# Logic and Languages of Higher-Dimensional Automata

Amazigh Amrane[1] Hugo Bazille[1] Uli Fahrenberg[1] Marie Fortin[2]

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DLT 2024

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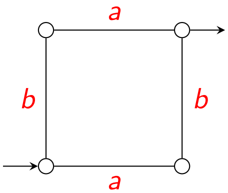
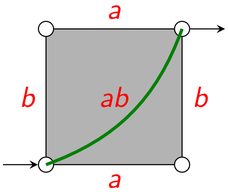
# Higher-dimensional automata

semantics of "*a* parallel *b*":




$a.b + b.a$




*a* and *b* are independent

# Higher-dimensional automata & concurrency

HDAs as a model for concurrency:

- points: states
- edges: transitions
- squares, cubes etc.: independency relations / concurrently executing events
- two-dimensional automata ≅ asynchronous transition systems [Bednarczyk]
- [Pratt 1991, POPL], [van Glabbeek 1991, email message]

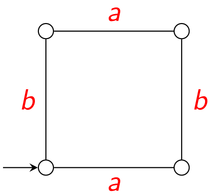
[van Glabbeek 2006, TCS]: Up to history-preserving bisimilarity, HDAs "generalize the main models of concurrency proposed in the literature" (notably, event structures and Petri nets)

Lots of recent activity on languages of HDAs:

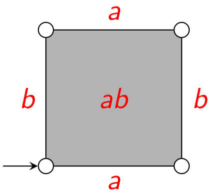
- Kleene theorem
- Myhill-Nerode theorem
- Büchi-Elgot-Trakhtenbrot theorem
- ...

# Higher-dimensional automata

$a|b$




$a.b + b.a$

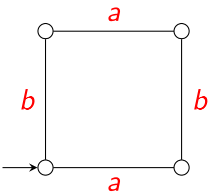


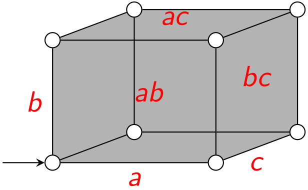

$a$ and $b$ are independent

# Higher-dimensional automata

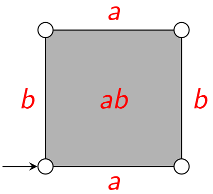
$a|b$

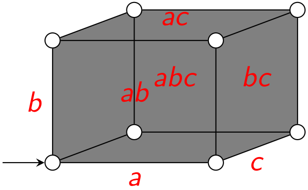
$a|b|c$




$a|b + a|c + b|c$




$\{a, b, c\}$ independent

# Higher-dimensional automata

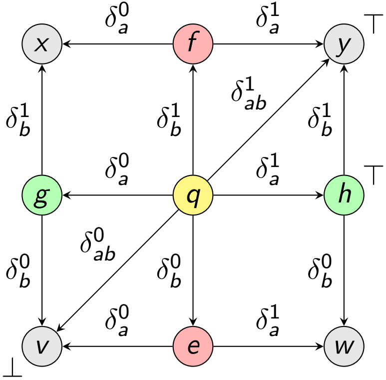
A conclist is a finite, ordered and $\Sigma$-labelled set. (a list of events)

A precubical set $X$ consists of:

- A set of cells $X$ (cubes)
- Every cell $x \in X$ has a conclist $\mathrm{ev}(x)$ (list of events active in $x$)
- We write $X[U] = \{x \in X \mid \mathrm{ev}(x) = U\}$ for a conclist $U$ (cells of type $U$)
- For every conclist $U$ and $A \subseteq U$ there are:
  upper face map $\delta^1_A : X[U] \to X[U \setminus A]$ (terminating events $A$)
  lower face map $\delta^0_A : X[U] \to X[U \setminus A]$ (unstarting events $A$)
- Precube identities: $\delta^\mu_A \delta^\nu_B = \delta^\nu_B \delta^\mu_A$ for $A \cap B = \emptyset$ and $\mu, \nu \in \{0, 1\}$

