A Myhill-Nerode Theorem for Higher-Dimensional Automata

Uli Fahrenberg Krzysztof Ziemiański

LRE & EPITA Rennes, France

University of Warsaw

Petri Nets 2023

Languages of higher-dimensional automata

- Languages of Higher-Dimensional Automata. MSCS 2021
- Posets with Interfaces as a Model for Concurrency. I&C 2022
- A Kleene Theorem for Higher-Dimensional Automata. CONCUR 2022
- A Myhill-Nerode Theorem for Higher-Dimensional Automata. PN 2023

Today:

1. What are HDAs?
2. What are languages of HDAs?
3. What can I do with languages of HDAs?

Nice people

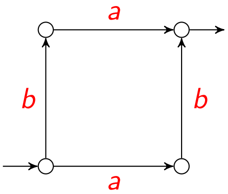
- Christian Johansen, NTNU
- Georg Struth, Sheffield
- Krzysztof Ziemiański, Warsaw

- Amazigh Amrane, Hugo Bazille, EPITA
- Safa Zouari, NTNU
- Eric Goubault, LIX

- See `https://ulifahrenberg.github.io/pomsetproject/` for more

Higher-dimensional automata

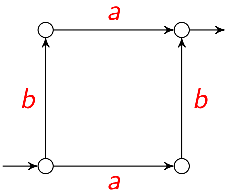
semantics of "*a* parallel *b*":



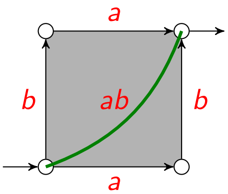
$a.b + b.a$

Higher-dimensional automata

semantics of "*a* parallel *b*":



$a.b + b.a$



a and *b* are independent

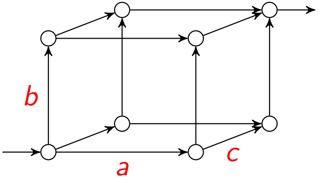
Higher-dimensional automata & concurrency

HDAs as a model for concurrency:

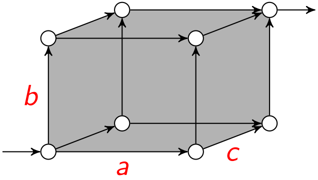
- points: states
- edges: transitions
- squares, cubes etc.: independency relations (concurrently executing events)
- two-dimensional automata ≅ asynchronous transition systems [Bednarczyk]

[van Glabbeek 2006, TCS]: Up to history-preserving bisimilarity, HDAs "generalize the main models of concurrency proposed in the literature" (notably, event structures and Petri nets)

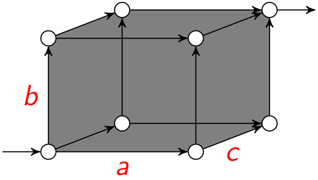
Examples



no concurrency



two out of three



full concurrency

Precubical sets and higher dimensional automata

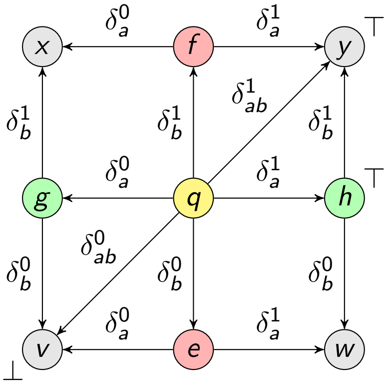
An loset is a finite, ordered and Σ-labelled set. (a list of events)

A precubical set X consists of:

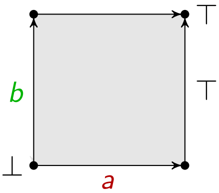
- A set of cells X
- Every cell $x \in X$ has an loset $\text{ev}(x)$ (list of events active in x)
- We write $X[U] = \{x \in X \mid \text{ev}(x) = U\}$ for an loset U (cells of type U)
- For every loset U and $A \subseteq U$ there are:
  upper face map $\delta^1_A : X[U] \to X[U - A]$ (terminating events A)
  lower face map $\delta^0_A : X[U] \to X[U - A]$ ("unstarting" events A)
- Precubical identities: $\delta^\mu_A \delta^\nu_B = \delta^\nu_B \delta^\mu_A$ for $A \cap B = \emptyset$ and $\mu, \nu \in \{0, 1\}$

