Directed Topology and Concurrency

A Personal View

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Nice People

Jérémy Dubut, Lisbeth Fajstrup, Philippe Gaucher, Eric Goubault, Emmanuel Haucourt, Christian Johansen, Jérémy Ledent, Samuel Mimram, Sergio Rajsbaum, Martin Raussen, Georg Struth, Rob van Glabbeek, Krzysztof Ziemiański

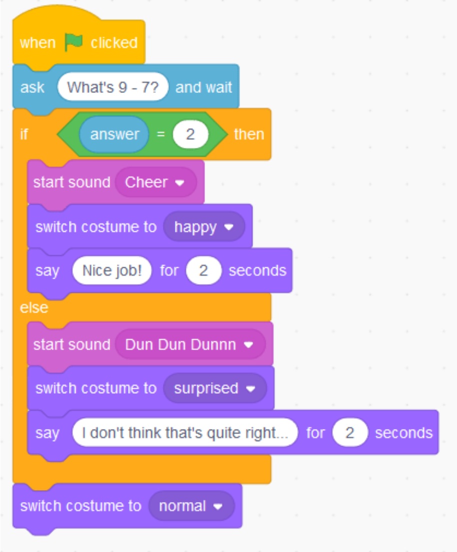
Algebraic View

A program is a sequence of instructions

- plus branches and loops

Kleene algebra:

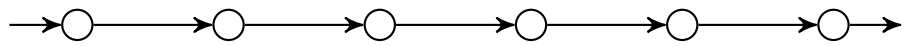
- set S with operations:
- concatenation $\otimes$
- choice $\oplus$
- repetition *
- idempotent semiring with unary * which computes fixed points
- (Kleene algebra with domain for conditional branches)



Geometric View

A program is a sequence of instructions

- ignoring branches and loops for now



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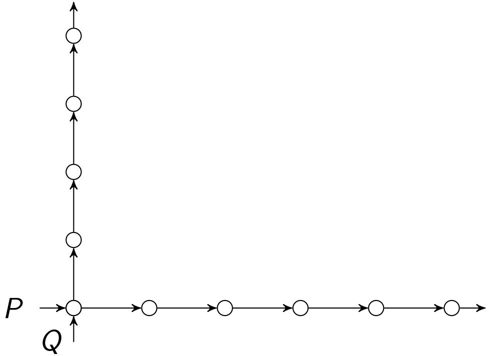

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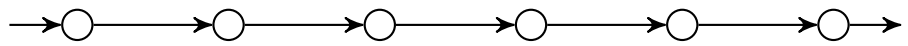

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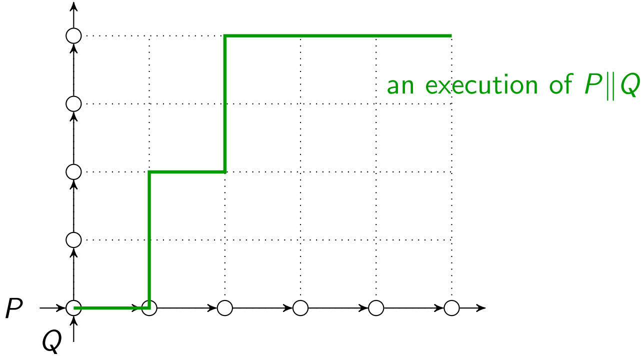

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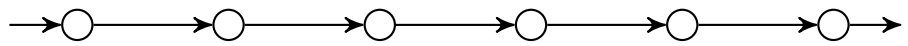

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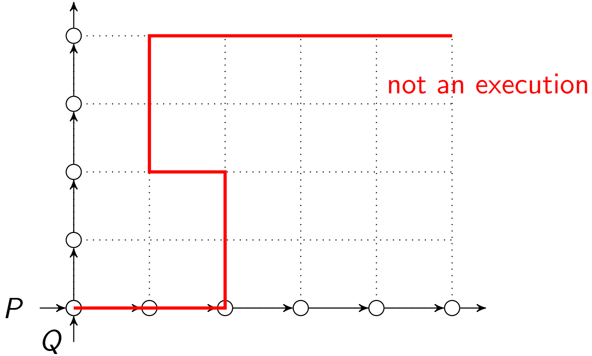