A higher dimensional automaton (HDA) is a precubical set $X$ with start cells $\bot \subseteq X$ and accept cells $\top \subseteq X$ (not necessarily vertices)

# Example




$$X[\emptyset] = \{v, w, x, y\}$$

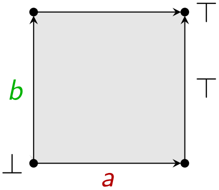
$$X[a] = \{e, f\}$$

$$X[b] = \{g, h\}$$

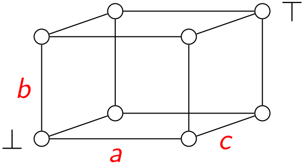
$$X[ab] = \{q\}$$

$$\bot_X = \{v\}$$

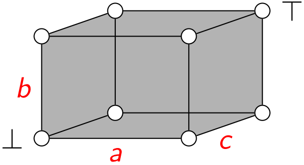
$$\top_X = \{h, y\}$$

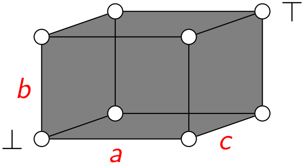


# Languages of HDAs: Examples



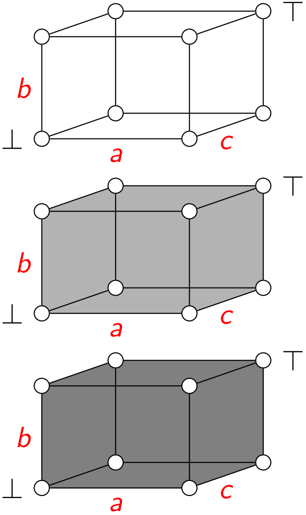

$$L_1 = \{abc, acb, bac, bca, cab, cba\}$$




$$L_3 = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \dots \right\}$$

# Languages of HDAs: Examples




$$L_1 = \{abc, acb, bac, bca, cab, cba\}$$

$$L_2 = \left\{ \begin{pmatrix} a \\ b \to c \end{pmatrix}, \begin{pmatrix} a \\ c \to b \end{pmatrix}, \begin{pmatrix} b \\ a \to c \end{pmatrix}, \begin{pmatrix} b \\ c \to a \end{pmatrix}, \begin{pmatrix} c \\ a \to b \end{pmatrix}, \begin{pmatrix} c \\ b \to a \end{pmatrix} \right\} \cup L_1 \cup \dots$$

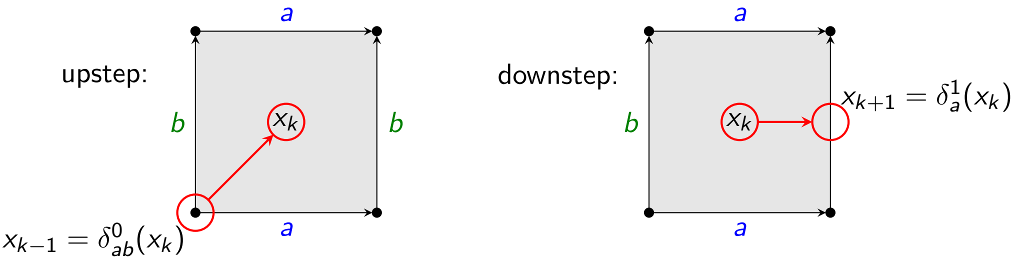
sets of pomsets

$$L_3 = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\} \cup L_2$$

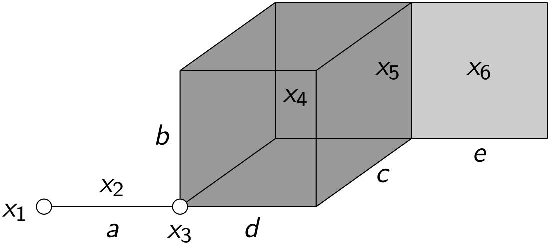
# Computations of HDAs

A path on an HDA $X$ is a sequence $(x_0, \varphi_1, x_1, \dots, x_{n-1}, \varphi_n, x_n)$ such that for every $k$, $(x_{k-1}, \varphi_k, x_k)$ is either

