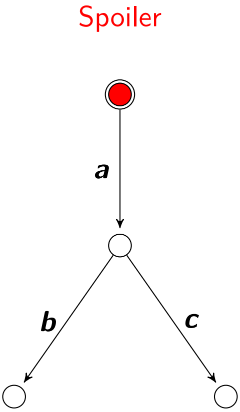

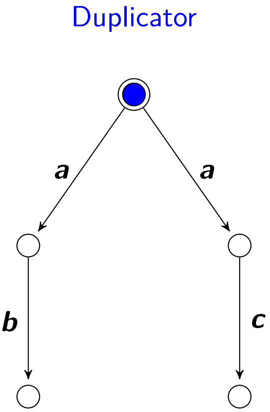

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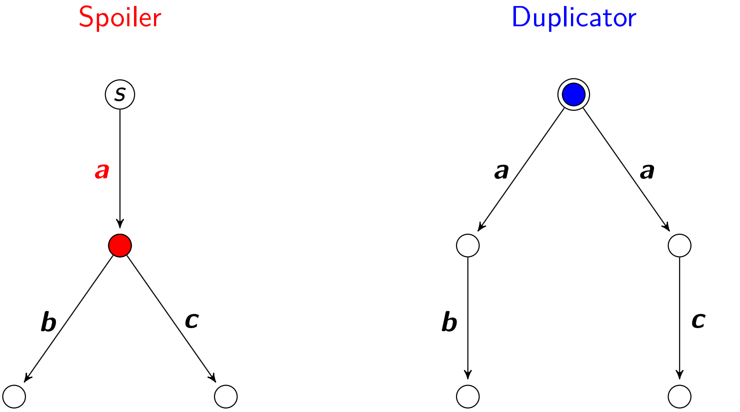

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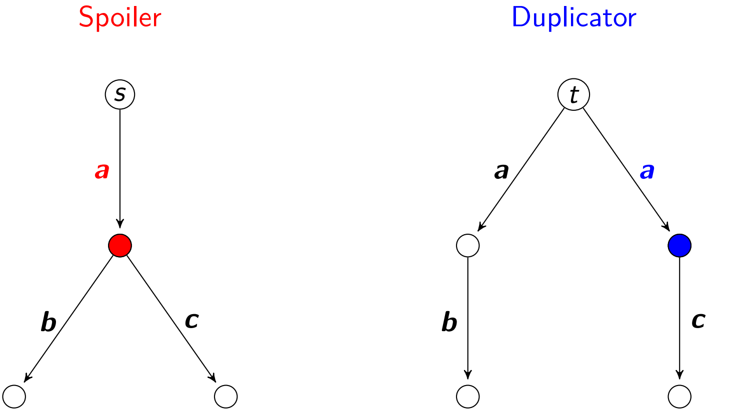

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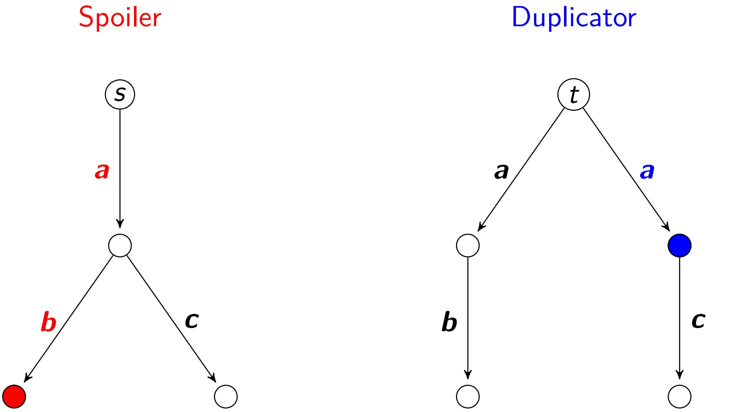

