ℓr-Multisemigroups, Modal Quantales and the Origin of Locality

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Abstract. We introduce ℓr-multisemigroups as duals of modal quantales and study modal correspondences between equations in these multisemigroups and the domain and codomain axioms of modal quantales. Our results yield new insights on the origin of locality in modal semirings and quantales. They also yield construction principles for modal powerset quantales that cover a wide range of models and applications.

1 Introduction

This work adds to a series on convolution semirings and quantales built over relational monoids and multimonoids [3, 8, 12]. It explains the structure of modal semirings and quantales [7, 11], not generally for convolution algebras [12], but specifically for modal powerset quantales—the standard setting for computational models in this context. We consider such quantales as boolean algebras with operators [19]. The quantalic composition is then a binary modality; the domain and codomain operations needed for defining modal operators are unary ones. We ask about the dual relational structure in the sense of Jónsson and Tarski [19] and its equational properties corresponding to the modal quantale axioms for domain and codomain [7, 11] in the sense of modal correspondence theory. For plain quantales, this is well known: the dual monoidal structure is a ternary relation equipped with a relational monoid structure and many units [3, 8]—a monoid in the category **Rel** with the standard tensor. Yet which relational structure corresponds to domain and codomain?

The standard models of modal semirings and quantales give us a hint: modal quantales of binary relations, for instance, are powerset liftings of pair groupoids; modal quantales of paths lift from path categories. We might therefore try to lift (object-free) categories [23, Chapter XII.5] to modal quantales so that their source and target maps match the domain and codomain operations of modal quantales. Categories, however, are partial monoids, whereas relational monoids

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are isomorphic to multimonoids, whose composition maps pairs of elements to sets, like the shuffle of words. Other examples, such as the lifting of partial abelian monoids of heaplets to assertion quantales of separation logic, do not fall into this lifting scheme with categories either. A generalisation is desirable.

We introduce ℓr-multisemigroups as relational structures in disguise and the appropriate dual structures to modal powerset quantales. Categories then arise as partial ℓr-semigroups (where the image of the multioperation is suitably restricted) that satisfy a locality property capturing the categorical composition pattern: two arrows are composable precisely if the target of the first equals the source of the second. Thus, ℓr-multisemigroups generalise object-free categories and related structures such as function systems [28], ordered semigroupoids [21] and modal semigroups [5] from (partial) operations to multioperations.

Our second main contribution lies in modal correspondences between identities in families of modal quantales with axioms of varying strength and those of families of ℓr-multisemigroups. The most intriguing one holds between the well studied locality axioms for domain and codomain in modal semirings and quantales and similar identities in ℓr-multisemigroups, which in turn are equivalent to the composition pattern for categories mentioned. This explains the origin of locality of domain and codomain in modal semirings and quantales in terms of this fundamental pattern. It also makes local ℓr-multisemigroups the algebras of choice for constructing modal quantales axiom by axiom.

Our results thus provide a generic construction recipe for modal quantales from simpler structures: every ℓr-multisemigroup gives us a modal powerset quantale for free—and even modal convolution quantales capturing weighted variants of the models presented in this text. This generalisation is briefly outlined at the end of this article, see [12] for details.

All results about ℓr-multisemigroups and the lifting to modal powerset quantales have been formalised with Isabelle/HOL[5]. The proofs for ℓr-multisemigroups are straightforward equational calculations that do not need to be shown on paper. The proof of the powerset lifting has been added because it yields an intuition for the more complex construction of modal convolution quantales. Additional proofs, definitions and explanations can be found in [12], including a glossary of the algebraic structures featured in this text.

2 ℓr-Multisemigroups and Object-Free Categories

As mentioned in the introduction, the dual of the binary composition of a quantale is a ternary relation. For powerset quantales it is defined on their atom structure of singleton sets. But instead of a ternary relation $R \subseteq X \times X \times X$ on a set X, say, we work with the isomorpic multioperation $\odot : X \times X \to \mathcal{P}X$ and the resulting multisemigroups. See [22] for an overview. Henceforth we are using "set" naively, so that we can speak, for instance, about the set of all posets and include large categories as examples.

[5] `https://github.com/gstruth/lr-multisemigroups`

We extend the multioperation $\odot$ to $\mathcal{P}X \times \mathcal{P}X \to \mathcal{P}X$ by

$$A \odot B = \bigcup \{x \odot y \mid x \in A \text{ and } y \in B\} \qquad \text{for all } A, B \subseteq X.$$

We write $x \odot B$ instead of $\{x\} \odot B$, $A \odot x$ instead of $A \odot \{x\}$, $f(A)$ for the image of A under f and drop $\odot$ when convenient. Finally, $\odot$ is a *partial operation* if $|x \odot y| \leq 1$ and a *(total) operation* if $|x \odot y| = 1$, for all $x, y \in X$.

A *multimagma* $(X, \odot)$ is a set X with a multioperation $\odot$ on X. A *multisemigroup* X is an associative multimagma, it satisfies $x \odot (y \odot z) = (x \odot y) \odot z$ for all $x, y, z \in X$. Partial semigroups and semigroups are defined by restricting the image of $\odot$ as just explained.

Object-free categories are obtained either by defining source and target maps on partial semigroups or by equipping partial semigroups with many units [23]. We explore both ways more generally for multisemigroups.

An *ℓr-multimagma* is a multimagma X with operations $\ell, r : X \to X$ that satisfy, for all $x, y \in X$,

$$x \odot y \neq \emptyset \Rightarrow r(x) = \ell(y), \qquad \ell(x) \odot x = \{x\}, \qquad x \odot r(x) = \{x\}.$$

An *ℓr-multisemigroup* is an associative ℓr-multimagma. We call ℓ the source operation and r the target operation of X. The letters indicate “left” and “right”.

Alternatively, a multimagma X is *unital* if there exists a set $E \subseteq X$ such that $E \odot x = \{x\} = x \odot E$ for all $x \in X$. A *multimonoid* is then a unital multisemigroup. See [12] for a more detailed discussion.

