Posets with Interfaces as a Model for Concurrency

Uli Fahrenberg

EPITA Rennes, France

68nqrt, 16 November 2021

Friends

- Christian Johansen, Gjøvik, Norway
- Georg Struth, Sheffield, UK
- Krzysztof Ziemiański, Warsaw, Poland

- Cameron Calk, Paris, France
- James Cranch, Sheffield, UK
- Eric Goubault, Paris, France


Uli Fahrenberg
Krzysztof Ziemiański
Georg Struth
Christian Johansen

The (i)Po(m)set Project

› What
› Who
› When
› How
› What
› Contact

What

The **(i)Po(m)set Project** is a research project at the crossroads of concurrency theory, algebra, and geometry. It aims to understand the basics of concurrency theory and develop its foundations.

Who

Members:

- Uli Fahrenberg, EPITA Rennes, France
- Christian Johansen, Norwegian University of Science and Technology, Gjøvik, Norway
- Georg Struth, University of Sheffield, UK
- Krzysztof Ziemiański, University of Warsaw, Poland

Associates:

- Cameron Calk, École polytechnique, Paris, France
- James Cranch, University of Sheffield, UK
- Eric Goubault, École polytechnique, Paris, France

Former members or associates:

How

Members and possibly associates meet about once a week on zoom to discuss research and papers and otherwise banter over big and small things.

What

Research published by the (i)Po(m)set Project:

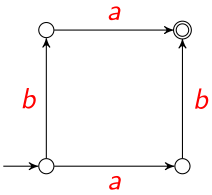
1. Uli Fahrenberg, Christian Johansen, Christopher Trotter, Krzysztof Ziemiański: Sculptures in Concurrency. Logical Methods in Computer Science 17(2) (2021)
2. Uli Fahrenberg, Christian Johansen, Georg Struth, Ratan Bahadur Thapa: Generating Posets Beyond N. RAMiCS 2020: 82-99
3. Uli Fahrenberg, Christian Johansen, Georg Struth, Krzysztof Ziemiański: Domain Semirings United. CoRR abs/2011.04704 (2020)
4. Uli Fahrenberg, Christian Johansen, Georg Struth, Krzysztof Ziemiański: Languages of Higher-Dimensional Automata. Mathematical Structures in Computer Science (2021)
5. Uli Fahrenberg, Christian Johansen, Georg Struth, Krzysztof Ziemiański: lr-Multisemigroups and Modal Convolution Algebras. CoRR abs/2105.00188 (2021)
6. Cameron Calk, Uli Fahrenberg, Christian Johansen, Georg Struth, Krzysztof Ziemiański: lr-Multisemigroups, Modal Quantales and the Origin of Locality. RAMiCS 2021
7. Uli Fahrenberg, Christian Johansen, Georg Struth, Krzysztof Ziemiański: Posets with Interfaces for Concurrent Kleene Algebra. CoRR abs/2106.10895 (2021)

Software and data published by the (i)Po(m)set Project:

- Python code related to [2]
- Julia code related to [6]
- Iposets and gp-iposets on up to 8 points, and forbidden substructures on up to 10 points

