Posets With and Without Interfaces

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Series-Parallel Posets

- a poset: *finite* set P plus partial order $\leq$: reflexive, transitive, antisymmetric
- parallel composition of posets $(P_1, \leq_1)$, $(P_2, \leq_2)$:

$$P_1 \otimes P_2 = (P_1 \sqcup P_2, \leq_1 \cup \leq_2)$$

$\upuparrows$ disjoint union

- serial composition:

$$P_1 * P_2 = (P_1 \sqcup P_2, \leq_1 \cup \leq_2 \cup P_1 \times P_2)$$

$\upuparrows$ P_1 before P_2

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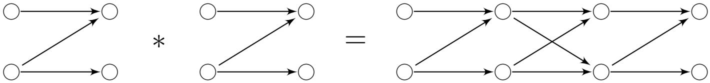


Series-Parallel Posets

Definition (Winkowski '77, Grabowski '81)

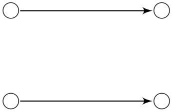
A poset is series-parallel (sp) if it is empty or can be obtained from the singleton poset by a finite number of serial and parallel compositions.

Theorem (Grabowski '81)

A poset is sp iff it does not contain N *as an induced subposet.*

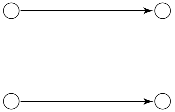
The equational theory of sp-posets is well-understood: [Gischer 1988, TCS], [Bloom-Esik 1996, MSCS]

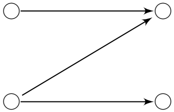
Interval Orders



- posets which are good for concurrency?
- already in [Wiener 1914], then [Winkowski '77], [Lamport '86], [van Glabbeek '90], [Vogler '91], [Janicky '93], etc.
- interval orders: posets which have representation as (real) intervals, ordered by $\max_1 \le \min_2$
- Lemma (Fishburn '70): A poset is interval iff it does not contain $2{+}2 = \left(\begin{smallmatrix} \cdot \longrightarrow \cdot \\ \cdot \longrightarrow \cdot \end{smallmatrix}\right)$ as induced subposet.
- intuitively: if $a \longrightarrow b$ and $c \longrightarrow d$, then also $a \longrightarrow d$ or $c \longrightarrow b$

Interval Orders vs Series-Parallel Posets





- interval orders are used in Petri net theory and distributed computing
- but have no algebraic representation (so far)
- sp-posets are used in concurrency theory & have nice algebraic theory
- Concurrent Kleene algebra
- int. orders are 2+2-free; sp-posets are **N**-free
- incomparable: 2+2 is sp; **N** is interval

⇒ [F.-Johansen-Struth-Thapa 2020, RAMiCS]

Posets With Interfaces

Definition

A poset with interfaces (iposet) is a poset P plus two injections

$$[n] \xrightarrow{s} P \xleftarrow{t} [m]$$

such that $s[n]$ is minimal and $t[m]$ is maximal in P.

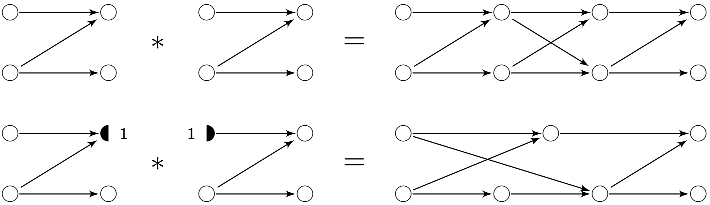
- $([n] = \{1, \dots, n\})$
- s: starting interface ; t: terminating interface
- events in $t[m]$ are *unfinished* ; events in $s[n]$ are "*unstarted*"

Gluing Composition

Definition

The gluing composition of iposets $s_1 : [n] \to (P_1, \leq_1) \leftarrow [m] : t_1$ and $s_2 : [m] \to (P_2, \leq_2) \leftarrow [k] : t_2$:

$$P_1 * P_2 = \begin{cases} (P_1 \sqcup P_2)/t_1(i) = s_2(i) \\ \leq_1 \cup \leq_2 \cup (P_1 \setminus t_1[m]) \times (P_2 \setminus s_2[m]) \end{cases}$$