A higher dimensional automaton (HDA) is a finite precubical set X with start cells $\bot \subseteq X$ and accept cells $\top \subseteq X$ (not necessarily vertices)

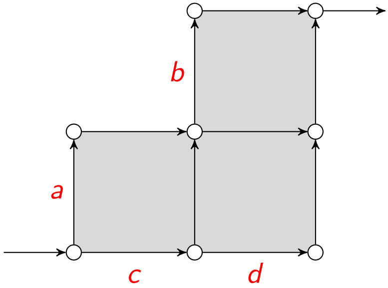
Example



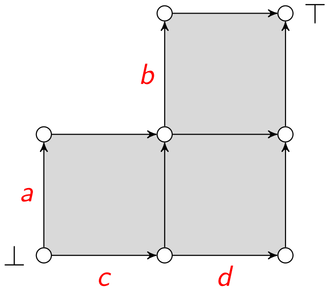
$$X[\emptyset] = \{v, w, x, y\}$$
$$X[a] = \{e, f\}$$
$$X[b] = \{g, h\}$$
$$X[ab] = \{q\}$$
$$\bot_X = \{v\}$$
$$\top_X = \{h, y\}$$



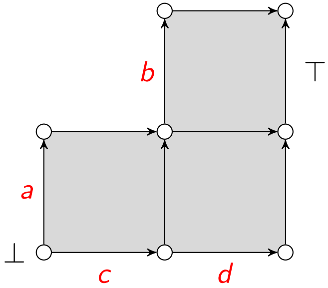
More interesting



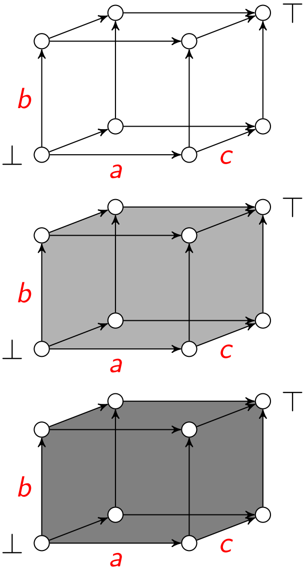
More interesting



More interesting



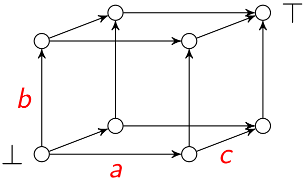
Examples



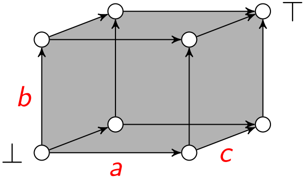
$$L_1 = \{abc, acb, bac, bca, cab, cba\}$$

$$L_3 = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \dots \right\}$$

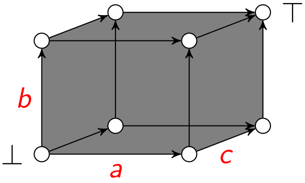
Examples



$$L_1 = \{abc, acb, bac, bca, cab, cba\}$$



$$L_2 = \left\{ \begin{pmatrix} a \\ b \to c \end{pmatrix}, \begin{pmatrix} a \\ c \to b \end{pmatrix}, \begin{pmatrix} b \\ a \to c \end{pmatrix}, \begin{pmatrix} b \\ c \to a \end{pmatrix}, \begin{pmatrix} c \\ a \to b \end{pmatrix}, \begin{pmatrix} c \\ b \to a \end{pmatrix} \right\} \cup L_1$$



sets of pomsets

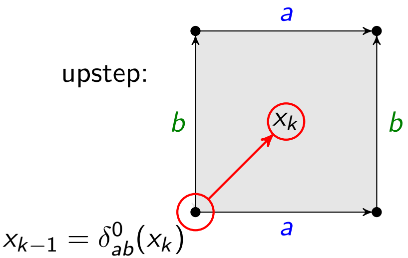
$$L_3 = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\} \cup L_2$$