- $(\delta^0_A(x_k), \nearrow^A, x_k)$ for $A \subseteq \mathrm{ev}(x_k)$ or (upstep: start $A$)
- $(x_{k-1}, \searrow_B, \delta^1_B(x_{k-1}))$ for $B \subseteq \mathrm{ev}(x_{k-1})$ (downstep: terminate $B$)

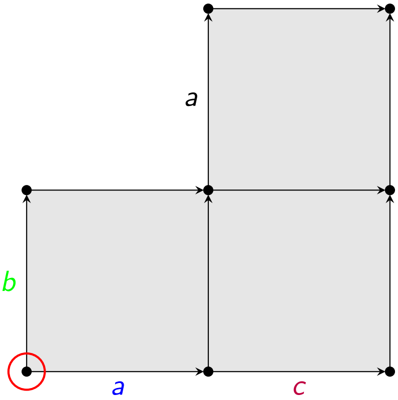


# Example

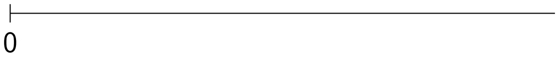



$$\left(x_1 \nearrow^{a} x_2 \searrow_{a} x_3 \nearrow^{\{b,c,d\}} x_4 \searrow_{\{c,d\}} x_5 \nearrow^{e} x_6\right)$$

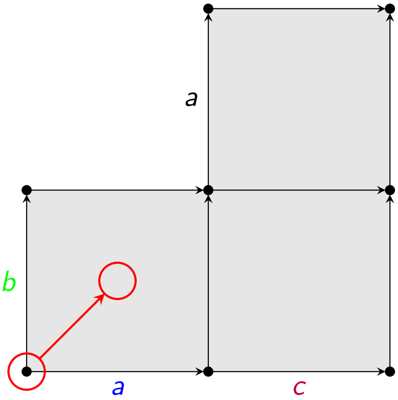
# Event ipomset of a path



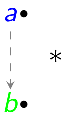

Lifetimes of events



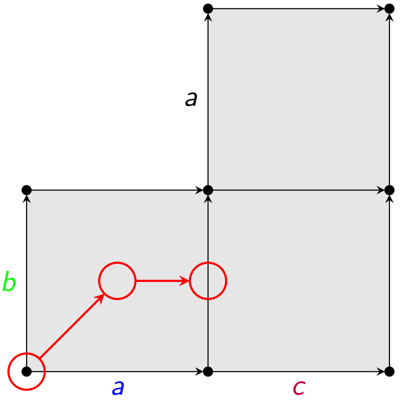
# Event ipomset of a path



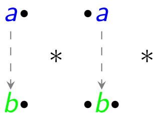

Lifetimes of events

# Event ipomset of a path




Lifetimes of events

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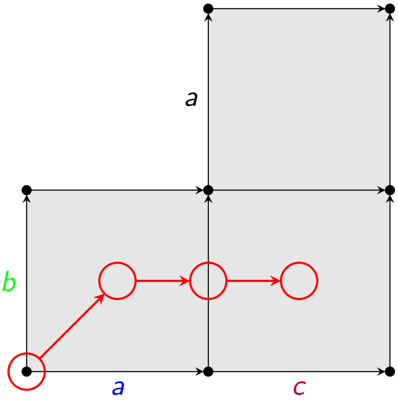
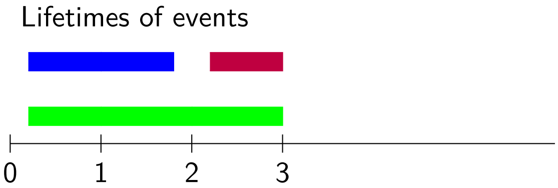
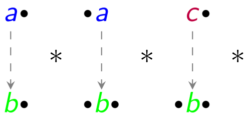
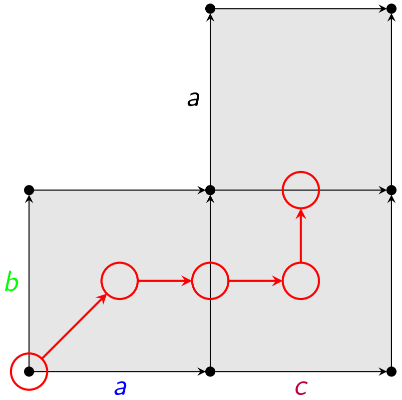
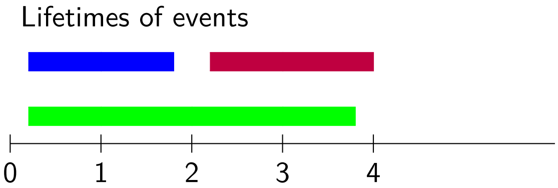