We briefly summarise the relationship between the two structures. First, in unital multimagmas, every $e \in E$ satisfies $e \odot e = \{e\}$ and, if $e, e' \in E$, then $e \odot e' \neq \emptyset \Leftrightarrow e = e'$. Units are thus “orthogonal” idempotents. In multimonoids, every element has therefore precisely one left and one right unit, and this allows defining source and target maps. Second, the set $\ell(X)$ of all source elements in any ℓr-multisemigroup X equals the set $r(X)$ of all target elements and the elements of those sets satisfy the unit axioms for multimonoids (see also Section 4). Third, ℓr-multisemigroups and multimonoids form categories with morphisms satisfying $f(x \odot_1 y) \subseteq f(x) \odot_2 f(y)$ for multisemigroups $(X_i, \odot_i)$ with $i \in \{1, 2\}$. For ℓr-multisemigroups, morphisms need to preserve ℓ and r as well; for multimonoids they need to preserve units. It is then easy to see that the categories of ℓr-multisemigroups and multimonoids are isomorphic [12].

Partial ℓr-semigroups are not yet (object-free) categories—see Examples 7 and 8 below. We need to impose the typical composition pattern of categories: two morphisms can be composed if the target of the first equals the source of the second. So we call an ℓr-multimagma *ℓr-local* if

$$r(x) = \ell(y) \Rightarrow x \odot y \neq \emptyset \qquad \text{for all } x, y \in X.$$

We relate this property with notions of locality known from modal semigroups and semirings in Section 4. Example 6 below shows a local ℓr-multisemigroup with a proper multioperation that does not form an object-free category.

An ℓr-multisemigroup X is ℓr-local if and only if

$$u \in x \odot y \ \wedge\ y \odot z \neq \emptyset \ \Rightarrow\ u \odot z \neq \emptyset \qquad \text{for all } u, x, y, z \in X.$$

This implication is expressible in any multimagma. The connection to the two equivalent formalisations of (object-free) categories in Mac Lane's book [23] is thus as follows.

Proposition 1 ([4]). *The categories of object-free categories [23, Chapter I.1] and those of local partial monoids are isomorphic.*

Proposition 2. *The categories of object-free categories [23, Chapter XII.5] and those of ℓr-local partial ℓr-semigroups are isomorphic.*

The morphisms used are those outlined above. Hence local partial ℓr-semigroups *are* categories (when these structures are defined over classes).

3 Examples of ℓr-Multisemigroups

We start with concrete instances of categories.

Example 3 (Monoids). Monoids are one-object categories. The monoid $1 \xrightarrow{a} 1$, for instance, corresponds to a partial monoid $X = \{1, a\}$ with composition defined by $11 = \{1\}$ and $1a = a1 = aa = \{a\}$. Obviously, $\ell(a) = 1 = r(a)$ and locality follows from totality of composition. □

Multimonoids must have precisely one unit if the multioperation is total (in the sense that images of compositions cannot be empty).

Example 4 (Pair Groupoids). The pair groupoid $(X \times X, \odot, Id_X)$ on set X (or the universal relation on X) is a local partial ℓr-semigroup with

$$(w, x) \odot (y, z) = \begin{cases} \{(w, z)\} & \text{if } x = y, \\ \emptyset & \text{otherwise,} \end{cases}$$

identity relation Id_X on X, $\ell((x, y)) = (x, x)$ and $r((x, y)) = (y, y)$. □

Pair groupoids lift to quantales of binary relations.

Example 5 (Matrix Theories). Elgot's matricial theories [9] consist of sets $\mathbb{M}S = \bigcup_{n,m \geq 0} S^{n \times m}$ of matrices over a semiring S with matrix multiplication as partial composition. These form a category with natural numbers as objects and $n \times m$-matrices as morphisms. Defining ℓ and r to map any $M \in S^{n \times m}$ to the identity matrices $\ell(M) = I_n$ and $r(M) = I_m$ of the appropriate dimensions, $\mathbb{M}S$ forms a local partial ℓr-semigroup. Matrix theories become categories of finite relations if S is the semiring of booleans. □

The next example presents a local proper ℓr-multisemigroup.

Example 6 (Shuffle Algebras). The shuffle multimonoid $(\Sigma^*, \|, \{\varepsilon\})$ over the free monoid Σ^* has the empty word ε as its unit, and the proper multioperation $\| : \Sigma^* \times \Sigma^* \to \mathcal{P}\Sigma^*$ models the standard interleaving of words that respects the orders of their letters. The shuffle multimonoid is local because $\|$ is total (defined everywhere) and $\ell(w) = \varepsilon = r(w)$. □

Finally, here are two non-local partial semigroups.

Example 7 (Broken Monoid). The monoid in Example 3 becomes a non-local partial ℓr-semigroup when composition is broken by imposing $aa = \emptyset$. □

Example 8 (Heaplets). The partial abelian monoid of heaplets $(H, \odot, \varepsilon)$ from separation logic is formed by the set of partial functions $X \rightharpoonup Y$. Its partial operation $f \odot g$ equals $f \cup g$ if $dom(f) \cap dom(g)$ is empty and $\emptyset$ otherwise. The unit is the empty partial function ε with empty domain. Locality fails because $\ell(f) = \varepsilon = r(g)$ always holds while $f \odot g = \emptyset$ if domains of f and g overlap. □

4 ℓr-Multisemigroups in Context

We have already seen that local partial ℓr-semigroups are categories. Here we relate them with Schweizer and Sklar's function systems [28] and modal semigroups [5]. The following property gives us half of our results for free.

Duality (by opposition) for ℓr-multimagmas arises by interchanging ℓ and r as well as the arguments of $\odot$. The classes of ℓr-multimagmas and ℓr-multisemigroups are closed under this transformation. Locality and partiality are self-dual. Hence the dual of any property that holds in any of these classes holds as well.