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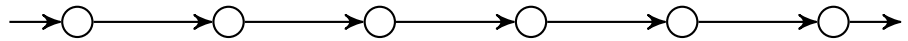

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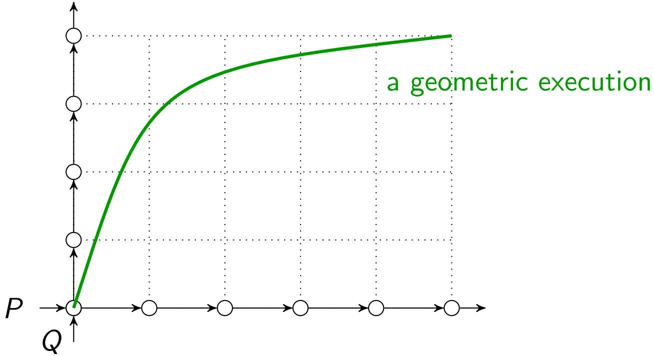

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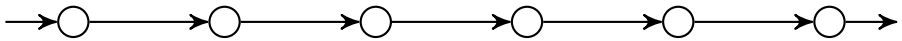

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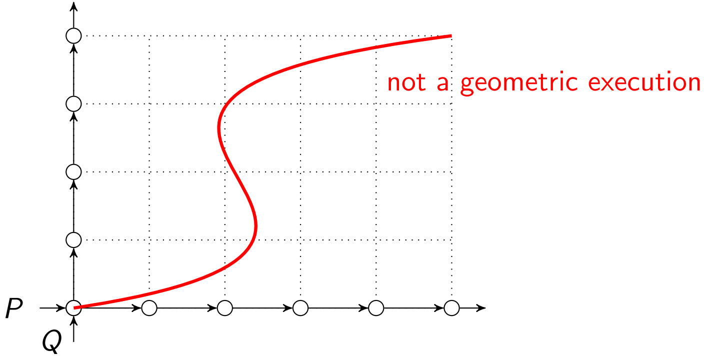
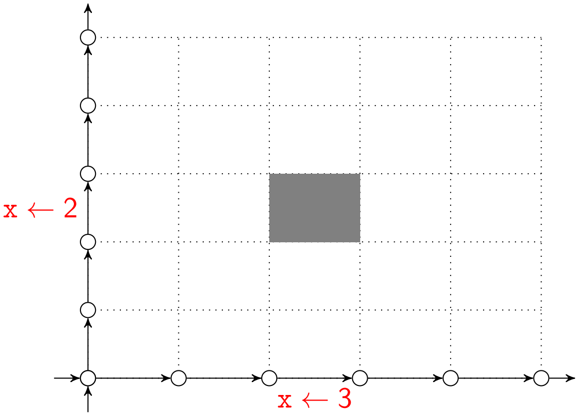


Holes

Adding mutual exclusion:



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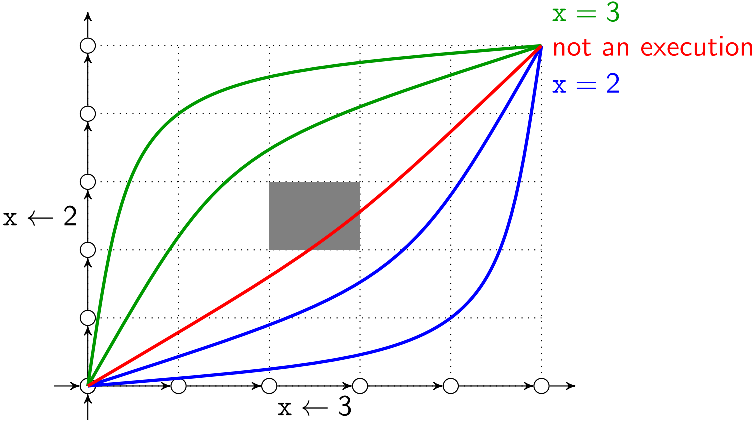
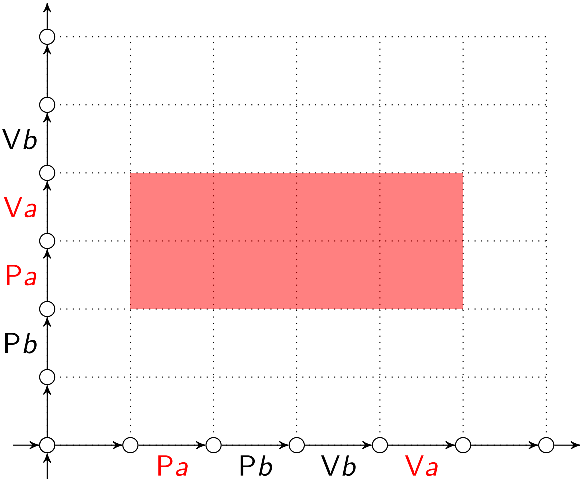


