

Languages of Higher-Dimensional Automata

Uli Fahrenberg

EPITA Research Laboratory (LRE), Paris, France

Highlights 2022



Developments in Higher-Dimensional Automata Theory

Uli Fahrenberg

EPITA Research Laboratory (LRE), Paris, France

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Languages of higher-dimensional automata

- Languages of Higher-Dimensional Automata [MSCS 2021]
- A Kleene Theorem for Higher-Dimensional Automata [CONCUR 2022]
- A Myhill-Nerode Theorem for Higher-Dimensional Automata [Petri Nets 2023]
- Decision and Closure Properties for Higher-Dimensional Automata [ICTAC 2023]
- Logic and Languages of Higher-Dimensional Automata [DLT 2024]

Today:

- ① What are HDAs (and why should I be interested)?
- ② What are languages of HDAs (and why should I be interested)?
- ③ What can I do with HDAs (that I cannot do with other models)?

Nice people

- Eric Goubault, Paris
- Christian Johansen, Oslo
- Georg Struth, Sheffield
- Krzysztof Ziemiański, Warsaw

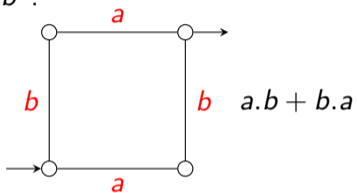
- Amazigh Amrane, Hugo Bazille, Emily Clement, Jérémy Dubut, Marie Fortin, Roman Kniazev, Jérémy Ledent, Safa Zouari, ...

- See also <https://ulifahrenberg.github.io/pomsetproject/>

Higher-dimensional automata

Filled-in diamonds are concurrency's best friends

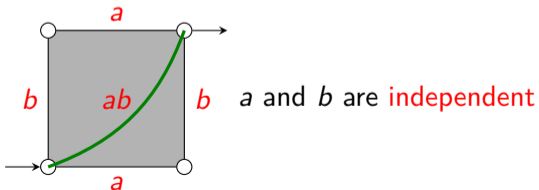
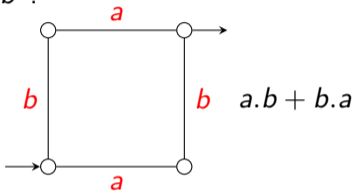
semantics of “ a parallel b ”:



Higher-dimensional automata

Filled-in diamonds are concurrency's best friends

semantics of “ a parallel b ”:



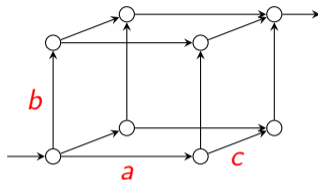
Higher-dimensional automata & concurrency

HDAs as a model for **concurrency**:

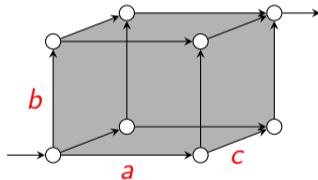
- points: **states**
- edges: **transitions**
- squares, cubes etc.: **independency** relations (concurrently executing events)
- **two**-dimensional automata \cong asynchronous transition systems [[Bednarczyk](#)]

[[van Glabbeek 2006, TCS](#)]: Up to history-preserving bisimilarity, HDAs “generalize the main models of concurrency proposed in the literature” (notably, event structures and Petri nets)

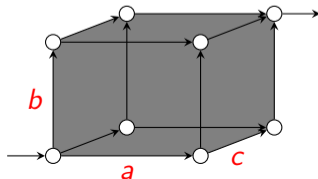
Examples



no concurrency



two out of three



full concurrency

Higher-dimensional automata

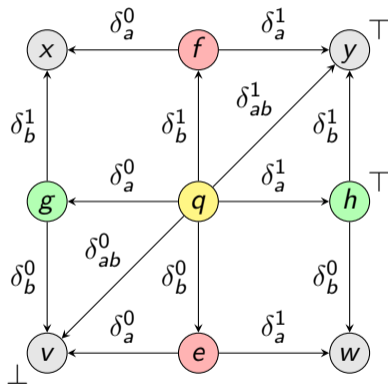
A **conclist** is a finite, ordered and Σ -labelled set. (a list of events)