Lemma 9. *In any ℓr-multimagma, the following laws hold:*

1. $\ell \circ r = r$, $r \circ \ell = \ell$ (compatibility),
2. $\ell \circ \ell = \ell$, $r \circ r = r$ (retraction),
3. $\ell(x)\ell(x) = \{\ell(x)\}$ (idempotency),
4. $r(x)\ell(y) = \ell(y)r(x)$ (commutativity),
5. $\ell(\ell(x)y) = \ell(x)\ell(y)$ *and* $r(xr(y)) = r(x)r(y)$ (export),
6. $\ell(xy)x \subseteq x\ell(y)$ *and* $xr(yx) \subseteq r(y)x$ (weak twisted).

All proofs have been checked with Isabelle. All laws in Lemma 9 correspond to axioms for Schweizer and Sklar's function systems [28] (see [12] for a detailed comparison), yet generalised to multioperations.

The compatibility laws imply that $\ell(x) = x \Leftrightarrow r(x) = x$ and further that

$$X_\ell = \{x \mid \ell(x) = x\} = \{x \mid r(x) = x\} = X_r.$$

Moreover, by the retraction laws, $X_\ell = \ell(X)$ and $X_r = r(X)$.

Lemma 9 also implies that $\ell(x)\ell(y) = \ell(y)\ell(x)$, $r(x)r(y) = r(y)r(x)$ and $r(x)r(x) = \{r(x)\}$. Further, the orthogonality law $\ell(x)\ell(y) \neq \emptyset \Leftrightarrow \ell(x) = \ell(y)$ and its dual hold. As ℓr-Multimagmas are unital, we may write E for X_ℓ or X_r.

Lemma 10. *In any ℓr-multisemigroup, the following laws hold:*

1. $\ell(xy) \subseteq \ell(x\ell(y))$ *and* $r(xy) \subseteq r(r(x)y)$ (weak locality),
2. $xy \neq \emptyset \Rightarrow \ell(xy) = \ell(x\ell(y))$ *and* $xy \neq \emptyset \Rightarrow r(xy) = r(r(x)y)$ (cond. locality),
3. $\ell(xy) \subseteq \{\ell(x)\}$ *and* $r(xy) \subseteq \{r(y)\}$,
4. $xy \neq \emptyset \Rightarrow \ell(xy) = \{\ell(x)\}$ *and* $xy \neq \emptyset \Rightarrow r(xy) = \{r(y)\}$,
5. $xy \neq \emptyset \Rightarrow \ell(xy)x = x\ell(y)$ *and* $xy \neq \emptyset \Rightarrow yr(xy) = r(x)y$ (cond. twisted).

Proofs have again been checked with Isabelle. The locality and twisted laws generalise the remaining axioms of function systems. Function systems without the twisted laws correspond to modal semigroups [5] and therefore semigroups of binary relations. The twisted laws are specific to semigroups of functions. ℓr-Multisemigroups thus generalise function systems and modal semigroups beyond totality. See [5] for a discussion of related structures studied in semigroups theory and applications.

5 ℓr-Locality in Context

Next we return to locality, the specific difference between object-free categories and partial ℓr-semigroups according to Section 2.

Lemma 11. *In any local ℓr-multisemigroup, the following laws hold:*

1. $\ell(xy) = \ell(x\ell(y))$ *and* $r(xy) = r(r(x)y)$ *(equational locality),*
2. $\ell(xy)x = x\ell(y)$ *and* $yr(xy) = r(x)y$ *(twisted).*

Once again, all proofs have been done with Isabelle. In fact, ℓr-locality, the composition pattern of categories, is an equational property. We henceforth refer to equational locality simply as *locality*.

Proposition 12. *An ℓr-multisemigroup is ℓr-local if and only if*

$$\ell(x\ell(y)) \subseteq \ell(xy) \qquad \text{and} \qquad r(r(x)y) \subseteq r(xy).$$

Proof. Isabelle confirms that the equational locality laws imply ℓr-locality in any ℓr-multimagma. Equality in ℓr-multisemigroups follows from Lemma 11. □

Locality and weak locality are known from (pre)domain and (pre)codomain operations for modal semirings [6]. Predomain and precodomain operations are weakly local, domain and codomain are local. Relative to ℓr-multisemigroups, these laws are at powerset level. Modal semirings are meant to model semirings of binary relations. These in turn are based on pair groupoids, as we shall see. Equational locality and the equivalent variant

$$xy \neq \emptyset \Leftrightarrow r(x) = \ell(y)$$

of ℓr-locality thus describe the origin of locality in categories and more generally ℓr-multisemigroups. The precise relationship to modal semirings and quantales is explained in the following sections.

Our final lemma on ℓr-multisemigroups yields a more fine-grained view on definedness conditions and ℓr-locality.

Lemma 13.

1. *In any ℓr-multimagma,*

$$r(x) = \ell(y) \Leftrightarrow r(x)\ell(y) \neq \emptyset \qquad \text{and} \qquad r(x)\ell(y) = \emptyset \Rightarrow xy = \emptyset.$$

2. *In any local ℓr-multisemigroup, $xy = \emptyset \Leftrightarrow r(x)\ell(y) = \emptyset$.*

A property analogous to Lemma 13(2) is well known from modal semirings [6]. An analogue to ℓr-locality fails already in the one-element modal semiring.

6 Modal Quantales

We have already extended $\odot : X \times X \to \mathcal{P}X$ to $\mathcal{P}X \times \mathcal{P}X \to \mathcal{P}X$ and the functions $\ell, r : X \to X$ to $\mathcal{P}X \to \mathcal{P}X$ by taking images. We wish to explore the algebraic structure of such powerset liftings over ℓr-multimagmas and related structures. Powerset liftings of relational monoids, and therefore those of ℓr-multisemigroups, yield unital quantales [8, 26]. But the precise lifting of source and target operations remains to be explored. This requires some preparation.

A *quantale* [25] $(Q, \leq, \cdot, 1)$ is a complete lattice $(Q, \leq)$ with a monoidal composition $\cdot$ with unit 1 that preserves all sups in both arguments. A quantale is *boolean* if its lattice reduct is a complete boolean algebra—a complete lattice and a boolean algebra. Some applications require weaker notions. A *prequantale* is a quantale where the associativity law is absent [25].