- homotopic paths ≙ equivalent executions

More Holes

Semaphores à la Dijkstra (P $\hat{=}$ acquire; V $\hat{=}$ release):



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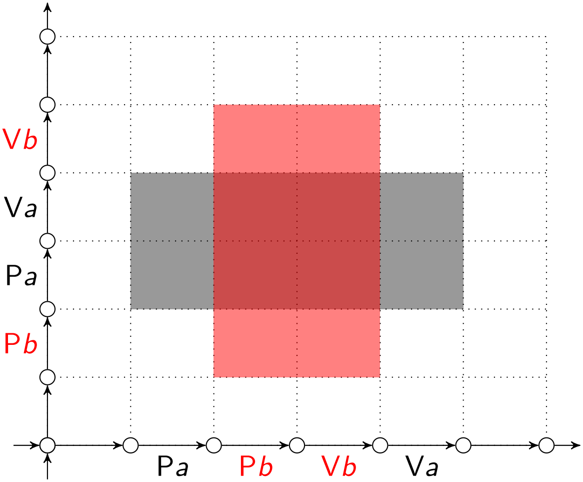

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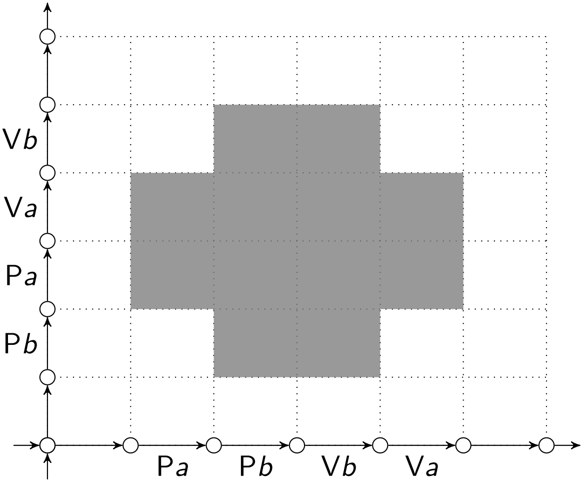

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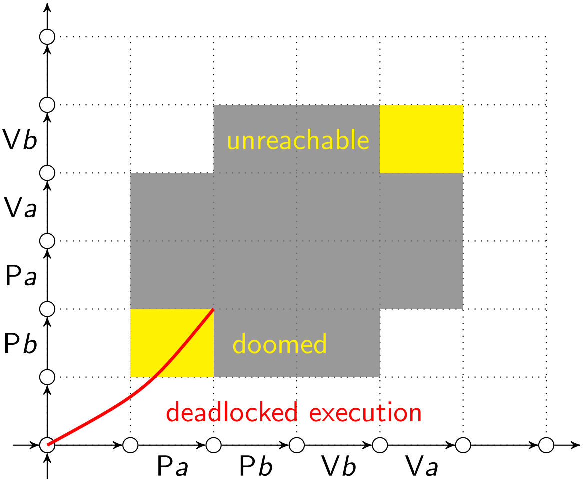
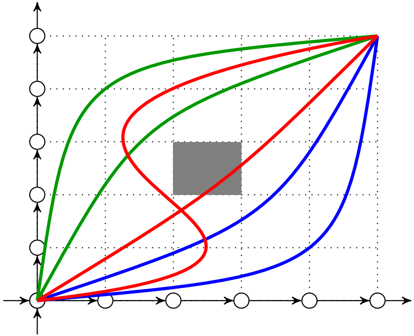
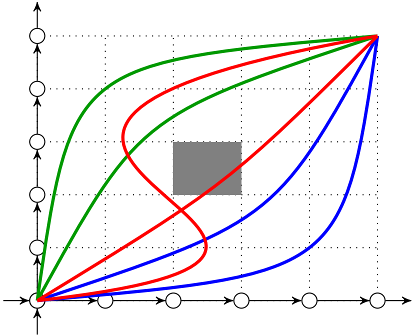