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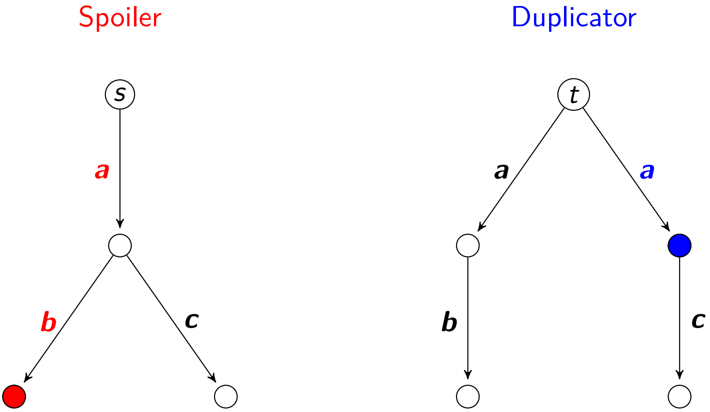


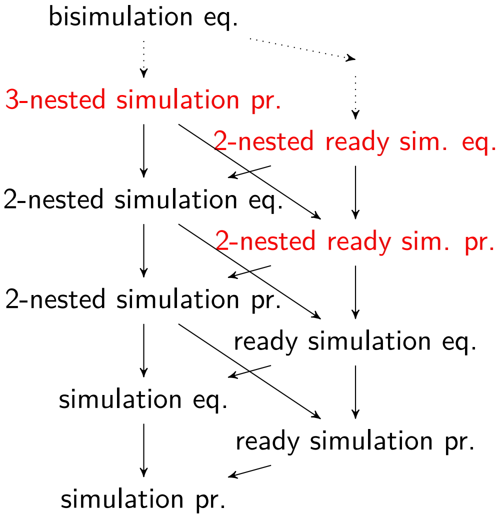
Spoiler wins

The Simulation Game

1. Player 1 ("Spoiler") chooses edge from s (leading to s')
2. Player 2 ("Duplicator") chooses matching edge from t (leading to t')
3. Game continues from configuration s', t'

ω. If Player 2 can always answer: YES, t simulates s.
Otherwise: NO

The Linear-Time–Branching-Time Spectrum, Reordered



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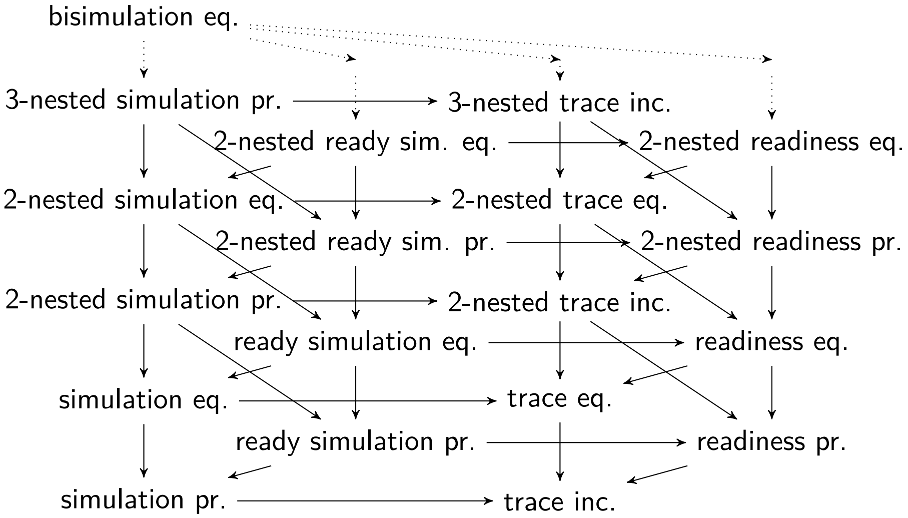


The Simulation Game, Revisited

1. Player 1 chooses edge from s (leading to s')
2. Player 2 chooses matching edge from t (leading to t')
3. Game continues from configuration s', t'

ω. If Player 2 can always answer: YES, t simulates s.
Otherwise: NO

Or, as an Ehrenfeucht-Fraïssé game:

1. Player 1 chooses edge from s (leading to s')
2. Player 2 chooses edge from t (leading to t')
3. Game continues from new configuration s', t'