A **precubical set** X consists of:

- A set of cells X (cubes)
- Every cell $x \in X$ has a conclist $\text{ev}(x)$ (list of events active in x)
- We write $X[U] = \{x \in X \mid \text{ev}(x) = U\}$ for a conclist U (cells of type U)
- For every conclist U and $A \subseteq U$ there are:
 - upper face map $\delta_A^1 : X[U] \rightarrow X[U \setminus A]$ (terminating events A)
 - lower face map $\delta_A^0 : X[U] \rightarrow X[U \setminus A]$ (unstarting events A)
- **Precube identities:** $\delta_A^\mu \delta_B^\nu = \delta_B^\nu \delta_A^\mu$ for $A \cap B = \emptyset$ and $\mu, \nu \in \{0, 1\}$

A **higher dimensional automaton (HDA)** is a precubical set X with **start cells** $\perp \subseteq X$ and **accept cells** $\top \subseteq X$ (not necessarily vertices)

Example



$$X[\emptyset] = \{v, w, x, y\}$$

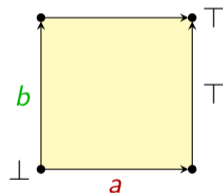
$$X[a] = \{e, f\}$$

$$X[b] = \{g, h\}$$

$$X[ab] = \{q\}$$

$$\perp_X = \{v\}$$

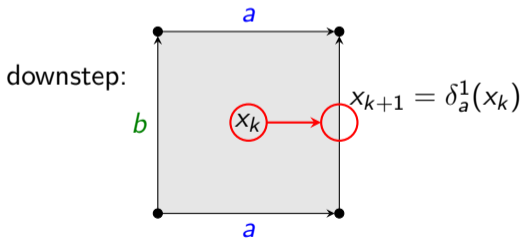
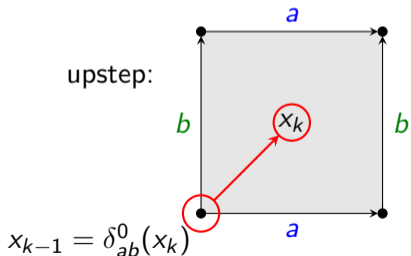
$$\top_X = \{h, y\}$$



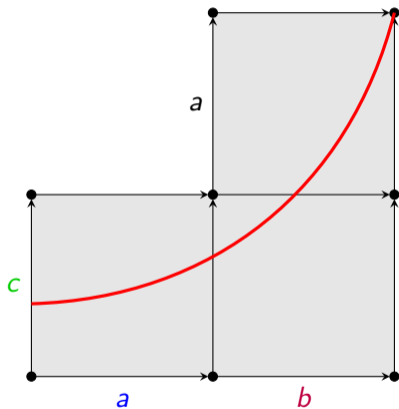
Computations of HDAs

A **path** on an HDA X is a sequence $(x_0, \varphi_1, x_1, \dots, x_{n-1}, \varphi_n, x_n)$ such that for every k , $(x_{k-1}, \varphi_k, x_k)$ is either

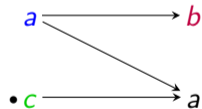
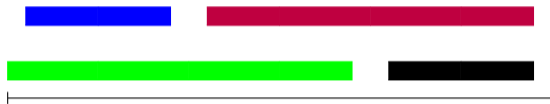
- $(\delta_A^0(x_k), \nearrow^A, x_k)$ for $A \subseteq \text{ev}(x_k)$ or (upstep: start A)
- $(x_{k-1}, \searrow_B, \delta_B^1(x_{k-1}))$ for $B \subseteq \text{ev}(x_{k-1})$ (downstep: terminate B)



Event ipomset of a path



Lifetimes of events



an interval order!