Summing Up



- A program is a topological space
- An execution is a path through said space
- Two executions are equivalent iff their paths are homotopic

Summing Up



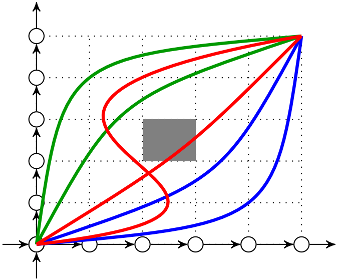
- A program is a directed topological space
- An execution is a directed path through said space
- Two executions are equivalent iff their dipaths are dihomotopic

Directed Spaces

Definition (po-space)

A partially ordered space is a topological space X together with a partial order $\leq$ on X such that $\leq \subseteq X \times X$ is *closed* in the product topology.
A morphism of po-spaces is a $\leq$-preserving continuous function.

- directed intervals; directed squares, cubes, etc.
- concatenation $\otimes$, branching $\oplus$
- no loops

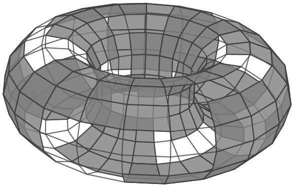


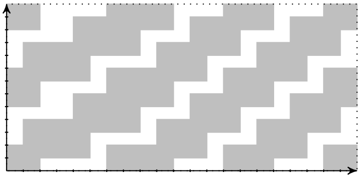
Directed Spaces

Definition (lpo-space)

A locally partially ordered space is a *Hausdorff* topological space X together with a relation $\leq$ on X in which any $x \in X$ has an open neighborhood $U \ni x$ such that the restriction of $\leq$ to U is a closed partial order.

A morphism of po-spaces is a continuous function which is *locally* $\leq$-preserving.





Directed Spaces

Definition (d-space)

A directed space is a topological space X together with a set $\vec{P}X$ of directed paths $I \to X$ such that

- all constant paths are directed,
- concatenations of directed paths are directed, and
- reparametrizations and restrictions of directed paths are directed.

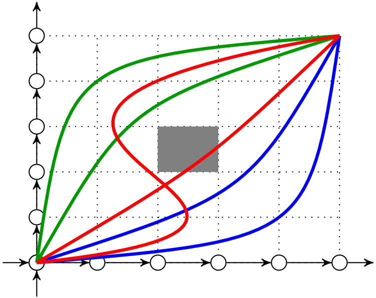
A morphism of d-spaces is a continuous function which preserves directed paths.

- po-spaces $\hookrightarrow$ lpo-spaces $\hookrightarrow$ d-spaces (not full)
- po-spaces are *loop-free*; lpo-spaces are *vortex-free*
- d-spaces are nice: axiomatize directly our objects of interest (dipaths); have good categorical properties

Directed Paths and Homotopies

- the directed interval $\vec{I}$:
  - $([0,1], \leq)$ (usual order): po-space; lpo-space
  - $([0,1], \vec{P}[0,1])$: all (weakly) increasing paths
- dipaths in X: morphisms $\vec{I} \to X$
  - for d-space $(X, \vec{P}X)$: dipaths $\hat{=}$ $\vec{P}X$
- a dihomotopy $H : I \times \vec{I} \to X$:
  - all $H(s, \cdot)$ dipaths
  - $H : I \times I \to X$ continuous
  - $H(\cdot, 0)$ and $H(\cdot, 1)$ constrained
  - (some variants exist)

Summing Up, Again

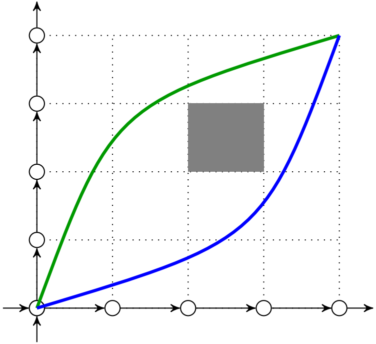


- A program is a directed topological space
  - po-space, lpo-space, d-space
  - (other models exist)
- An execution is a directed path through said space
- Two executions are equivalent iff their dipaths are dihomotopic

Combinatorial Model

Transition Systems?