We write $\bigvee$ for the sup and $\bigwedge$ for the inf operator, and $\vee$, $\wedge$ for their binary variants. We also write $\bot = \bigwedge Q = \bigvee \emptyset$ for the least and $\top = \bigvee Q = \bigwedge \emptyset$ for the greatest element of Q, and $-$ for boolean complementation (both unary and binary) if Q is boolean. We write $Q_1 = \{\alpha \in Q \mid \alpha \leq 1\}$ for the set of *subidentities* of Q. In a boolean quantale, Q_1 is a complete boolean subalgebra with complementation $\lambda x.\ 1 - x$ and composition coinciding with meet [11].

We lift the source and target operations of ℓr-multisemigroups to domain and codomain operations at powerset level. Modal quantales of relations, which are formally lifted from pair groupoids below, provide some intuition:

$$\mathit{dom}(R) = \{(a, a) \mid \exists b.\ (a, b) \in R\}, \qquad \mathit{cod}(R) = \{(b, b) \mid \exists a.\ (a, b) \in R\}$$

and hence $\mathit{dom}(R) = \ell(R)$ and $\mathit{cod}(R) = r(R)$.

More generally, a *domain quantale* [11] is a quantale $(Q, \leq, \cdot, 1)$ equipped with a domain operation $\mathit{dom} : Q \to Q$ that satisfies, for all $\alpha, \beta \in Q$,

$$\alpha \leq \mathit{dom}(\alpha) \cdot \alpha, \qquad \mathit{dom}(\alpha \cdot \mathit{dom}(\beta)) = \mathit{dom}(\alpha \cdot \beta), \qquad \mathit{dom}(\alpha) \leq 1,$$
$$\mathit{dom}(\bot) = \bot, \qquad \mathit{dom}(\alpha \vee \beta) = \mathit{dom}(\alpha) \vee \mathit{dom}(\beta).$$

We call these equations the *absorption*, *locality*, *subidentity*, *strictness* and *(binary) sup-preservation* axiom, respectively. Absorption can be strengthened to $\mathit{dom}(\alpha)\alpha = \alpha$. These domain axioms are precisely those of domain semirings [7]. Domain quantales are thus quantales that are also domain semirings. Properties of domain semirings therefore translate [11,12]. Interestingly, domain axioms for $\bigvee$ are not needed in domain quantales [11] because *dom* preserves arbitrary sups. The interaction of *dom* with $\bigwedge$ is weaker and not our concern.

Much of the structure of the domain algebra induced by *dom* is inherited from domain semirings as well. In particular, $Q_{dom} = \{x \mid dom(x) = x\} = dom(Q)$, and it follows that the *domain algebra* $(Q_{dom}, \leq, \cdot, 1)$ is a subquantale of Q that forms a bounded distributive lattice with $\cdot$ as binary inf [7]. The elements of Q_{dom} are called *domain elements* of Q. Yet, by contrast to modal semirings, the lattice Q_{dom} is complete [11], and if Q is boolean, then $Q_{dom} = Q_1$ is a complete boolean algebra. For powerset quantales, this complete boolean algebra is atomic.

Quantales are closed under opposition: interchanging the order of composition in Q yields the quantale Q^{op}; properties translate under this duality. The dual of *dom* on a domain quantale is of course a codomain operation *cod*.

A *codomain quantale* (Q, cod) is thus simply a domain quantale (Q^{op}, dom). It satisfies the dual domain axioms. A *modal quantale* is a domain and codomain quantale $(Q, \leq, \cdot, 1, dom, cod)$ that satisfies the *compatibility* axioms

$$dom \circ cod = cod \qquad \text{and} \qquad cod \circ dom = dom.$$

These force $Q_{dom} = Q_{cod}$.

Some ℓr-structures of interest fail to yield associativity or locality laws when lifted. This requires more general notions.

- A *modal prequantale* is a prequantale in which the locality axioms for *dom* and *cod* are replaced by the export axiom $dom(dom(\alpha)\beta) = dom(\alpha)dom(\beta)$ and its dual for *cod*. Then $Q_{dom} = dom(Q) = cod(Q) = Q_{cod}$ is still a complete distributive lattice, but locality laws for *dom* and *cod* are not even derivable as inequalities.
- A *weakly local modal quantale* is a modal quantale that satisfies the previous axioms for *dom* and *cod*. The *weak locality* law $dom(\alpha\beta) \leq dom(\alpha\, dom(\beta))$ and its dual for *cod* are now derivable, but not the equational laws.

7 Constructing Modal Powerset Quantales

We now construct modal powerset quantales from ℓr-multisemigroups in the spirit of modal correspondence theory for boolean algebras with operators. First we recall the quantalic part.

Proposition 14. *Let $(X, \odot, \ell, r)$ be an ℓr-multisemigroup. Then $(\mathcal{P}X, \subseteq, \odot, E)$ forms a boolean quantale whose underlying lattice is boolean atomic.*

Proof. If $(X, \odot, \ell, r)$ is an ℓr-multisemigroup, then it is isomorphic to a multimonoid and further to a relational monoid, and its powerset algebra forms a quantale [8,26]. The complete lattice on $\mathcal{P}X$ is trivially boolean atomic. □

Similarly, ℓr-multimagmas lift to prequantales.

Example 15 (Powerset Quantales over ℓr-Semigroups). The powerset lifting of any category yields a powerset quantale. It is boolean and has the arrows of the category as atoms. The pair groupoid on set X lifts to the quantale of binary relations over X. Its elements are possibly infinite-dimensional boolean-valued square matrices in which the quantalic composition is matrix multiplication. □

The fact that groupoids can be lifted to algebras of binary relations with an additional operation of converse was known to Jónsson and Tarski [20].

Proposition 14 combines source and target elements into the unit E of the powerset quantale. The lifting to modal quantales is more refined. In the following theorems, we identify $dom(A)$ with $\ell(A)$ and $cod(A)$ with $r(A)$ for $A \subseteq X$. We develop our main theorem step-by-step from ℓr-multimagmas.