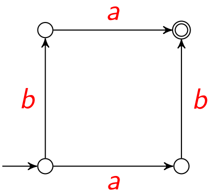
Higher-dimensional automata

$a \| b$

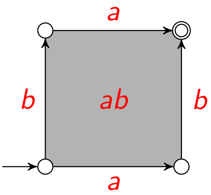


Higher-dimensional automata

$a|b$



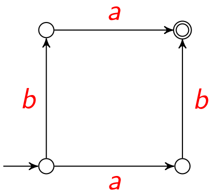
$a.b + b.a$

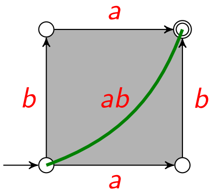


a and b are independent

Higher-dimensional automata

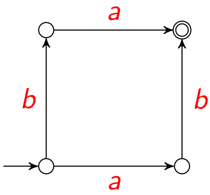
$a|b$

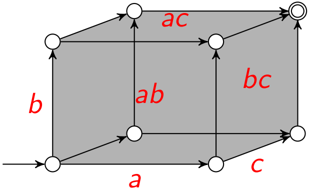




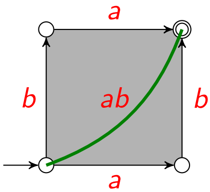
Higher-dimensional automata

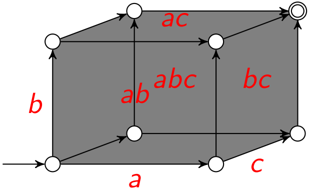
$a|b$ $a|b|c$





pairwise independent





$\{a, b, c\}$ independent

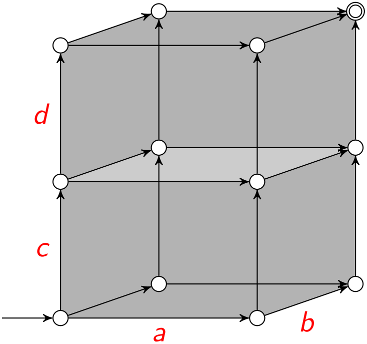
Higher-dimensional automata & concurrency

HDA as a model for concurrency:

- points: states
- edges: transitions
- squares, cubes etc.: independency relations (concurrently executing events)
- two-dimensional automata $\cong$ asynchronous transition systems [Bednarczyk 1987, PhD]

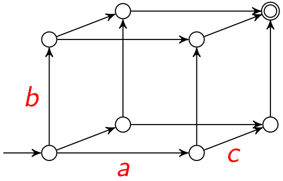
[van Glabbeek 2006, TCS]: Up to history-preserving bisimilarity, HDA "generalize the main models of concurrency proposed in the literature"

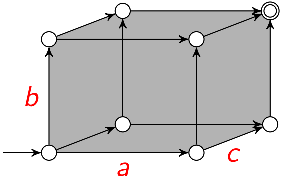
Another example

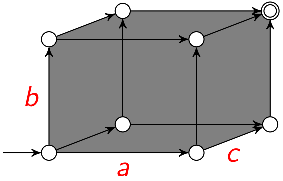


- no cubes, all faces except middle horizontal
- *a* and *b* independent; *c* introduces conflict; *d* releases conflict

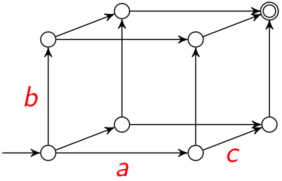
Languages of HDA

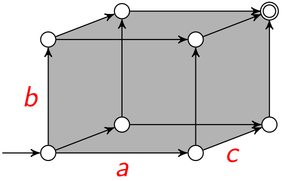


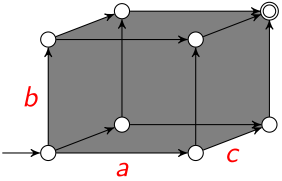




Languages of HDA

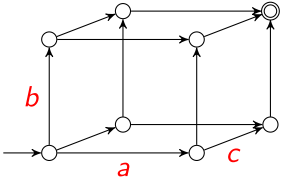






$$L_1 = \{abc, acb, bac, bca, cab, cba\}$$

Languages of HDA



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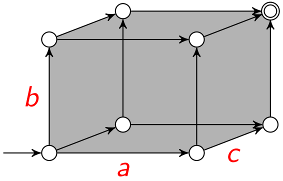
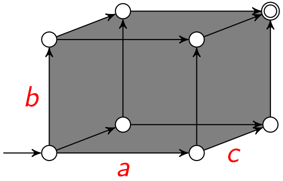
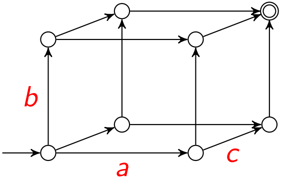




$$L_3 = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \dots \right\}$$

Languages of HDA



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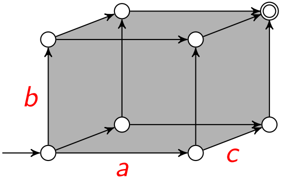
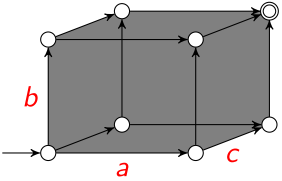


$$L_2 = \left\{ \begin{pmatrix} a \\ b \to c \end{pmatrix}, \begin{pmatrix} a \\ c \to b \end{pmatrix}, \begin{pmatrix} b \\ a \to c \end{pmatrix}, \begin{pmatrix} b \\ c \to a \end{pmatrix}, \begin{pmatrix} c \\ a \to b \end{pmatrix}, \begin{pmatrix} c \\ b \to a \end{pmatrix}, \dots \right\}$$