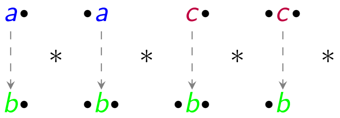



# Event ipomset of a path



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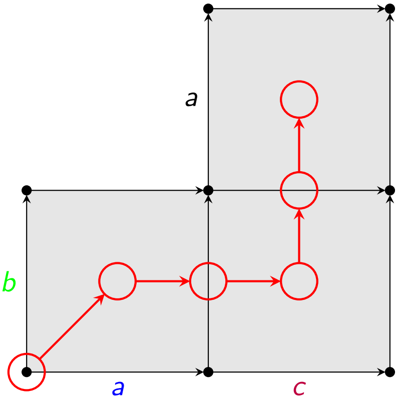
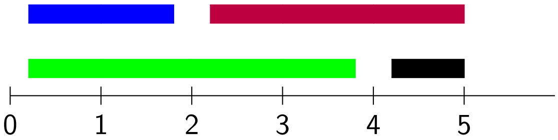
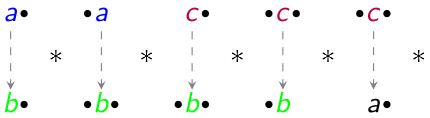



Lifetimes of events



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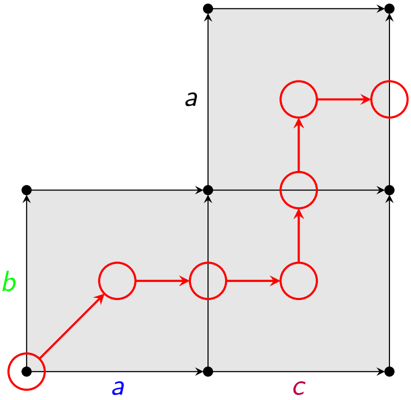
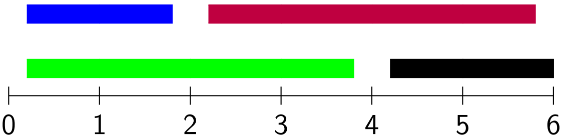
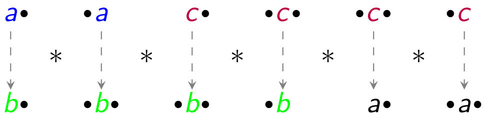



Lifetimes of events



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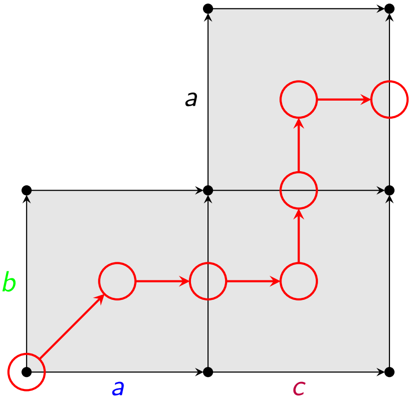
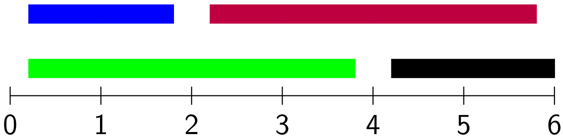
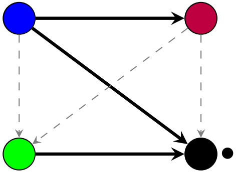



Lifetimes of events




Event ipomset



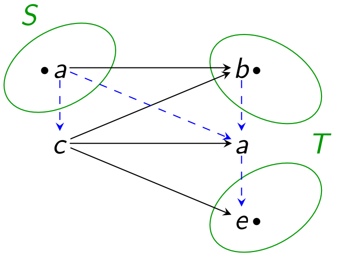
(not series-parallel!)