Lemma 16. *Let X be an ℓr-multimagma. For $A, B \subseteq X$ and $\mathcal{A} \subseteq \mathcal{P}X$,*

1. *$\ell(r(A)) = r(A)$ and $r(\ell(A)) = \ell(A)$ (compatibility),*
2. *$\ell(A) \cdot A = A$ and $A \cdot r(A) = A$ (absorption),*
3. *$\ell\left(\bigcup \mathcal{A}\right) = \bigcup_{A \in \mathcal{A}} \ell(A)$ and $r\left(\bigcup \mathcal{A}\right) = \bigcup_{A \in \mathcal{A}} r(A)$ (sup-preservation),*
4. *$f(A)g(B) = g(B)f(A)$ hold for $f, g \in \{\ell, r\}$ (commutativity),*
5. *$\ell(A) \subseteq X_\ell$ and $r(A) \subseteq X_r$ (subidentity),*
6. *$\ell(\ell(A) \cdot B) = \ell(A)\ell(B)$ and $r(A \cdot r(B)) = r(A)r(B)$ (export).*

Proof. We show proofs up-to duality.

1. $\ell(r(A)) = \{\ell(r(x)) \mid x \in A\} = \{r(x) \mid x \in A\} = r(A)$.
2.
$$\begin{aligned}
\ell(A)A &= \bigcup\{\ell(x)y \mid x, y \in A \text{ and } \ell(x)y \neq \emptyset\} \\
&= \bigcup\{\ell(x)y \mid x, y \in A,\ \ell(x)y \neq \emptyset \text{ and } r(\ell(x)) = \ell(y)\} \\
&= \bigcup\{\ell(x)y \mid x, y \in A,\ \ell(x)y \neq \emptyset \text{ and } \ell(x) = \ell(y)\} \\
&= \bigcup\{\ell(y)y \mid y \in A\} \\
&= \bigcup\{\{y\} \mid y \in A\} = A.
\end{aligned}$$
3. $\ell\left(\bigcup \mathcal{A}\right) = \{\ell(x) \mid x \in \bigcup \mathcal{A}\} = \bigcup\{\ell(A) \mid A \in \mathcal{A}\}$.
4. We only prove the identity for $\ell(A)r(B)$. The others then follow from (1).
$$\begin{aligned}
\ell(A)r(B) &= \bigcup\{\ell(x)r(y) \mid x \in A \text{ and } y \in B\} \\
&= \bigcup\{r(y)\ell(x) \mid x \in A \text{ and } y \in B\} \\
&= r(B)\ell(A).
\end{aligned}$$
5. $\ell(A) = \{\ell(x) \mid x \in A\} \subseteq \{\ell(x) \mid x \in X\} = \{x \mid \ell(x) = x\} = E$.
6.

$$\begin{aligned}
\ell(\ell(A)B) &= \bigcup\{\ell(\ell(x)y) \mid x \in A,\ y \in B \text{ and } \ell(x)y \neq \emptyset\} \\
&= \bigcup\{\ell(x)\ell(y) \mid x \in A,\ y \in B,\ \ell(x)y \neq \emptyset \text{ and } r(\ell(x)) = \ell(y)\} \\
&= \bigcup\{\ell(x)\ell(y) \mid x \in A,\ y \in B,\ \ell(x)y \neq \emptyset \text{ and } \ell(x) = \ell(y)\} \\
&= \bigcup\{\ell(x)\ell(y) \mid x \in A,\ y \in B \text{ and } \ell(y)y \neq \emptyset\} \\
&= \bigcup\{\ell(x)\ell(y) \mid x \in A \text{ and } y \in B\} \\
&= \ell(A)\ell(B). \qquad \square
\end{aligned}$$

The proof has also been checked with Isabelle. And now for locality.

Lemma 17. *Let X be an ℓr-multisemigroup and $A, B \subseteq X$. Then*

$$\ell(AB) \subseteq \ell(A\ell(B)) \qquad \textit{and} \qquad r(AB) \subseteq r(r(A)B).$$

The converse inclusions of these weak locality laws hold if X is local.

Proof. The inclusions hold in any quantale that satisfies the laws of Lemma 16. For the opposite direction, suppose that X is a local ℓr-multisemigroup. Then, writing $r(x) = \ell(y)$ in place of $x \odot y \neq \emptyset$ owing to locality,

$$\begin{aligned}
\ell(A\ell(B)) &= \bigcup\{\ell(x\ell(y)) \mid x \in A,\ y \in B \text{ and } r(x) = \ell(\ell(y))\} \\
&= \bigcup\{\ell(xy) \mid x \in A,\ y \in B \text{ and } r(x) = \ell(y)\} = \ell(AB)
\end{aligned}$$

and the opposite result for r is obvious. $\square$

The proofs have again been checked with Isabelle. We can now summarise.

Theorem 18. *Let X be an ℓr-multimagma.*

1. *Then $(\mathcal{P}X, \subseteq, \odot, E, dom, cod)$ is a boolean modal prequantale, and the complete boolean algebra is atomic.*
2. *It is a weakly local modal quantale if X is an ℓr-multisemigroup.*
3. *It is a modal quantale if X is a local ℓr-multisemigroup.*

This result highlights the role of weak locality and locality in the three stages of lifting. Its construction follows one direction of Jónsson-Tarski duality between relational structures and boolean algebras with operators [13,19], which it refines. Like in modal logic, it leads to correspondences between identities in relational structures and boolean algebras with operators. Those lifted in Lemma 16 and 17 are one direction of these. Their converses are explored in Section 8.

Example 19 (Modal Powerset Quantales over ℓr-Semigroups).

1. Any category as a local partial ℓr-semigroup can be lifted to a modal powerset quantale. The domain algebra is the entire boolean subalgebra of subidentities of the quantale, the set of all objects of the category (or its identity arrows). A modal quantale can thus be obtained from any category.