"Programs are topological spaces"?!?



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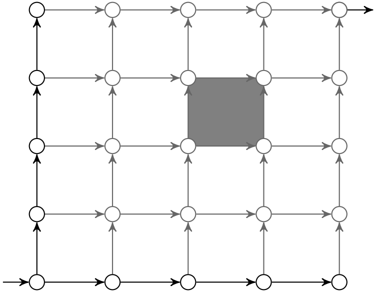
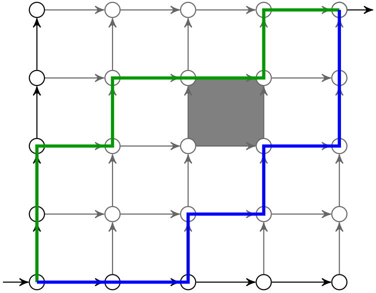


Transition Systems?

"Programs are topological spaces"?!?

Programs are transition systems!

- have lost info on "forbidden squares"



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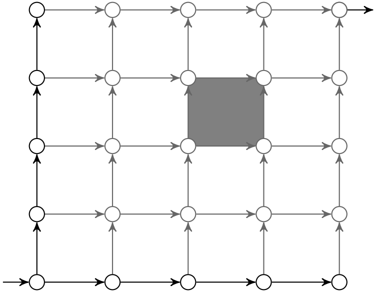
Higher-dimensional automata:

- transition systems
- plus info on concurrency



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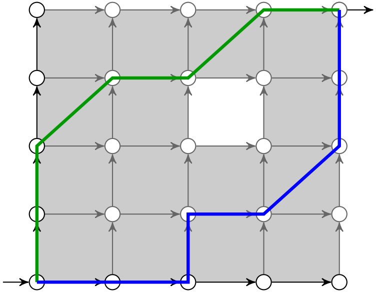
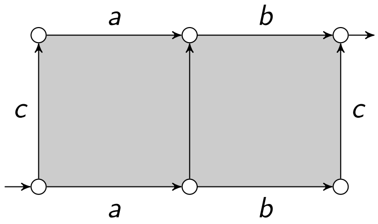
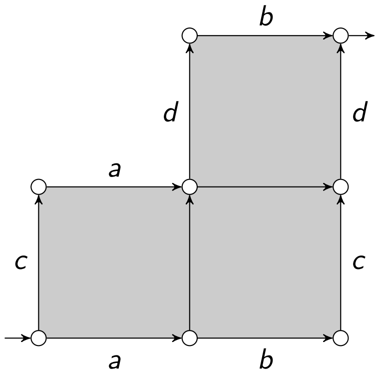


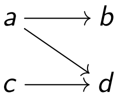
Examples



first a, then b; all in parallel with c: $ab \parallel c$

Examples





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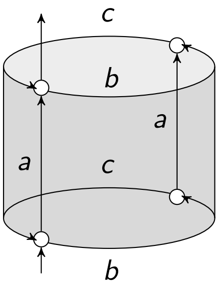

$a \parallel (bc)^*$

Geometric Realization

Definition

The geometric realization of a precubical set X is the d-space $|X| = \bigsqcup_{n\geq 0} X_n \times \vec{I}^n /_\sim$, where $\sim$ is the equivalence generated by $(\delta_i^\nu x, (t_1, \dots, t_{n-1})) \sim (x, (t_1, \dots, t_{i-1}, \nu, t_{i+1}, \dots, t_{n-1}))$.

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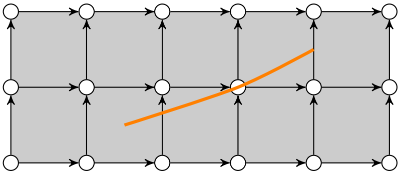
- (No, we haven't properly introduced precubical sets (and HDAs, for that sake).)
- (Please wait.)

Dipaths in Geometric Realizations

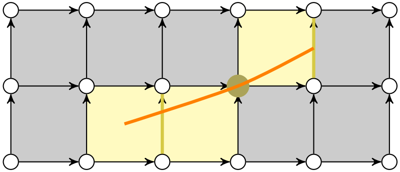


Let $p : \vec{I} \to |X|$ be a dipath in the geometric realization of precubical set X.