# Pomsets with interfaces

## Definition

A pomset with interfaces (ipomset): $(P, <, \dashrightarrow, S, T, \lambda)$:

- finite set $P$;
- two partial orders $<$ (precedence order), $\dashrightarrow$ (event order)
  - s.t. $< \cup \dashrightarrow$ is a *total relation*;
- $S, T \subseteq P$ source and target interfaces
  - s.t. $S$ is $<$-minimal and $T$ is $<$-maximal.

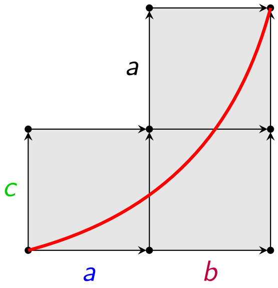


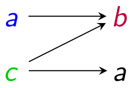
# Interval orders

## Definition

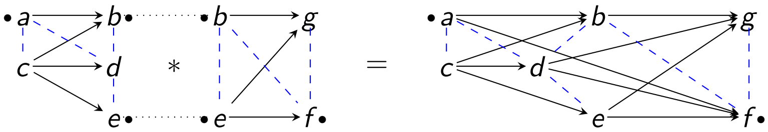
An ipomset $(P, <_P, \dashrightarrow, S, T, \lambda)$ is interval if $(P, <_P)$ has an interval representation: functions $b, e : P \to \mathbb{R}$ s.t.

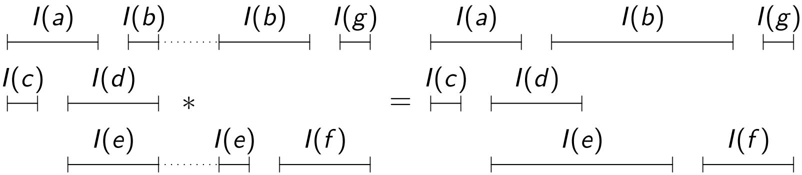
- $\forall x \in P : b(x) \leq_{\mathbb{R}} e(x)$;
- $\forall x, y \in P : e(x) <_{\mathbb{R}} b(y) \iff x <_P y$



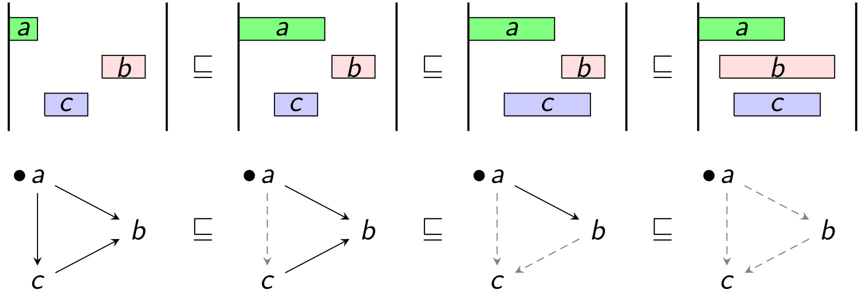
# Gluing composition




- Gluing $P * Q$: $P$ before $Q$, except for interfaces (which are identified)
- (also have parallel composition $P \parallel Q$: disjoint union)

# Subsumption




$P$ refines $Q$ / $Q$ subsumes $P$ / $P \sqsubseteq Q$ iff

- $P$ and $Q$ have same interfaces
- $P$ has more $<$ than $Q$
- $Q$ has more $\dashrightarrow$ than $P$

# Languages of HDAs

## Definition

The language of an HDA $X$ is the set of event ipomsets of all accepting paths:

$$L(X) = \{\text{ev}(\pi) \mid \pi \in \text{Paths}(X), \text{src}(\pi) \in \bot_X, \text{tgt}(\pi) \in \top_X\}$$

- $L(X)$ contains only interval ipomsets,
- is closed under subsumption,
- and has finite width

## Definition

A language $L \subseteq$ iiPoms is regular if there is an HDA $X$ with $L = L(X)$.