2. An instance is the modal powerset quantale of binary relations over X lifted from the pair groupoid on X. Domain and codomain elements are precisely the subidentity relations below Id_X. In the associated matrix algebras, these correspond to (boolean-valued) sub-diagonal matrices (which may have zeros along the diagonal) and further to predicates.
3. Recall that the partial ℓr-semigroup of the broken monoid is only weakly local. The powerset quantale is only weakly local, too. To check this, we simply replay the non-locality proof for the partial ℓr-semigroup with $A = \{a\}$: $dom(AA) = dom(\emptyset) = \emptyset \subset \{1\} = dom(A\{1\}) = dom(A\,dom(A))$. Locality of codomain is ruled out by duality. □

Most models of domain and modal semirings considered in the literature are powerset structures lifted from categories. Theorem 18 yields a uniform construction recipe for all of them. The final example of this section shows that the twisted laws for function systems do not lift to powersets.

Example 20. The category $1 \stackrel{a}{\to} 2$ is a partial local ℓr-semigroup with $X = \{1, a, 2\}$, ℓ and r defined by $\ell(1) = r(1) = 1 = \ell(a)$ and $\ell(2) = r(2) = 2 = r(a)$ and composition $11 = 1$, $1a = a = a2$ and $22 = 2$. Then, for $A = \{1, a\}$ and $B = \{2\}$, $A \cdot dom(B) = A \cdot B = \{a\} \subset A = \{1\} \cdot A = dom(A \cdot B) \cdot A$ refutes the twisted law in $\mathcal{P}X$. The opposite law for cod is refuted by a dual example. □

8 Recovering ℓr-Multisemigroups

We know from Jónsson-Tarski duality that one can find an ℓr-multisemigroup within each modal powerset quantale, using its atom structure. Here we prove correspondence results in this direction. These strengthen the relationship between locality in modal quantales and ℓr-multisemigroups further. Parts of these results are special cases of more general theorems for convolution algebras [3,12].

Proposition 21.

1. *If $\mathcal{P}X$ is a prequantale in which $\emptyset \neq E$, then X is an ℓr-multimagma.*
2. *If $\mathcal{P}X$ is a quantale in which $\emptyset \neq E$, then X is an ℓr-multisemigroup.*

Proof. The results are known for unital relational magmas and relational monoids [3, Proposition 4.1 and Corollary 4.7]. They thus hold for ℓr-multimagmas and multisemigroups via the isomorphisms. □

The ℓr-semigroup X is thus completely determined by the subidentites below E in $\mathcal{P}X$. We calculate the absorption law for ℓ explicitly as an example:

$$\ell(x) \odot x = \{\ell(x)\} \odot \{x\} = dom(\{x\}) \odot \{x\} = \{x\},$$

where the second step uses domain absorption in modal quantales. The fact that dom appears in the calculation does not go beyond Proposition 21 because $dom(\{x\}) = \{\ell(x)\} \subseteq E$ in $\mathcal{P}X$.

The next statement adds locality to the picture.

Theorem 22. *If $\mathcal{P}X$ is a modal quantale in which $\emptyset \neq E$, then X is a local ℓr-multisemigroup.*

Proof. Relative to Proposition 21 it remains to consider locality:

$$\ell(x \odot \ell(y)) = dom(\{x\}) \odot dom(\{y\}) = dom(\{x\} \odot \{y\}) = \ell(x \odot y).$$

Locality of r follows by duality. □

In light of Jónsson-Tarski duality, these results extend to atomic boolean quantales. With the lifting results from Section 7 they yield in particular a correspondence between locality in ℓr-multisemigroups and modal powerset quantales. To construct such a quantale one should therefore look for the underlying ℓr-multisemigroup, and often, more specifically, the underlying category.

9 Further Examples

We apply our construction to further examples of ℓr-multisemigroups and modal convolution quantales. We start with those based on categories.

Path quantales. A quiver (or digraph) K is formed by a set V_K of vertices, a set E_K of edges and source/target maps $s, t : E_K \to V_K$. The path category of K has vertices as objects and sequences $\pi = (v_1, e_1, v_2, \dots, v_{n-1}, e_{n-1}, v_n) : v_1 \to v_n$ as arrows in which vertices and edges alternate. Composition $\pi_1 \cdot \pi_2$ of $\pi_1 : v_3 \to v_4$ and $\pi_2 : v_1 \to v_2$ is defined if $v_2 = v_3$. It concatenates the two paths while gluing the common end $v_2 = v_3$. Sequences (v) of length 0 are identities. Path categories are local partial ℓr-semigroups, with $\ell(\pi) = (v_1)$ and $r(\pi) = (v_n)$ for π as above. Theorem 18 shows that the powerset algebra over the path category of any quiver is a modal quantale—a modal quantale of path languages.

The path category generated by the one-point quiver with n arrows represents the free monoid with n generators. The ℓr-structure and hence the modal structure is then trivial. Lifting along Theorem 18 yields the quantale of formal languages. Path categories are relevant to computing: they capture execution sequences of programs, automata or transition systems.

Interval quantales. Pair groupoids over X become poset categories in which pairs represent (closed) segments or intervals when the universal relations used for pair groupoids are generalised to partially or totally ordered relations. Segments or intervals can be composed like the elements of the pair groupoid; the units are the singleton intervals. Modal powerset quantales over such categories yield algebraic semantics for interval logics [17] and interval temporal logics [24] via the isomorphism between sets and predicates [8]. The modalities lifted from source and target maps express properties of endpoints of segments and intervals.

Pomset quantales. Finite posets form partial ℓr-multisemigroups with respect to serial composition, which is the disjoint union of posets with all elements of the first poset preceding that of the second one in the order of the composition.

The only unit is the empty poset, the algebra is therefore non-local and the modal structure of the powerset quantale trivial.

Partial words [14] or pomsets are isomorphism classes of finite node-labelled posets. The serial composition becomes total on pomsets, which yields a monoid and establishes locality. Pomsets and pomset languages, obtained by powerset lifting, form a standard model of concurrency.

Pomsets can be equipped with interfaces [29]. The source interface of a pomset consists of its minimal elements (with their labels); its target interface of its maximal elements (again with their labels). Pomsets with interfaces form partial ℓr-semigroups with ℓ and r mapping pomsets to their source and target interfaces, and composition defined by gluing pomsets on their interfaces whenever they match and extending the order as in serial composition. The partial ℓr-semigroups are local and therefore categories. The modal structure at powerset level is no longer trivial.