ω. At the end (maybe after infinitely many rounds!),
compare the chosen traces:
If the trace chosen by t matches the one chosen by s: YES
Otherwise: NO

Quantitative Ehrenfeucht-Fraïssé Games

The quantitative setting:

- Assume we have a way, possibly application-determined, to measure distances of (finite or infinite) traces
- a hemimetric $D : (\sigma, \tau) \mapsto D(\sigma, \tau) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$

The quantitative Ehrenfeucht-Fraïssé game:

1. Player 1 chooses edge from s (leading to s')
2. Player 2 chooses edge from t (leading to t')
3. Game continues from new configuration s', t'

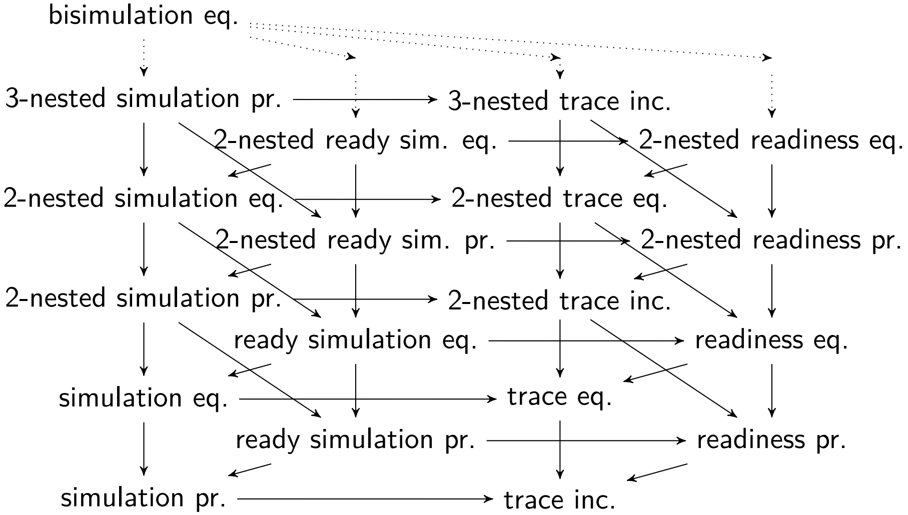
ω. At the end, compare the chosen traces σ, τ:
The simulation distance from s to t is defined to be $D(\sigma, \tau)$

- Player 1 plays to maximize $D(\sigma, \tau)$; Player 2 plays to minimize $D(\sigma, \tau)$

This can be done for all the games in the LTBT spectrum.

The Quantitative Linear-Time–Branching-Time Spectrum

For any trace distance $D : (\sigma, \tau) \mapsto D(\sigma, \tau) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$:



Quantitative EF Games: The Gory Details – 1

- Configuration of the game: (π, ρ): π the Player-1 choices up to now; ρ the Player-2 choices
- Strategy: mapping from configurations to next moves
  - Θ_i: set of Player-i strategies
- Simulation strategy: Player-1 moves allowed from end of π
- Bisimulation strategy: Player-1 moves allowed from end of π or end of ρ
  - (hence π and ρ are generally not paths – "mingled paths")
- Pair of strategies $\Longrightarrow$ (possibly infinite) sequence of configurations
- Take the limit; unmingle $\Longrightarrow$ pair of (possibly infinite) traces (σ, τ)
- Bisimulation distance: $\sup_{\theta_1 \in \Theta_1} \inf_{\theta_2 \in \Theta_2} d_T(\sigma, \tau)$
- Simulation distance: $\sup_{\theta_1 \in \Theta_1^0} \inf_{\theta_2 \in \Theta_2} d_T(\sigma, \tau)$ (restricting Player 1's capabilities)