# Monadic Second-Order Logics over Ipomsets

## Definition

MSO-iiPoms is built using the grammar

$$\psi ::= a(x) \mid \mathsf{s}(x) \mid \mathsf{t}(x) \mid x < y \mid x \dashrightarrow y \mid x \in X \mid$$
$$\exists x.\,\psi \mid \forall x.\,\psi \mid \exists X.\,\psi \mid \forall X.\,\psi \mid \psi_1 \wedge \psi_2 \mid \psi_1 \vee \psi_2 \mid \neg\psi$$

- signature $\{<, \dashrightarrow, (a)_{a\in\Sigma}, \mathsf{s}, \mathsf{t}\}$
- $x \prec y := x < y \wedge \neg(\exists z. x < z < y)$
- for $\psi \in$ MSO with free variables $x_1, \dots, x_n, X_1, \dots, X_m$ and $P \in$ iiPoms, a valuation is $\nu = (\nu_1, \nu_2)$ with $\nu_1 : \{x_1, \dots, x_n\} \to P$ and $\nu_2 : \{X_1, \dots, X_m\} \to 2^P$
- $P \models_\nu \psi$ if $\psi$ holds with $x_i$ and $X_j$ interpreted as $\nu(x_i)$ and $\nu(X_j)$
- $P \models \psi$ if $P \models_{(\emptyset,\emptyset)} \psi$ (no free variables: a sentence)
- $L(\psi) = \{P \in \text{iiPoms} \mid P \models \psi\}$
- $L$ is MSO-definable if $\exists \psi \in \text{MSO} : L = L(\psi)$

# Example

$$\varphi = \exists x\, \exists y.\, a(x) \land b(y) \land \neg(x < y) \land \neg(y < x)$$

- satisfied by all $P$ which contain an $a$ in parallel with a $b$
- in particular any conclists $\begin{bmatrix} \vdots \\ a \\ \vdots \\ b \\ \vdots \end{bmatrix}$ or $\begin{bmatrix} \vdots \\ b \\ \vdots \\ a \\ \vdots \end{bmatrix}$
- but not by $ab$ nor $ba$

$\Longrightarrow$ $L(\varphi)$ has infinite width and is not closed under subsumption

# Results

## Theorem

*For all* $L \subseteq \text{iiPoms}$,

- *if L is MSO-definable, then* $L_{\leq k}{\downarrow}$ *is regular for all* $k \in \mathbb{N}$;
  *(subsumption closure of width restriction of L)*
- *if L is regular, then L is MSO-definable.*

*The constructions are effective in both directions.*

## Corollary

*For all* $k \in \mathbb{N}$, *a language* $L \subseteq \text{iiPoms}_{\leq k}$ *with* $L = L{\downarrow}$ *is regular iff it is MSO-definable.*

# Results

## Corollary

*For all $k \in \mathbb{N}$ and $\varphi \in \mathrm{MSO}$ with $L(\varphi) = L(\varphi)_{\le k}{\downarrow}$, satisfiability and model checking are decidable.*

- also implied by Courcelle's theorem, even more generally for $L(\varphi) = L(\varphi)_{\le k}$, because $\mathrm{iiPoms}_{\le k}$ has bounded treewidth

## Corollary

*For all $k \in \mathbb{N}$ and $L \subseteq \mathrm{iiPoms}_{\le k}$, if $L$ is MSO-definable, then so is $L{\downarrow}$.*

- does not hold for general (non-interval) ipomsets [Fanchon-Morin 2009]