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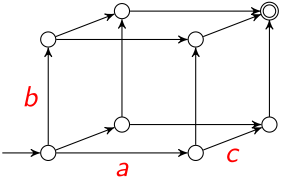

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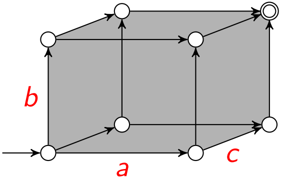
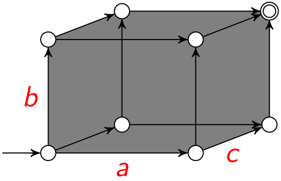


$$L_2 = \left\{ \begin{pmatrix} a & \\ b \to c \end{pmatrix}, \begin{pmatrix} a & \\ c \to b \end{pmatrix}, \begin{pmatrix} b & \\ a \to c \end{pmatrix}, \begin{pmatrix} b & \\ c \to a \end{pmatrix}, \begin{pmatrix} c & \\ a \to b \end{pmatrix}, \begin{pmatrix} c & \\ b \to a \end{pmatrix} \right\} \cup L_1$$

sets of pomsets



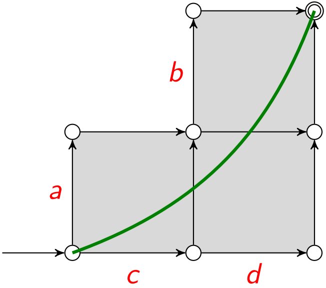
$$L_3 = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\} \cup L_2$$

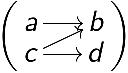
Pomsets

A (finite) pomset ("partially ordered multiset") $(P, \leq, \ell)$:

- a finite partially ordered set $(P, \leq)$
- with labeling $\ell : P \to \Sigma$
- (AKA labeled partial order)
- (up to isomorphism: don't care about identity of points)
- [Winkowski 1977, IPL], [Lamport 1978, CACM], etc.

Example





- an **N**-shaped pomset, which is not series-parallel

Are all pomsets generated by HDA?

No, only (labeled) interval orders

- Poset $(P, \leq)$ is an interval order iff it does not contain $\left(\begin{matrix}\circ \longrightarrow \circ \\ \circ \longrightarrow \circ\end{matrix}\right)$
  - (iff it is "2+2-free")
- iff it has an interval representation:
  - a set $I = \{[l_i, r_i]\}$ of real intervals
  - with order $[l_i, r_i] \preceq [l_j, r_j]$ iff $r_i \leq l_j$
  - and an order isomorphism $(P, \leq) \leftrightarrow (I, \preceq)$
- [Fishburn 1970, J. Math. Psych.]

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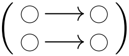
- Poset $(P, \leq)$ is an interval order iff it does not contain 
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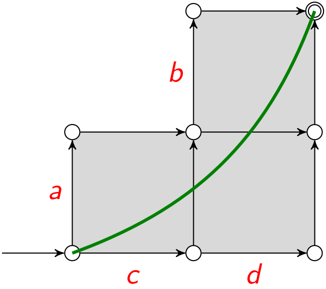
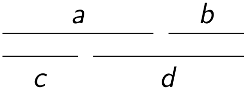
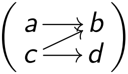
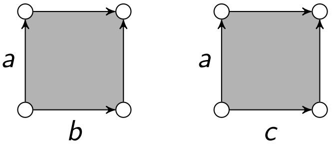






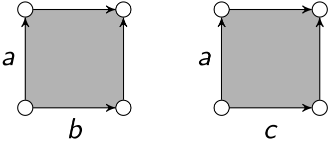
Concatenation of HDA



$$\begin{pmatrix} a \\ b \end{pmatrix} \qquad \begin{pmatrix} a \\ c \end{pmatrix}$$