Quantitative EF Games: The Gory Details – 2

- Blind Player-1 strategies: depend only on the end of ρ
  - ("cannot see Player-2 moves")
  - $\tilde{\Theta}_1$: set of blind Player-1 strategies
- Trace inclusion distance: $\sup_{\theta_1 \in \tilde{\Theta}_1^0} \inf_{\theta_2 \in \Theta_2} d_T(\sigma, \tau)$
- For nesting: count the number of times Player 1 choses edge from end of ρ
  - Θ_1^k: k choices from end of ρ allowed
- Nested simulation distance: $\sup_{\theta_1 \in \Theta_1^1} \inf_{\theta_2 \in \Theta_2} d_T(\sigma, \tau)$
- Nested trace inclusion distance: $\sup_{\theta_1 \in \tilde{\Theta}_1^1} \inf_{\theta_2 \in \Theta_2} d_T(\sigma, \tau)$
- For ready: allow extra "I'll see you" Player-1 transition from end of ρ

Transfer Theorem

- Given two equivalences or preorders in the *qualitative* setting for which there is a *counter-example* which separates them, then the two corresponding distances are topologically inequivalent
- (under certain mild conditions for the trace distance)
- (The proof uses precisely the same counter-example)

Recursive Characterization

- If the trace distance $D : (\sigma, \tau) \mapsto d(\sigma, \tau)$ has a decomposition $d = g \circ f : \mathsf{Tr} \times \mathsf{Tr} \to L \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ through a complete lattice L,
- and f has a recursive formula
- *i.e.* such that $f(\sigma, \tau) = F(\sigma_0, \tau_0, f(\sigma^1, \tau^1))$ for some $F : \Sigma \times \Sigma \times L \to L$ (which is *monotone* in the third coordinate)
- (where $\sigma = \sigma_0 \cdot \sigma^1$ is a split of σ into first element and tail)
- then all distances in the QLTBT are given as least fixed points of some functionals using F

All trace distances we know can be expressed recursively like this.

Recursive Characterization: One of Four Theorems

The endofunction I on $(\mathbb{N}_+ \cup \{\infty\}) \times \{1,2\} \to L^{S \times S}$ defined by

$$I(h_{m,p})(s,t) = \begin{cases} \max \begin{cases} \sup\limits_{s \xrightarrow{x} s'} \inf\limits_{t \xrightarrow{y} t'} F(x, y, h_{m,1}(s', t')) \\ \sup\limits_{t \xrightarrow{y} t'} \inf\limits_{s \xrightarrow{x} s'} F(x, y, h_{m-1,2}(s', t')) \end{cases} & \text{if } m \geq 2, p = 1 \\ \sup\limits_{s \xrightarrow{x} s'} \inf\limits_{t \xrightarrow{y} t'} F(x, y, h_{m,1}(s', t')) & \text{if } m = 1, p = 1 \\ \max \begin{cases} \sup\limits_{t \xrightarrow{y} t'} \inf\limits_{s \xrightarrow{x} s'} F(x, y, h_{m,2}(s', t')) \\ \sup\limits_{s \xrightarrow{x} s'} \inf\limits_{t \xrightarrow{y} t'} F(x, y, h_{m-1,1}(s', t')) \end{cases} & \text{if } m \geq 2, p = 2 \\ \sup\limits_{t \xrightarrow{y} t'} \inf\limits_{s \xrightarrow{x} s'} F(x, y, h_{m,2}(s', t')) & \text{if } m = 1, p = 2 \end{cases}$$

has a least fixed point $h^* : (\mathbb{N}_+ \cup \{\infty\}) \times \{1,2\} \to L^{S \times S}$, and if the LTS (S, T) is finitely branching, then $d^{k\text{-sim}} = g \circ h^*_{k,1}$ for all $k \in \mathbb{N}_+ \cup \{\infty\}$.

Path-Building Games

- Have seen how branching distances can be defined using a type of "double path-building game"
- But now, how to compute them?
- Nothing in the literature about computing values of double path-building games . . .
- On the other hand, people know how to compute values of (single) path-building games!
    - reachability games; discounted games; mean-payoff games, . . .
- So, let's convert our double path-building games to single path-building games

From Double to Single Path-Building Games

- Let $D : (\sigma, \tau) \mapsto D(\sigma, \tau) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ be a hemimetric on traces
- Assume that we have functions val_D and f_D such that always,

$$D(\sigma, \tau) = \mathsf{val}_D(0, f_D(\sigma_0, \tau_0), 0, f_D(\sigma_1, \tau_1), 0, \dots)$$