Computations of HDAs

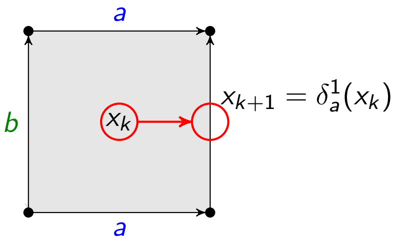
A path on an HDA X is a sequence $(x_0, \phi_1, x_1, \dots, x_{n-1}, \phi_n, x_n)$ such that for every k, (x_{k-1}, ϕ_k, x_k) is either

- $(\delta^0_A(x_k), \nearrow^A, x_k)$ for $A \subseteq \text{ev}(x_k)$ or (upstep: start A)
- $(x_{k-1}, \searrow_B, \delta^1_B(x_{k-1}))$ for $B \subseteq \text{ev}(x_{k-1})$ (downstep: terminate B)

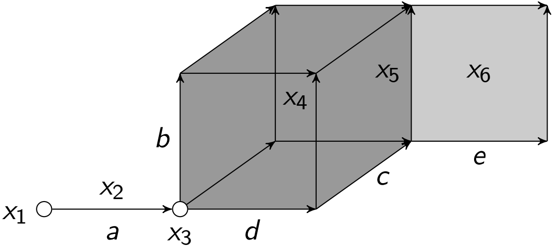
upstep:



downstep:

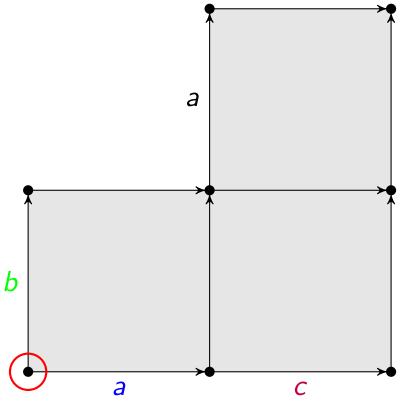


Example

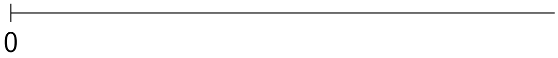


$$\left(x_1 \nearrow^{a} x_2 \searrow_{a} x_3 \nearrow^{\{b,c,d\}} x_4 \searrow_{\{c,d\}} x_5 \nearrow^{e} x_6\right)$$