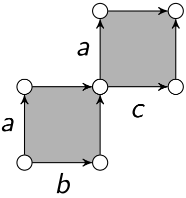
Two possible compositions:

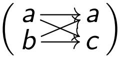
Concatenation of HDA

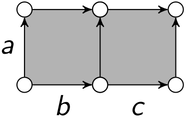


$\begin{pmatrix} a \\ b \end{pmatrix}$ $\begin{pmatrix} a \\ c \end{pmatrix}$

Two possible compositions:

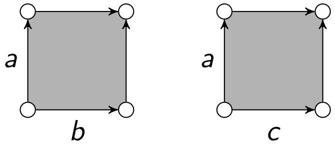






$\begin{pmatrix} a & \\ b \longrightarrow & c \end{pmatrix}$

Concatenation of HDA

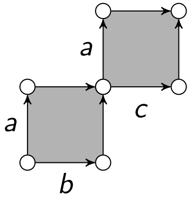


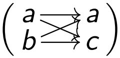
$\begin{pmatrix} a \\ b \end{pmatrix}$ $\begin{pmatrix} a \\ c \end{pmatrix}$

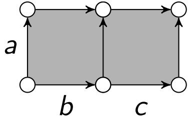
Two possible compositions:

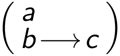
- not clear whether the two *a* are the same event
- idea: let the objects specify how they may be composed

⇒ pomsets with interfaces









Series-Parallel Posets

- a poset: *finite* set P plus partial order $\leq$: reflexive, transitive, antisymmetric
- parallel composition of posets $(P_1, \leq_1)$, $(P_2, \leq_2)$:

$$P_1 \otimes P_2 = (P_1 \sqcup P_2, \leq_1 \cup \leq_2)$$

$\upuparrows$ disjoint union

- serial composition:

$$P_1 * P_2 = (P_1 \sqcup P_2, \leq_1 \cup \leq_2 \cup P_1 \times P_2)$$

$\upuparrows$ P_1 before P_2

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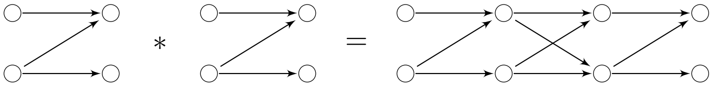


Series-Parallel Posets

Definition (Winkowski '77, Grabowski '81)

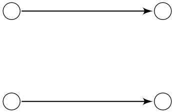
A poset is series-parallel (sp) if it is empty or can be obtained from the singleton poset by a finite number of serial and parallel compositions.

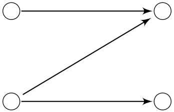
Theorem (Grabowski '81)

A poset is sp iff it does not contain N *as an induced subposet.*

The equational theory of sp-posets is well-understood: [Gischer 1988, TCS], [Bloom-Esik 1996, MSCS]

Interval Orders vs Series-Parallel Posets





- interval orders are used in Petri net theory and distributed computing
- but have no algebraic representation (so far)
- sp-posets are used in concurrency theory & have nice algebraic theory
- Concurrent Kleene algebra
- int. orders are 2+2-free; sp-posets are **N**-free
- incomparable: 2+2 is sp; **N** is interval

⇒ [F.-Johansen-Struth-Thapa 2020, RAMiCS]

Posets with interfaces

Definition

A poset with interfaces (iposet) is a poset P plus two injections

$$[n] \xrightarrow{s} P \xleftarrow{t} [m]$$

such that $s[n]$ is minimal and $t[m]$ is maximal in P.