- Let $\mathcal{S} = (S, i, T)$ and $\mathcal{S}' = (S', i', T')$ be LTS
- Construct a game $\mathcal{U} = \mathcal{U}(\mathcal{S}, \mathcal{S}') = (U_1 \cup U_2, u_0, \rightharpoonup)$ by

$$U_1 = S \times S' \qquad U_2 = S \times S' \times \Sigma \qquad u_0 = (i, i')$$

$$\rightharpoonup = \{(s, s') \xrightarrow{0} (t, s', a) \mid (s, a, t) \in T\}$$

$$\cup \{(t, s', a) \xrightarrow{f_D(a, a')} (t, t') \mid (s', a', t') \in T'\}$$

- Path-building game: players alternate to build path π
- Player 1 plays to maximize $\mathsf{val}_D(\pi)$; Player 2 plays to minimize $\mathsf{val}_D(\pi)$

Computing Distances Using Path-Building Games

$\mathcal{U}(\mathcal{S}, \mathcal{S}') = (U_1 \cup U_2, u_0, \multimap)$:

$$U_1 = S \times S' \qquad U_2 = S \times S' \times \Sigma \qquad u_0 = (i, i')$$

$$\begin{aligned} \multimap = \{(s, s') \xrightarrow{0} (t, s', a) \mid (s, a, t) \in T\} \\ \cup \{(t, s', a) \xrightarrow{f_D(a,a')} (t, t') \mid (s', a', t') \in T'\} \end{aligned}$$

Theorem

The value of $\mathcal{U}(\mathcal{S}, \mathcal{S}')$ is the simulation distance from $\mathcal{S}$ to $\mathcal{S}'$.

Computing Distances Using Path-Building Games, contd.

$\mathcal{V}(\mathcal{S}, \mathcal{S}') = (V_1 \cup V_2, v_0, \twoheadrightarrow)$:

$$V_1 = S \times S' \qquad V_2 = S \times S' \times \Sigma \times \{1,2\} \qquad v_0 = (i, i')$$

$$\begin{aligned}
\twoheadrightarrow = &\{(s,s') \xrightarrow{0} (t, s', a, 1) \mid (s,a,t) \in T\} \\
&\cup \{(s,s') \xrightarrow{0} (s, t', a', 2) \mid (s', a', t') \in T'\} \\
&\cup \{(t, s', a, 1) \xrightarrow{f_D(a,a')} (t, t') \mid (s', a', t') \in T'\} \\
&\cup \{(s, t', a', 2) \xrightarrow{f_D(a,a')} (t, t') \mid (s, a, t) \in T\}
\end{aligned}$$

Theorem

The value of $\mathcal{V}(\mathcal{S}, \mathcal{S}')$ is the bisimulation distance between $\mathcal{S}$ and $\mathcal{S}'$.

- Similar constructions for all distances in the linear-time–branching-time spectrum

Coda: Computing the Values of Path-Building Games

$\mathcal{U}(\mathcal{S}, \mathcal{S}') = (U_1 \cup U_2, u_0, \multimap)$:

$$U_1 = S \times S' \qquad U_2 = S \times S' \times \Sigma \qquad u_0 = (i, i')$$

$$\multimap = \{(s, s') \xrightarrow{0} (t, s', a) \mid (s, a, t) \in T\}$$
$$\cup \{(t, s', a) \xrightarrow{f_D(a, a')} (t, t') \mid (s', a', t') \in T'\}$$

- discrete distance: reachability game — PTIME
- point-wise distance: weighted reachability game — PTIME
- discounted distance: discounted game — NP ∩ coNP
- limit-average distance: mean-payoff game — NP ∩ coNP
- maximum-lead distance: energy game — NEXPTIME
- Cantor distance: iterated reachability game — PTIME

Conclusion & Further Work

- A general method to define linear and branching system distances using double path-building games
- A general method to compute linear and branching system distances using (single) path-building games

- Application to real-time and hybrid systems
- Quantitative specification theories
- Quantitative LTBT with silent moves?
- What about probabilistic systems?