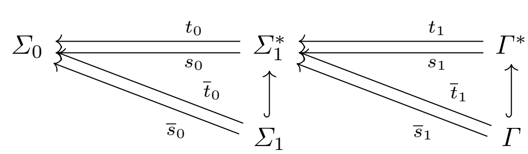
Path quantales in topology. A *path* in a topological space X is a continuous map $f : [0,1] \to X$. The source of path f is $\ell(f) = f(0)$, its target $r(f) = f(1)$. Paths f and g in X can be composed whenever $r(f) = \ell(g)$, and then

$$(f \cdot g)(x) = \begin{cases} f(2x) & \text{if } 0 \leq x \leq \frac{1}{2}, \\ g(2x-1) & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

The parameterisation destroys associativity; $(X^{[0,1]}, \cdot, \ell, r)$ is therefore only a local partial ℓr-magma. The powerset lifting to $\mathcal{P}(X^{[0,1]})$ satisfies the properties of Lemma 16, but even weak locality fails due to the absence of associativity in $X^{[0,1]}$ and, accordingly, $\mathcal{P}(X^{[0,1]})$. This leads to modal prequantales.

Yet path composition is associative up-to homotopy. The associated local partial ℓr-semigroup then lifts to a modal quantale like any other category. Alternatively, categories of paths can be defined on intervals of arbitrary length [1].

Higher path algebras. *A* 2*-polygraph* is a quiver $\Sigma = (\bar{s}_0, \bar{t}_0 : \Sigma_1 \to \Sigma_0)$, whose edges (or 1-*cells*) are equipped with a *cellular extension*. This consists of a quiver $(\bar{s}_1, \bar{t}_1 : \Gamma \to \Sigma^*)$, where Σ^* is the free category generated by Σ and Γ is a set of globular 2-*cells* relating parallel 1-cells. A 2-polygraph generates a 2-category pictured in the following diagram:



Here, s_i, t_i are the source and target maps induced by the free category construction, and the globular equations $s_0 t_1 = s_0 s_1$ and $t_0 t_1 = t_0 s_1$ hold, see [16] for details. In the example of abstract rewriting systems, Σ_0 is a carrier set, Σ_1 a set of generating rewrite rules, Γ a set of relations between these rules.

For $i \in \{0,1\}$, the set Γ^* of freely generated 2-cells forms a local partial ℓr-semigroup $(\Gamma^*, \odot_i, \ell_i, r_i)$, where $\odot_i$ is forward i-composition of 2-cells and

$\ell_i = s_i$, $r_i = t_i$. By Theorem 18, $(\mathcal{P}\Gamma^*, \subseteq, \odot_i, E_i, dom_i, cod_i)$ is a modal quantale with $E_0 = \{1_{1_x} \mid x \in \Sigma_0\}$ and $E_1 = \{1_u \mid u \in \Sigma_1\}$. Beyond Theorem 18, we get a *globular* 2*-quantale* [2] when combining the two structures. For all $A, A', B, B' \in \mathcal{P}\Gamma^*$, a lax interchange law $(A \odot_1 B) \odot_0 (A' \odot_1 B') \subseteq (A \odot_0 A') \odot_1 (B \odot_0 B')$ holds, and also $E_1 \odot_0 E_1 = E_1$. The absorption laws $dom_1 \circ dom_0 = dom_0$ and $cod_1 \circ cod_0 = cod_0$ hold as well. Finally, we recover the globularity conditions that $dom_0 \circ cod_1 = dom_0$, that $cod_0 \circ dom_1 = cod_0$ and that dom_1 as well as cod_1 are morphisms for $\odot_0$. This construction generalises to n-polygraphs and *globular* n*-quantales* [2]. Applications include higher dimensional algebraic rewriting [16].

Δ-sets. *A presimplicial set* [27] K is a sequence of sets $(K_n)_{n\geq 0}$, called *simplices*, equipped with face maps $d_{i,n} : K_n \to K_{n-1}$, $i \in \{0, \dots, n\}$, satisfying the simplicial identities $d_{i,n-1} \circ d_{j,n} = d_{j-1,n-1} \circ d_{i,n}$ for all $i < j \leq n$ (we omit the extra indices n from now). The set $K = \bigsqcup_{n\geq 0} K_n$ forms an ℓr-multisemigroup $(K, \odot, \ell, r)$ with

$$x \in y \odot z \Leftrightarrow \exists i.\ y = s_i(x) \wedge z = t_{n-i}(x)$$

and $\ell(x) = s_0(x)$, $r(x) = t_0(x)$, where $s_i(x) = (d_{i+1} \circ d_{i+2} \circ \dots \circ d_n)(x)$ and $t_i(x) = (d_0 \circ d_1 \circ \dots \circ d_{n-i-1})(x)$ stand for the initial and the final i–face of $x \in K_n$, respectively. In general, $(K, \odot, \ell, r)$ is neither local nor partial. Locality and partiality hold if K is the nerve of a category (we omit degeneracies).

Also, the set of triples $(s_i(x), x, t_j(x))$ ($x \in K_n$, $0 \leq i, j \leq n$), called *simplices with interfaces*, forms an ℓr-multisemigroup $\mathrm{Int}(K)$ with

$$\begin{aligned}&(s_p(x), x, t_q(x)) \in (s_i(y), y, t_j(y)) \odot (s_k(z), z, t_l(z))\\ &\Leftrightarrow s_p(x) = s_i(y) \wedge t_j(y) = s_k(z) \wedge t_q(x) = t_l(z) \wedge y = s_u(x) \wedge z = t_{n-u+j}(x),\end{aligned}$$

for $x \in K_n$, $y \in K_u$, $z \in K_{n-u+j}$. There is an obvious embedding $K \ni x \mapsto (s_0(x), x, t_0(x)) \in \mathrm{Int}(K)$ of ℓr-multisemigroups. Hence $\mathrm{Int}(K)$ is again neither partial nor local in general.