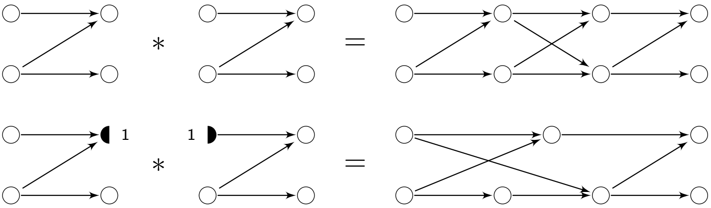
- $([n] = \{1, \dots, n\})$
- s: starting interface ; t: terminating interface
- events in $t[m]$ are *unfinished* ; events in $s[n]$ are “*unstarted*”

Gluing composition

Definition

The gluing composition of iposets $s_1 : [n] \to (P_1, \leq_1) \leftarrow [m] : t_1$ and $s_2 : [m] \to (P_2, \leq_2) \leftarrow [k] : t_2$:

$$P_1 * P_2 = \begin{cases} (P_1 \sqcup P_2) / t_1(i) = s_2(i) \\ \leq_1 \cup \leq_2 \cup (P_1 \setminus t_1[m]) \times (P_2 \setminus s_2[m]) \end{cases}$$



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- iposets form category (with gluing as composition)

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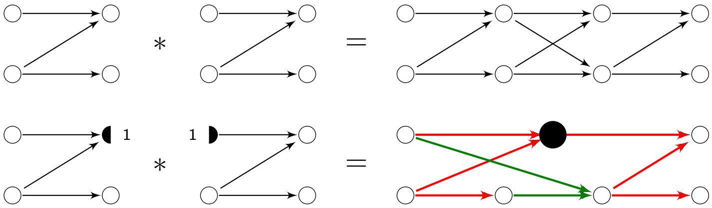

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Gluing-Parallel Iposets

- recall *sp-posets*: freely generated from $\bigcirc$ using $*$ and $\otimes$
- the four singleton iposets:

$$\bigcirc \qquad 1\,\RIGHTcircle \qquad \LEFTcircle\,1 \qquad 1\,\Bowtie\,1$$

- gp-iposets: generated from $\bigcirc$, $1\,\RIGHTcircle$, $\LEFTcircle\,1$, $1\,\Bowtie\,1$ using $*$ and $\otimes$

Fact

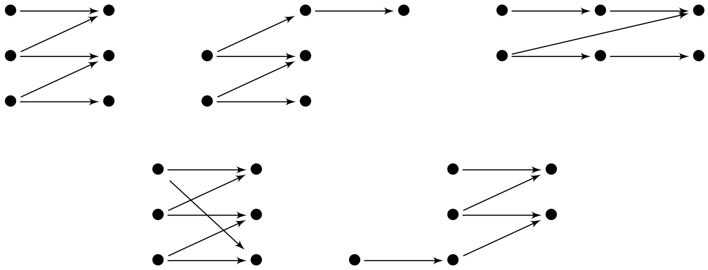
Gp-iposets are not freely generated, for example:

$$\begin{pmatrix} \LEFTcircle\,1 \\ P \end{pmatrix} * \begin{pmatrix} 1\,\RIGHTcircle \\ Q \end{pmatrix} = \begin{pmatrix} \bigcirc \\ P * Q \end{pmatrix} \qquad \begin{pmatrix} \LEFTcircle\,1 \\ P \end{pmatrix} * \begin{pmatrix} 1\,\Bowtie\,1 \\ Q \end{pmatrix} = \begin{pmatrix} \LEFTcircle\,1 \\ P * Q \end{pmatrix}$$

$$\begin{pmatrix} 1\,\Bowtie\,1 \\ P \end{pmatrix} * \begin{pmatrix} 1\,\RIGHTcircle \\ Q \end{pmatrix} = \begin{pmatrix} 1\,\RIGHTcircle \\ P * Q \end{pmatrix} \qquad \begin{pmatrix} 1\,\Bowtie\,1 \\ P \end{pmatrix} * \begin{pmatrix} 1\,\Bowtie\,1 \\ Q \end{pmatrix} = \begin{pmatrix} 1\,\Bowtie\,1 \\ P * Q \end{pmatrix}$$

Forbidden substructures

- sp-posets $\hat{=}$ **N**-free
- interval orders $\hat{=}$ 2+2-free
- gluing-parallel posets $\Rightarrow$ free of



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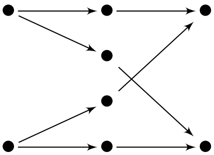