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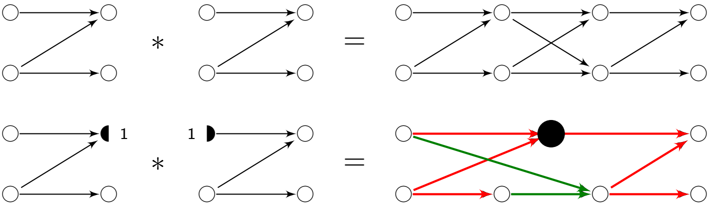

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Gluing-Parallel Iposets

- recall *sp-posets*: freely generated from ○ using $*$ and $\otimes$
- the four singleton iposets:

○ 1◗ ◖1 1⧓1

- gp-iposets: generated from ○, 1◗, ◖1, 1⧓1 using $*$ and $\otimes$

Fact

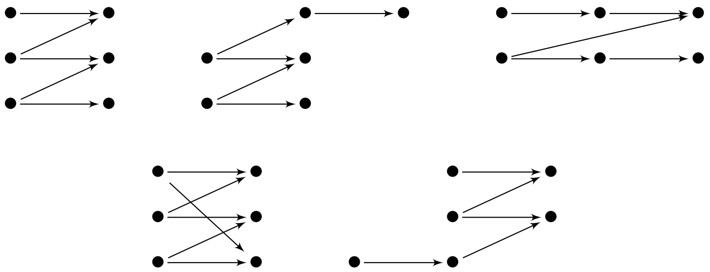
Gp-iposets are not freely generated, for example:

$$\begin{pmatrix} \text{◖1} \\ P \end{pmatrix} * \begin{pmatrix} \text{1◗} \\ Q \end{pmatrix} = \begin{pmatrix} \text{○} \\ P * Q \end{pmatrix} \qquad \begin{pmatrix} \text{◖1} \\ P \end{pmatrix} * \begin{pmatrix} \text{1⧓1} \\ Q \end{pmatrix} = \begin{pmatrix} \text{◖1} \\ P * Q \end{pmatrix}$$

$$\begin{pmatrix} \text{1⧓1} \\ P \end{pmatrix} * \begin{pmatrix} \text{1◗} \\ Q \end{pmatrix} = \begin{pmatrix} \text{1◗} \\ P * Q \end{pmatrix} \qquad \begin{pmatrix} \text{1⧓1} \\ P \end{pmatrix} * \begin{pmatrix} \text{1⧓1} \\ Q \end{pmatrix} = \begin{pmatrix} \text{1⧓1} \\ P * Q \end{pmatrix}$$

Forbidden Substructures

- sp-posets $\widehat{=}$ **N**-free
- interval orders $\widehat{=}$ 2+2-free
- gluing-parallel posets $\Rightarrow$ free of



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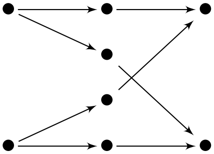

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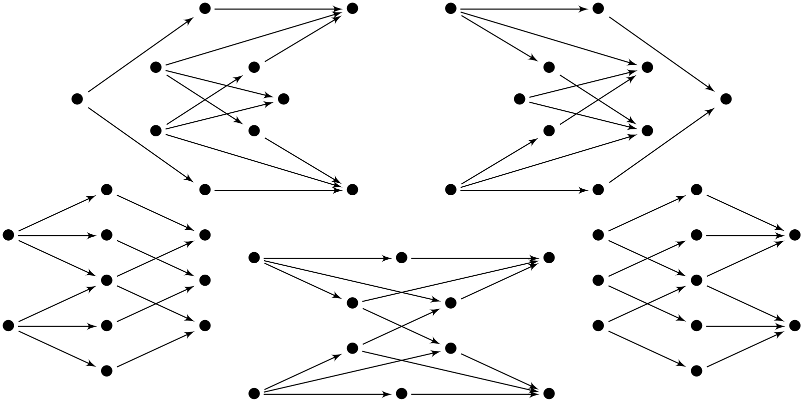
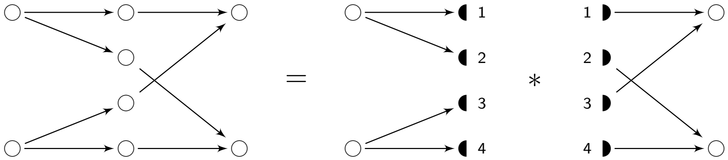


The 8-Point Forbidden Substructure is Decomposable?!?