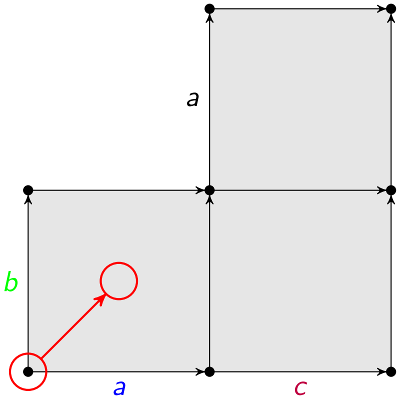
Event ipomset of a path



Lifetimes of events

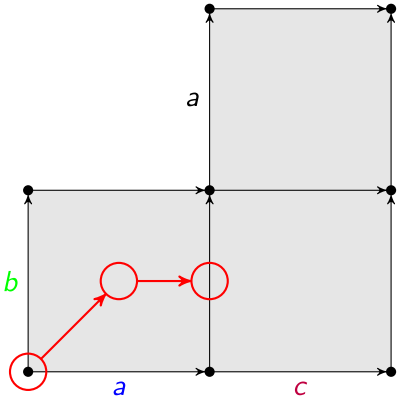


Event ipomset of a path

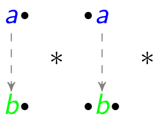


Lifetimes of events

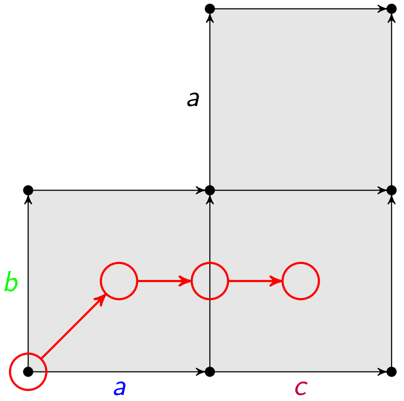
Event ipomset of a path



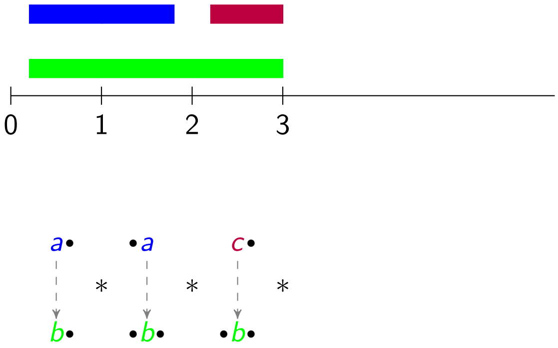
Lifetimes of events



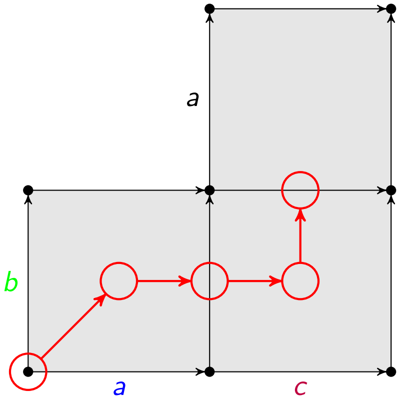
Event ipomset of a path

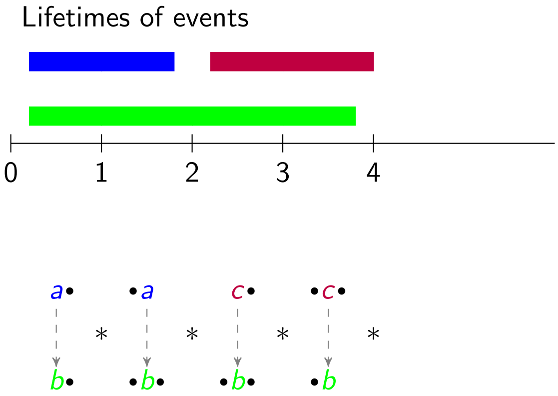


Lifetimes of events

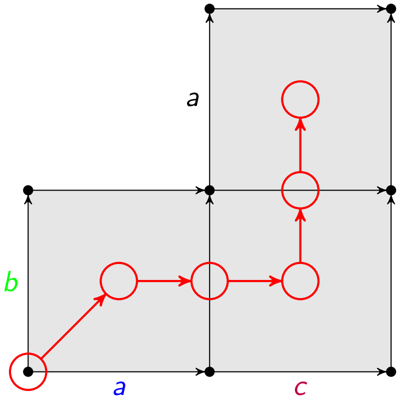


Event ipomset of a path

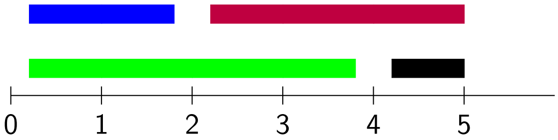


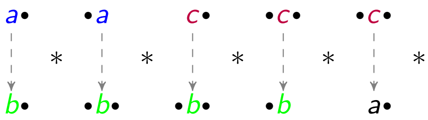


Event ipomset of a path

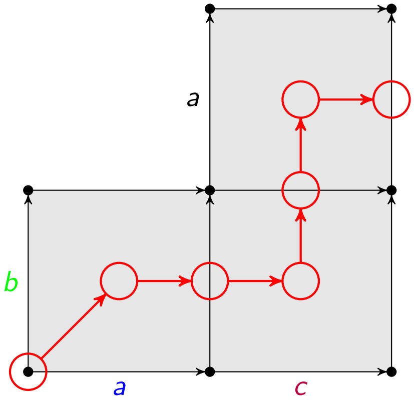


Lifetimes of events

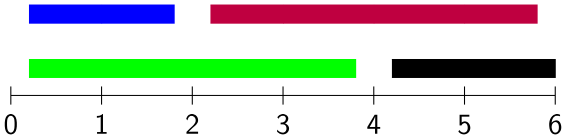


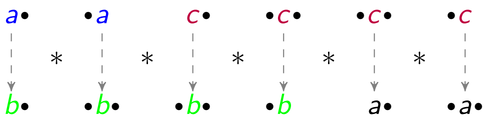


Event ipomset of a path

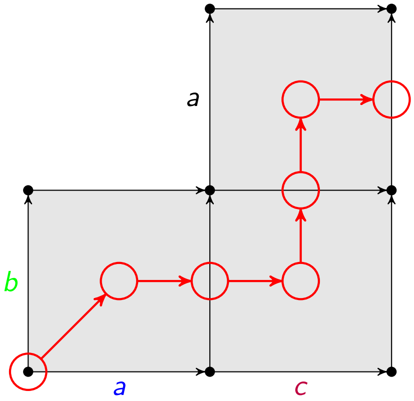


Lifetimes of events

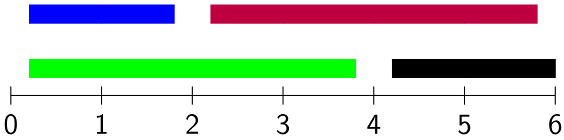




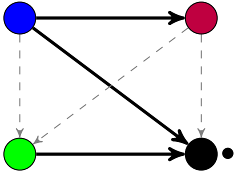
Event ipomset of a path