# Proofs: Special ipomsets

## Definition

An ipomset $(P, <, \dashrightarrow, S, T, \lambda)$ is

- discrete if $<$ is empty (hence $\dashrightarrow$ is total) $\begin{bmatrix} \bullet a \\ \bullet b \bullet \\ a \bullet \end{bmatrix}$
- a starter if it is discrete and $T = P$ $\begin{bmatrix} \bullet a \bullet \\ \bullet b \bullet \\ a \bullet \end{bmatrix}$
- a terminator if it is discrete and $S = P$ $\begin{bmatrix} \bullet a \\ \bullet b \bullet \\ \bullet a \bullet \end{bmatrix}$
- an identity if it is both a starter and a terminator $\begin{bmatrix} \bullet a \bullet \\ \bullet b \bullet \\ \bullet a \bullet \end{bmatrix}$

## Lemma

*Any interval ipomset has a decomposition as a sequence of starters and terminators.*

$$\begin{bmatrix} a & \longrightarrow & b \\ & \nearrow & \\ c & \longrightarrow & a \end{bmatrix} = \begin{bmatrix} a\bullet \\ c \end{bmatrix} * \begin{bmatrix} \bullet a \\ a\bullet \end{bmatrix} * \begin{bmatrix} b \\ \bullet a \end{bmatrix} = \begin{bmatrix} a\bullet \\ c\bullet \end{bmatrix} * \begin{bmatrix} \bullet a\bullet \\ \bullet c \end{bmatrix} * \begin{bmatrix} \bullet a\bullet \\ a\bullet \end{bmatrix} * \begin{bmatrix} \bullet a \\ \bullet a\bullet \end{bmatrix} * \begin{bmatrix} b\bullet \\ \bullet a\bullet \end{bmatrix} * \begin{bmatrix} \bullet b \\ \bullet a \end{bmatrix}$$

# Unique decompositions

Notation: St: set of starters $(S, U, U)$
Te: set of terminators $(U, U, T)$
$\mathsf{Id} = \mathsf{St} \cap \mathsf{Te}$: set of identities $(U, U, U)$
$\Omega = \mathsf{St} \cup \mathsf{Te}$

## Definition

A word $w = (S_1, U_1, T_1) \dots (S_n, U_n, T_n) \in \Omega^+$ is coherent if $T_i = S_{i+1}$ for all $i$.
The set of coherent words is denoted Coh.

## Definition

A coherent word is sparse if proper starters and proper terminators are alternating.

- additionally, all $w \in \mathsf{Id} \subseteq \Omega^+$ are sparse
- so that's $\mathsf{Id} \cup (\mathsf{St} \setminus \mathsf{Id})\big((\mathsf{Te} \setminus \mathsf{Id})(\mathsf{St} \setminus \mathsf{Id})\big)^* \cup (\mathsf{Te} \setminus \mathsf{Id})\big((\mathsf{St} \setminus \mathsf{Id})(\mathsf{Te} \setminus \mathsf{Id})\big)^*$

## Lemma

*Any interval ipomset $P$ has a unique decomposition $P = P_1 * \cdots * P_n$ such that $P_1 \dots P_n \in \Omega^+$ is sparse.*

# From MSO to HDAs

Idea: translate ipomsets to decompositions in $\Omega^+$ and MSO-iiPoms to MSO over $\Omega$

- problem: $\Omega$ is infinite $\Longrightarrow$ need to restrict width

## Lemma

*For every* $k \in \mathbb{N}$ *and* $\varphi \in \mathrm{MSO}_{\mathrm{iiPoms}}$ *there exists* $\hat{\varphi} \in \mathrm{MSO}_{\Omega_{\leq k}}$ *such that* $P_1 \dots P_n \models \hat{\varphi}$ *iff* $P_1 \dots P_n$ *is coherent and* $P_1 * \dots * P_n \models \varphi$.