Languages of HDAs

Definition

The **language** of an HDA X is the set of event ipomsets of all accepting paths:

$$L(X) = \{\text{ev}(\pi) \mid \pi \in \text{Paths}(X), \text{src}(\pi) \in \perp_X, \text{tgt}(\pi) \in \top_X\}$$

- $L(X)$ contains only **interval orders**,
- is **closed under order extension**,
- and has **finite width**

Definition

A language $L \subseteq \text{iiPoms}$ is **regular** if there is an HDA X with $L = L(X)$.

Theorems

Definition (Rational Languages over Σ)

- Generated by \emptyset , $\{\epsilon\}$, and all $\{[a]\}$, $\{[\bullet a]\}$, $\{[a\bullet]\}$, $\{[\bullet a\bullet]\}$ for $a \in \Sigma$
- under operations \cup , $*$, \parallel and (Kleene plus) $^+$

Definition (Monadic Second-Order Logics over Ipomsets)

$$\psi ::= a(x) \mid s(x) \mid t(x) \mid x < y \mid x \dashrightarrow y \mid x \in X \mid$$

$$\exists x. \psi \mid \forall x. \psi \mid \exists X. \psi \mid \forall X. \psi \mid \psi_1 \wedge \psi_2 \mid \psi_1 \vee \psi_2 \mid \neg \psi$$

Theorem (à la Kleene [CONCUR 2022])

A language is *rational* iff it is *regular*.

Theorem (à la Büchi-Elgot-Trakhtenbrot [DLT 2024])

A language is *rational* iff it is *MSO-definable*, of finite width, and subsumption-closed.

More theorems

Theorem (à la Myhill-Nerode [Petri Nets 2023])

A language is *rational* iff it has finite *prefix quotient*.

Theorem (Closure properties [ICTAC 2023])

Rational languages are *closed* under *intersection* but *not* under *complement*.

Theorem (Determinizability & ambiguity [Petri Nets 2023])

Not all HDAs are *determinizable*.

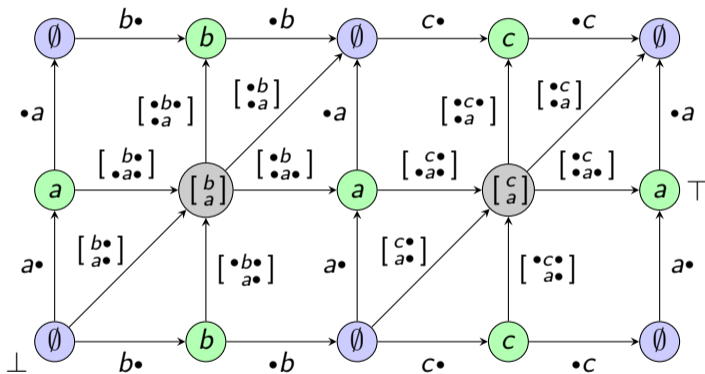
There is a rational language which is *inherently infinitely ambiguous*.

Theorem (Decidability [ICTAC 2023])

Inclusion of rational languages is *decidable*.

Important tool: ST-automata

Operational semantics of HDAs



- Takes care of half of all theorems: regular \Rightarrow rational; MSO-definable \Rightarrow regular; regular \Rightarrow finite prefix quotient; decidability of inclusion

Conclusion

- An automaton-like model for non-interleaving concurrency
- With a **nice** language theory!
- The trifecta Kleene–Myhill–Nerode–Büchi–Elgot–Trakhtenbrot is now complete for HDAs
[CONCUR 2022]–[Petri Nets 2023]–[DLT 2024]

Ongoing & Future work:

- **First-order** logic for HDAs PhD Enzo Erlich
- **Branching-time** logics for HDAs PhD Safa Zouari; [ICTAC 2024]
- Higher-dimensional **timed** automata [Petri Nets 2024]
- HDAs over **infinite** ipomsets Luc Passemard

See poster

... et l'EPITA Toulouse recrute !