Lifetimes of events



Event ipomset

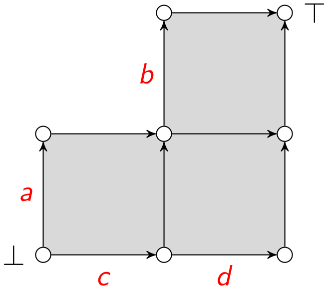


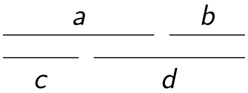
not series-parallel!

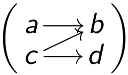
Are all pomsets generated by HDAs?

No, only (labeled) interval orders

- Poset $(P, \leq)$ is an interval order iff it has an interval representation:
  - a set $I = \{[l_i, r_i]\}$ of real intervals
  - with order $[l_i, r_i] \preceq [l_j, r_j]$ iff $r_i \leq l_j$
  - and an order isomorphism $(P, \leq) \leftrightarrow (I, \preceq)$
- [Fishburn 1970]







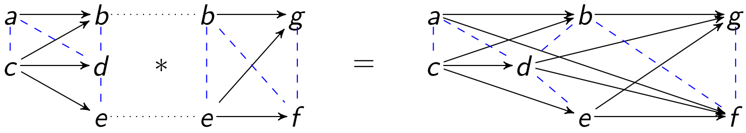
Pomsets with interfaces

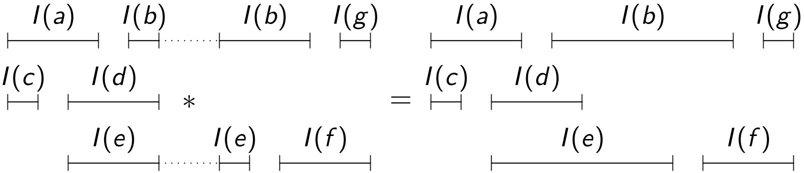
Definition (Ipomset)

A pomset with interfaces (and event order): $(P, <, \dashrightarrow, S, T, \lambda)$:

- finite set P;
- two partial orders $<$ (precedence order), $\dashrightarrow$ (event order)
  - s.t. $< \cup \dashrightarrow$ is a *total relation*;
- $S, T \subseteq P$ source and target interfaces
  - s.t. S is $<$-minimal, T is $<$-maximal.