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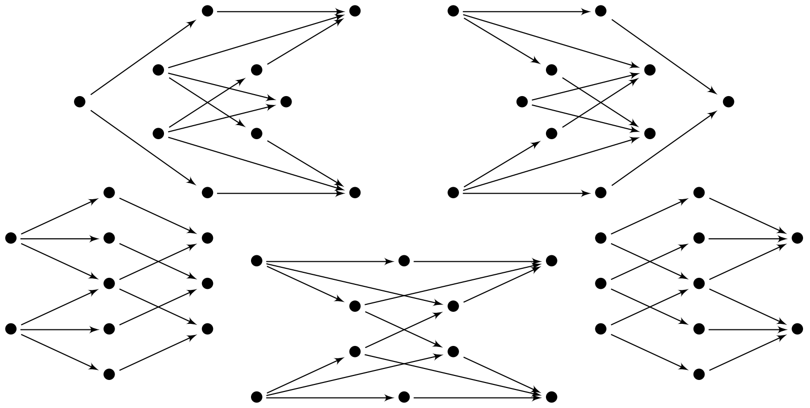


Some numbers

n	$\mathrm{P}(n)$	$\mathrm{SP}(n)$	$\mathrm{IO}(n)$	$\mathrm{GP}(n)$	$\mathrm{IP}(n)$	$\mathrm{GPI}(n)$
0	1	1	1	1	1	1
1	1	1	1	1	4	4
2	2	2	2	2	17	16
3	5	5	5	5	86	74
4	16	15	15	16	532	419
5	63	48	53	63	4068	2980
6	318	167	217	313	38.933	26.566
7	2045	602	1014	1903	474.822	289.279
8	16.999	2256	5335	13.943	7.558.620	3.726.311
9	183.231	8660	31.240	120.442		
10	2.567.284	33.958	201.608	1.206.459		
11	46.749.427	135.292	1.422.074			
EIS	112	3430	22493	345673	331158	331159

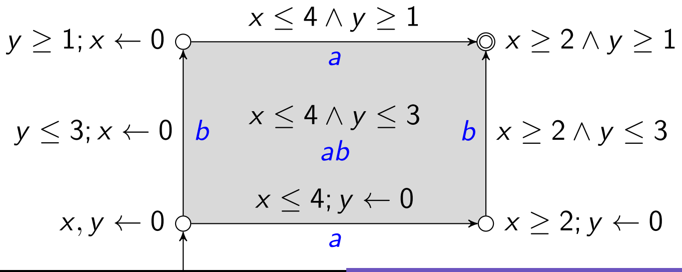
Higher-Dimensional Timed Automata

Definition

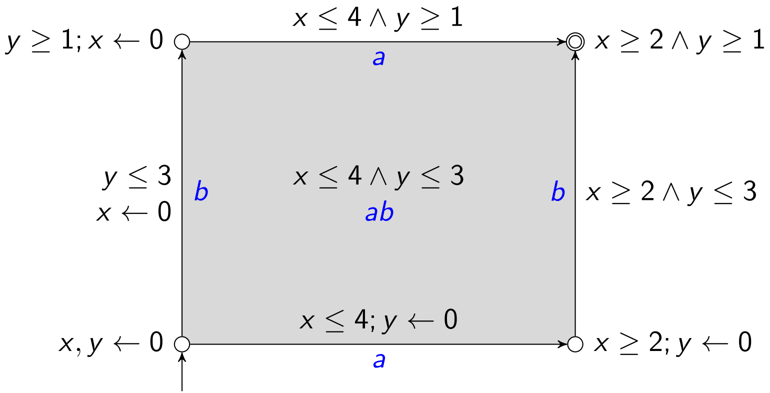
A HDTA is a structure $(L, l^0, L^f, \lambda, C, \text{inv}, \text{exit})$, where (L, l^0, L^f, λ) is a finite HDA, C is a finite set of clocks, and $\text{inv} : L \to \Phi(C)$, $\text{exit} : L \to 2^C$ give invariant and exit conditions for each n-cube.

Intuition:

- $\text{inv}(l)$: conditions on the clock values while delaying in l
- $\text{exit}(l)$: clocks to be reset to 0 when leaving l.