- second iposet is not gp "for the wrong reasons"
- interfaces "permuted wrong"

- same for all 10-point forbidden substructures: all "decomposable up to interface permutation"

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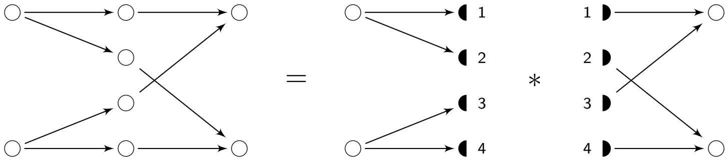

- second iposet is not gp "for the wrong reasons"
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- same for all 10-point forbidden substructures: all "decomposable up to interface permutation"
  - what does that even mean?

Gluing-Parallel-Symmetric Iposets

- recall gp-iposets: generated from ○ , 1 ◗ , ◖ 1 , and 1 ⧓ 1 (using $*$ and $\otimes$)
- let (1 ⧓ 2 / 2 ⧓ 1) $= (s, [2], t) : 2 \to 2$ be the non-trivial symmetry on 2
- gps-iposets: generated from ○ , 1 ◗ , ◖ 1 , 1 ⧓ 1 , and (1 ⧓ 2 / 2 ⧓ 1) (using $*$ and $\otimes$)

First lemma

An iposet is gps iff its underlying poset is.

- so all interface permutations included!

Some numbers

n	$P(n)$	$GP(n)$	$GPS(n)$	$IP(n)$	$GPI(n)$	$GPSI(n)$
0	1	1	1	1	1	1
1	1	1	1	4	4	4
2	2	2	2	17	16	17
3	5	5	5	86	74	86
4	16	16	16	532	419	532
5	63	63	63	4068	2980	4068
6	318	313	313	38.933	26.566	38.447
7	2045	1903	1903	474.822	289.279	
8	16.999	13.943	13.944	7.558.620	3.726.311	
9	183.231	120.442	120.465			
10	201.608	1.206.459				

Gps-Posets Without Interfaces

Definition

Let P_1 and P_2 be posets.

- The right-interior gluing composition $P_1 \triangleright^{\mathrm{i}} P_2$: carrier set $P_1 \sqcup P_2$, $(p,i) < (q,j) \Leftrightarrow \big(i = j \wedge p <_i q\big) \vee \big(i < j \wedge q \notin P_2^{\min}\big)$
- The left-interior gluing composition $P_1 \,{}^{\mathrm{i}}\!\triangleright P_2$: carrier set $P_1 \sqcup P_2$, $(p,i) < (q,j) \Leftrightarrow \big(i = j \wedge p <_i q\big) \vee \big(i < j \wedge p \notin P_1^{\max}\big)$
- The Winkowski multi-composition $P_1 \gtrdot P_2$: defined if $|P_1^{\max}| = |P_2^{\min}|$, and then $P_1 \gtrdot P_2 = \big\{P_1 \gtrdot_f P_2 \mid f \text{ bijection } P_1^{\max} \to P_2^{\min}\big\}$, where $P_1 \gtrdot_f P_2$ is the poset with carrier set $(P_1 \sqcup P_2)_{/x=f(x)}$ and order $(p,i) < (q,j) \Leftrightarrow \big(i = j \wedge p <_i q\big) \vee \big(i < j \wedge p \notin P_1^{\max} \wedge q \notin P_2^{\min}\big)$

Proposition

Gps-posets are generated from $\circ$ using $\otimes$, $*$, $\triangleright^{\mathrm{i}}$, ${}^{\mathrm{i}}\!\triangleright$, and $\gtrdot$.

Gps = Sp-Intervals

Definition

Let P, V be posets. An interval representation of P in V is a pair of functions $f, g : P \to V$ such that:

1. $f(p) \leq g(p)$ for all $p \in P$,
2. $p < q$ iff $g(p) < f(q)$ for all $p, q \in P$.

Proposition

A poset is gps iff it admits an interval representation in an sp-poset.

Conjecture

A poset is gps iff it does not contain any of the Small Forbidden Five.