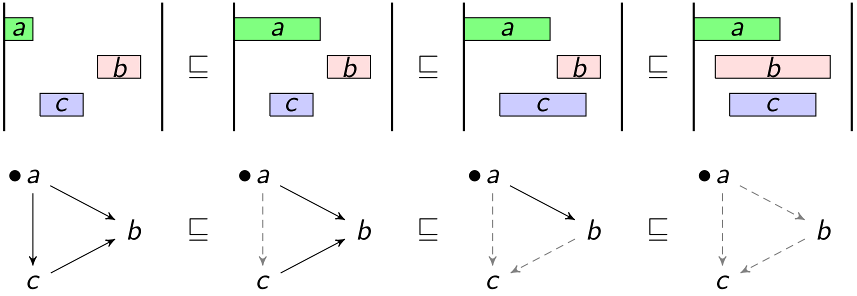
Composition of ipomsets





- Gluing $P * Q$: P before Q, except for interfaces (which are identified)
- Parallel composition $P \parallel Q$: P above Q (disjoint union)

Subsumption



P refines Q / Q subsumes P / $P \sqsubseteq Q$ iff

- P and Q have same interfaces
- P has more $<$ than Q
- Q has more $\dashrightarrow$ than P

Languages of HDAs

Definition

The language of an HDA X is the set of event ipomsets of all accepting paths:

$$L(X) = \{\text{ev}(\pi) \mid \pi \in \text{Paths}(X), src(\pi) \in \bot_X, tgt(\pi) \in \top_X\}$$

- $L(X)$ contains only interval-order ipomsets
- and is closed under subsumption

Path objects

Important tool:

Proposition

For any interval-order ipomset P there exists an HDA $\square^P$ for which $L(\square^P) = \{P\}{\downarrow}$.

Lemma

For any HDA X and ipomset P, $P \in L(X)$ iff $\exists f : \square^P \to X$.

Theorems

Definition (Rational Languages over Σ)

- Generated by $\emptyset$, $\{\epsilon\}$, and all $\{[a]\}$, $\{[\bullet\, a]\}$, $\{[a\, \bullet]\}$, $\{[\bullet\, a\, \bullet]\}$ for $a \in \Sigma$
- under operations $\cup$, $*$, $\|$ and (Kleene plus) $^+$

Theorem (à la Kleene)

A language is rational iff it is recognized by an HDA.

Theorem (à la Myhill-Nerode)

A language is rational iff it has finite prefix quotient.

Myhill-Nerode

Prefix quotients:

- $P \backslash L := \{Q \in \text{iiPoms} \mid PQ \in L\}$
- $\text{suff}(L) := \{P \backslash L \mid P \in \text{iiPoms}\}$

Theorem

L is rational iff $\text{suff}(L)$ is finite.

Proof $\Rightarrow$: Let $L = L(X)$ be rational.

1. For $x \in X$ denote $\text{Pre}(x) = L(X_\perp^x)$ and $\text{Post}(x) = L(X_x^\top)$.
2. Lemma: for all P, $P \backslash L = \bigcup \{\text{Post}(x) \mid x \in X, P \in \text{Pre}(x)\}$.
3. And then $\{P \backslash L \mid P \in \text{iiPoms}\} \subseteq \{\bigcup_{x \in Y} \text{Post}(x) \mid Y \subseteq X\}$ which is finite. □

Myhill-Nerode $\Longleftarrow$

Assume suff(L) finite. Construct HDA $M(L)$:

- Write $P \sim_L Q$ if $P \backslash L = Q \backslash L$
  - standard Myhill-Nerode equivalence: doesn't work for us
  - but implies $S_P = S_Q$ and $T_P = T_Q$
- Write $P \approx_L Q$ if $P \sim_L Q$ and $\forall A \subseteq T_P - S_P : (P - A) \backslash L = (Q - A) \backslash L$
- cells of $M(L)$: $M(L)[U] = \text{iiPoms}_U / {\approx_L} \cup \{w_U\}$ $\longleftarrow$ subsidiary "completion" cells
- face maps:
  - $\delta^1_A(\langle P \rangle) = \langle P * U{\downarrow}_A \rangle$ (terminate A)
  - $\delta^0_A(\langle P \rangle) = \langle P - A \rangle$ if $A \subseteq T_P - S_P$ (unstart A)
  - $\delta^0_A(\langle P \rangle) = w_{U-A}$ otherwise; $\delta^0_A(w_U) = \delta^1_A(w_U) = w_{U-A}$
- $\bot_{M(L)} = \{\langle \text{id}_U \rangle\}_{U \in \square}$ and $\top_{M(L)} = \{\langle P \rangle \mid P \in L\}$

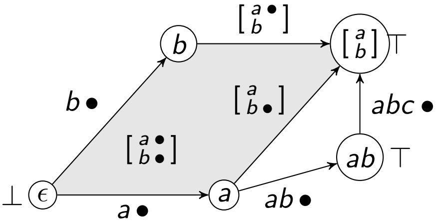
Proposition

The essential part of $M(L)$ is finite and $L(M(L)) = L$.

- essential part: reachable and co-reachable cells plus all their faces

Example

$L = \{ \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right], ab, ba, abc \}$



$M(L)[\emptyset]$

P	$P \backslash L$
ϵ	L
a	$\{b, bc\}$
b	$\{a\}$
ab	$\{\epsilon, c\}$
$\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$	$\{\epsilon\}$

$M(L)[a]$

P	$P \backslash L$
$a\,\bullet$	$\{ \left[\begin{smallmatrix} \bullet\, a \\ b \end{smallmatrix}\right], \bullet\, ab, \bullet\, abc \}$
$ba\,\bullet$	$\{\bullet\, a\}$

$M(L)[b]$

P	$P \backslash L$
$b\,\bullet$	$\{ \left[\begin{smallmatrix} a \\ \bullet\, b \end{smallmatrix}\right], \bullet\, ba \}$
$ab\,\bullet$	$\{\bullet\, b, \bullet\, bc\}$
$\left[\begin{smallmatrix} a \\ b\,\bullet \end{smallmatrix}\right]$	$\{\bullet\, b\}$

$M(L)[c]$

P	$P \backslash L$
$abc\,\bullet$	$\{\bullet\, c\}$

$M(L)[\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]]$

P	$P \backslash L$
$\left[\begin{smallmatrix} a\,\bullet \\ b\,\bullet \end{smallmatrix}\right]$	$\{ \left[\begin{smallmatrix} \bullet\, a \\ \bullet\, b \end{smallmatrix}\right] \}$

Properties

- $M(L)$ may be non-deterministic
- if L is determinizable, then $M(L)$ is deterministic (and minimal (?))
- but there exist non-determinizable ipomset languages
- in fact, there are languages of unbounded ambiguity
  - for example $L = ([\,{}^{a}_{b}\,]\, cd + ab\, [\,{}^{c}_{d}\,])^{+}$

Further:

- regular languages are closed under ($\cup$, $*$, $\|$, ${}^{+}$, and) $\cap$
- but not under complement
  - L regular $\Rightarrow$ L has finite width $\Rightarrow$ $(\text{iiPoms} - L){\downarrow}$ has infinite width
- width-bounded complement: $\overline{L}^{k} = \{P \in \text{iiPoms} - L \mid \text{wid}(P) \le k\}{\downarrow}$
- regular languages are closed under $\overline{}^{k}$ (for all k)

Further:

- emptiness and inclusion of regular languages are decidable

Conclusion & Further Work

Higher-Dimensional Automata Theory for Fun and Profit!

- Kleene and Myhill-Nerode: a good start
- are HDAs learnable?
- trouble with determinization and non-ambiguity: residual automata?

- logical characterization? Büchi-Elgot theorem?
- relation to interval semantics for Petri nets?
- relation to trace theory?

- higher-dimensional timed automata
- higher-dimensional omega-automata
- Distributed Hybrid Systems