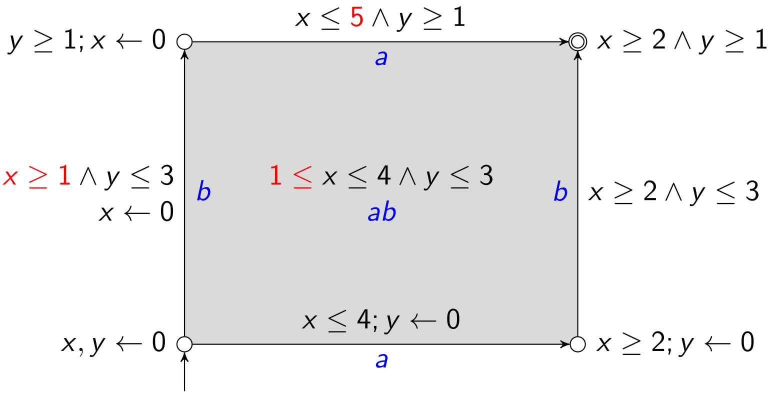


Examples



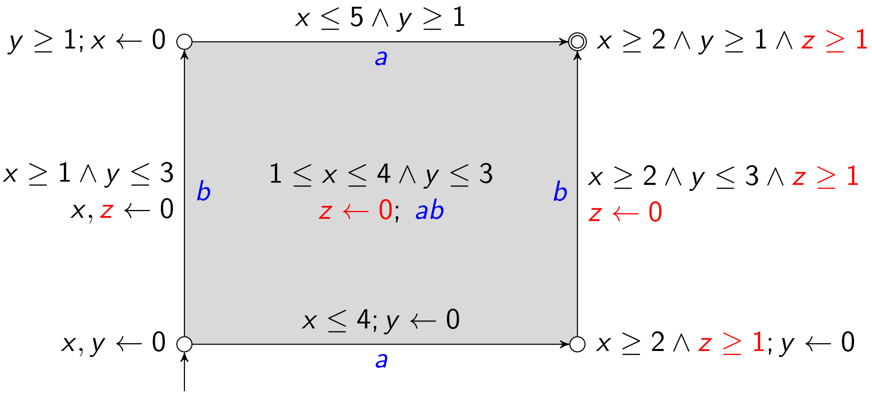
- a takes $[2, 4]$ time units, b takes $[1, 3]$ time units

Examples



- *a* takes $[2, 4]$ time units, *b* takes $[1, 3]$ time units
- unless *b* is done before *a*
- *b* can only start 1 time unit after *a*

Examples



- a takes $[2, 4]$ time units, b takes $[1, 3]$ time units
- b can only start 1 time unit after a
- b has to finish 1 time unit before a

Good News

- Reachability for HDTA is PSPACE-complete
- and can be checked using zone-based algorithms
- (Everything works like for timed automata)
- Universality probably still undecidable

⇒ [F. 2021, LITES]

Actions Take Time?

- [Cardelli 1982, ICALP]: Actions take time.
  - 'We read $p \xrightarrow[t]{a} q$ as "p moves to q performing a for an interval t"'
- since [Alur-Dill 1994, TCS] (even before?): Actions are immediate.
  - $(l, v) \overset{d}{\leadsto} (l, v + d) \overset{s}{\leadsto} (l', v + d)$
- Kim G. Larsen (many personal discussions): Actions are immediate because of technical resaons only. ("We know how to do.")
- [Chatain-Jard 2013, FORMATS]: In the concurrent semantics for time Petri nets, time has to (locally) be allowed to run backwards??
- [F. 2018, ADHS]: In real-time concurrency, actions cannot be immediate.
  - and it appears that the "technical reasons" argument is quite weak!

Conclusion

- Higher-dimensional automata: nice model for concurrency
- Languages of HDA: sets of labeled interval orders
  - [F.-Johansen-Struth-Ziemiański 2021, MSCS]
- Po(m)sets with interfaces for compositionality / algebra

Open / coming up:

- Higher-dimensional regular languages
- 2-categories with lax tensors: algebraic setting for iposets
- Combinatorial characterization of gluing-parallel iposets
- Languages of higher-dimensional timed automata
- ...