Precubical sets. *A precubical set* X [15] is a sequence of sets $(X_n)_{n\geq 0}$ equipped with face maps $d_i^\varepsilon : X_n \to X_{n-1}$, $1 \leq i \leq n$, $\varepsilon \in \{0, 1\}$, satisfying the identities $d_i^\varepsilon \circ d_j^\eta = d_{j-1}^\eta \circ d_i^\varepsilon$ for $i < j$ and $\varepsilon, \eta \in \{0, 1\}$. Denote $d_A^\varepsilon = d_{a_1}^\varepsilon \circ \dots \circ d_{a_k}^\varepsilon$ for $A = \{a_1 < \dots < a_k\} \subseteq [n]$ and $\varepsilon \in \{0, 1\}$. The precubical set X forms an ℓr-semigroup $(X, \odot, \ell, r)$ with

$$x \in y \odot z \Leftrightarrow \exists A \subseteq [n].\ y = d_A^0(x) \wedge z = d_{[n]\setminus A}^1(x),$$

$\ell(x) = d_{[n]}^0(x) \in X_0$, $r(x) = d_{[n]}^1(x) \in X_0$ for all $x \in X_n$. Like the previous example, the ℓr-multisemigroup X is neither partial nor local.

A special case of this example is the shuffle multimonoid (Example 6). Let Σ be a finite alphabet, X_n the set of all words of length n, and $d_i^\varepsilon : X_n \to X_{n-1}$ the map that removes the i–th letter. Then $X = (X_n, d_i^\varepsilon)$ is a precubical set and the associated ℓr-multisemigroup $(X, \odot, \ell, r)$ is the shuffle multimonoid on Σ. The domain/codomain structure of the quantale of shuffle languages is trivial, as there is no element between $\emptyset$ and the set containing the empty word.

Assertions quantales of separation logic. The non-local partial ℓr-semigroups of heaplets lift to weakly local modal powerset quantales, but once again with trivial domain/codomain structure. The set $\{\varepsilon\}$ containing the empty heaplet is the only unit. These form the assertion quantales of separation logic. The modal structure is again trivial as there is no element between $\emptyset$ and $\{\varepsilon\}$.

10 Discussion

We summarise some additional results and generalisations in this section. See [12] for details.

Extension to convolution algebras. The powerset lifting in Section 7 can be seen as a lifting to function spaces $X \to 2$ and generalised to $X \to Q$ for an arbitrary (modal) quantale Q. The composition $\odot : 2^X \times 2^X \to 2^X$ then generalises to a convolution $* : Q^X \to Q^X \to Q^X$ with $\bigvee$ and $\cdot$ taken in Q:

$$(f * g)(x) = \bigvee_{x \in y \odot z} f(y) \cdot g(z).$$

Theorem 18 also generalises: if X is a local ℓr-multisemigroup and Q a modal quantale, then Q^X is a modal quantale with

$$Dom(f) = \bigvee_{x \in X} dom(f(x)) \cdot \delta_{\ell(x)},$$

where $\delta_x(y)$ is 1 if $x = y$ and $\bot$ otherwise, and Cod given by duality. The monoidal identity in Q^X, $id_E(x)$ is 1 if $x \in E$ and $\bot$ otherwise. Beyond lifting, there is now a triangle of correspondences between identities in X, Q and Q^X. The results in this text thus generalise to modal quantales of weighted languages or weighted relations, and towards group, incidence or category algebras.

Finite decomposability. Some ℓr-multisemigroups in our examples are *finitely decomposable*: for every x the fiber $\odot^{-1}(x) = \{(y, z) \mid x \in y \odot z\}$ is finite. Examples are shuffle quantales, where each word can only be decomposed into finitely many pairs of words, or quantales of $n \times m$-matrices, where multiplications sum over finitely many indices. The sups in convolutions can then be replaced by sums and quantales by semirings. In modal settings, one can then use modal semirings [7] and, if X is a finitely decomposable local ℓr-multisemigroup and S a modal semiring, then S^X forms again a modal semiring.

Modal concurrent quantales. Concurrent semirings and quantales [18] can be constructed as convolution algebras from concurrent relational semigroups [3], hence from concurrent local ℓr-multisemigroups equipped with two multioperations that satisfy a weak interchange law. In combination with the corresponding results for modal structures we can construct modal concurrent semirings and quantales as convolution algebras. Target models are categories of pomsets with interfaces, with applications in concurrency theory [10,29], and n-polygraphs [2].

Algebras of modalities. The domain and codomain operations in convolution algebras support definitions of modal box and diamond operators along the lines of modal semirings [7] as $|f\rangle\pi = Dom(f * \pi)$, where $f \in Q^X$ and $\pi \in (Q^X)_{Dom}$, and dually $\langle f|\pi = Cod(\pi * f)$. In modal quantales, diamonds preserve arbitrary sups and box operators exist as right adjoints, even if $(Q^X)_{Dom}$ is not a boolean algebra. For box and diamond modalities, locality in ℓr-multisemigroups is crucial. Without it, the action laws $|f * g\rangle = |f\rangle \circ |g\rangle$, $\langle f * g| = \langle g| \circ \langle f|$ and their analogues for boxes would not exist. Our results thus lead to uniform construction principles for dynamic algebras and predicate transformer algebras based on more general semantics than Kripke frames, including arbitrary categories, and weighted variants.

11 Conclusion

We have introduced ℓr-multisemigroups, related them with categories, and shown how their source and target operations give rise to the domain and codomain operations studied previously in the contexts of function systems, modal semigroups, modal semirings and modal quantales. In particular, we have explained how the typical composition pattern of categories corresponds to the well-known locality axioms that appear in such modal algebras. This analysis is based on a generic lifting construction from ℓr-multisemigroups to modal quantales and the modal correspondences to which it leads. It captures most known models of computational interest of modal semirings and quantales, and explains how additional models for them could be built, including modal-concurrent ones. For every local ℓr-multisemigroup we find, we get a dual modal quantale for free. The approach extends to modal convolution algebras that seem relevant to quantitative verification, but this requires concepts and proofs beyond these pages [12].

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