- $\mathrm{coherent}_{\leq k} := \forall x\, \forall y.\, x \prec y \implies \bigvee_{P_1 P_2 \in \mathrm{Coh}_{\leq k} \cap \Omega^2_{\leq k}} P_1(x) \wedge P_2(y)$
- ...
- Let $K = \{P \in \mathrm{iiPoms}_{\leq k} \mid P \models \varphi\}$

$\Longrightarrow$ $L'' = \{P_1 \dots P_n \in \Omega^+_{\leq k} \mid P_1 * \dots * P_n \in K\}$ is $\mathrm{MSO}_{\Omega_{\leq k}}$-definable (lemma)

- standard Büchi & Kleene $\Longrightarrow$ $L''$ is $\Omega_{\leq k}$-rational
- replace concatenation by gluing $\Longrightarrow$ $L = \{P \in \mathrm{iiPoms}_{\leq k} \mid P \models \varphi\}{\downarrow}$ is rational

# From HDAs to MSO

Idea: encode accepting paths into MSO

- (similar to classical construction)
- for $P \in$ iiPoms, use sparse decomposition $P = P_1 * \cdots * P_n$ to define $st : P \to \{-\infty\} \cup \mathbb{N}$, $te : P \to \mathbb{N} \cup \{+\infty\}$ to denote where events start and terminate in $P_1 \ldots P_n$

$\Longrightarrow$ $e \in P_i$ iff $st(e) \leq i \leq te(e)$

- and then map *st* and *te* to upsteps and downsteps in an accepting path

## Lemma

*For $f, g \in \{st, te\}$ and $\bowtie \in \{=, <, >\}$, the relations $f(x) \bowtie g(y)$, $\min(f)$ and $\max(f)$ are MSO-definable.*

# From HDAs to MSO

## Lemma

*Let $X$ be an HDA and $P \in \text{iiPoms}$ with sparse decomposition $P = P_1 * \cdots * P_n$. Then $P \in L(X)$ iff there exist $\rho_\nearrow : P \setminus S_P \to \text{ups}(X)$ and $\rho_\searrow : P \setminus T_P \to \text{downs}(X)$ such that for all $e_1, e_2 \in P$,*

- $st(e_1) = st(e_2) \implies \rho_\nearrow(e_1) = \rho_\nearrow(e_2)$ *and* $te(e_1) = te(e_2) \implies \rho_\searrow(e_1) = \rho_\searrow(e_2)$
- $st(e_2) = te(e_1) + 1 \implies \text{src}(\rho_\nearrow(e_2)) = \text{tgt}(\rho_\searrow(e_1))$
- $te(e_2) = st(e_1) + 1 \implies \text{src}(\rho_\searrow(e_2)) = \text{tgt}(\rho_\nearrow(e_1))$
- $\rho_\nearrow(e_1) = (p, \nearrow^A, q) \implies A = \{e \mid st(e) = st(e_1)\} \,\&\, \text{ev}(q) = \{e \mid st(e) \leq st(e_1) < te(e)\}$
- $\rho_\searrow(e_1) = (p, \searrow_A, q) \implies A = \{e \mid te(e) = te(e_1)\} \,\&\, \text{ev}(q) = \{e \mid st(e) < te(e_1) \leq te(e)\}$
- $st(e_1) = 1 \implies \text{src}(\rho_\nearrow(e_1)) \in \bot_X$ *and* $te(e_1) = 1 \implies \text{src}(\rho_\searrow(e_1)) \in \bot_X$
- $st(e_1) = n \implies \text{tgt}(\rho_\nearrow(e_1)) \in \top_X$ *and* $te(e_1) = n \implies \text{tgt}(\rho_\searrow(e_1)) \in \top_X$

... and now encode these into MSO

# Conclusion

Büchi-Elgot-Trakhtenbrot theorem for higher-dimensional automata:

## Theorem

- *If* $L \subseteq \text{iiPoms}$ *is regular, then* $L$ *is effectively MSO-definable.*
- *If* $L$ *is MSO-definable, then* $L_{\leq k}\downarrow$ *is effectively regular for all* $k \in \mathbb{N}$*.*

The trifecta Kleene–Myhill-Nerode–Büchi-Elgot-Trakhtenbrot is now complete for HDAs.

- [CONCUR 2022]–[Petri Nets 2023]–[DLT 2024]

Further work:

- First-order logic for HDAs $\longleftarrow$ Internship Enzo Erlich
- HDAs over infinite ipomsets $\longleftarrow$ Internship Luc Passemard
- Branching-time logics for HDAs $\longleftarrow$ PhD Safa Zouari