- let $C_p = \{x \in X \mid \text{im}(p) \cap |x| \neq \emptyset\}$ – all cells touched by p
- organize C_p into a sequence $c_p = (x_1, \dots, x_m)$ s.t. $\forall i$:

$$x_i = \delta_+^0 x_{i+1} \qquad \text{or} \qquad x_{i+1} = \delta_+^1 x_i \quad \text{(iterated face maps)}$$

Dipaths in Geometric Realizations



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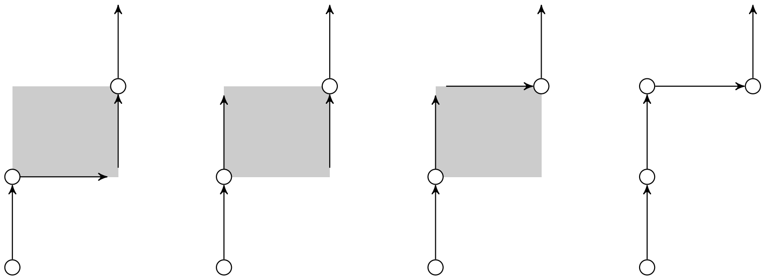
$\Rightarrow$ the combinatorial path of p

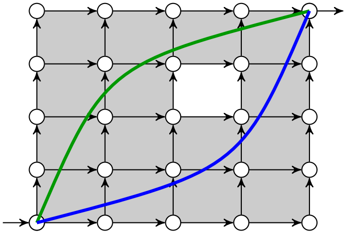
- any combinatorial path c gives rise to dipath p_c (non-unique) with $c_{p_c} = c$
- if $c_p = c_q$, then p and q are *dihomotopic*

Combinatorial Homotopy



- generated by local replacements
- dipaths p, q are dihomotopic iff c_p and c_q are homotopic
- combinatorial paths c, d are homotopic iff p_c and p_d are dihomotopic

Summing Up



- precubical sets / higher-dimensional automata: combinatorial models of directed spaces
- natural extension of transition systems
- closely linked to directed spaces via geometric realization:
  - dipaths $\hat{=}$ combinatorial paths $\hat{=}$ executions
  - dihomotopy $\hat{=}$ combinatorial homotopy $\hat{=}$ equivalence of executions

Languages of Higher-Dimensional Automata

More Nice People

Jérémy Dubut, Eric Goubault, Christian Johansen, Jérémy Ledent, Sergio Rajsbaum, Georg Struth, Krzysztof Ziemiański

Samy Abbes, Amazigh Amrane, Hugo Bazille, Emily Clement, Thomas Colcombet, Marie Fortin, Ryszard Janicki, Roman Kniazev, Łukasz Mikulski, Safa Zouari

Precubical sets and higher dimensional automata

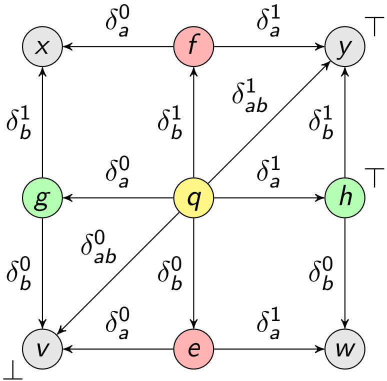
A conclist is a finite, ordered and Σ-labelled set. (a list of concurrent events)

A precubical set X consists of:

- A set of cells X
- Every cell $x \in X$ has a conclist $\text{ev}(x)$ (list of events active in x)
- We write $X[U] = \{x \in X \mid \text{ev}(x) = U\}$ for a conclist U (cells of type U)
- For every conclist U and $A \subseteq U$ there are:
  upper face map $\delta_A^1 : X[U] \to X[U - A]$ (terminating events A)
  lower face map $\delta_A^0 : X[U] \to X[U - A]$ ("unstarting" events A)
- Precubical identities: $\delta_A^\mu \delta_B^\nu = \delta_B^\nu \delta_A^\mu$ for $A \cap B = \emptyset$ and $\mu, \nu \in \{0, 1\}$

A higher dimensional automaton (HDA) is a finite precubical set X with start cells $\bot \subseteq X$ and accept cells $\top \subseteq X$ (not necessarily vertices)

Example



$X[\emptyset] = \{v, w, x, y\}$

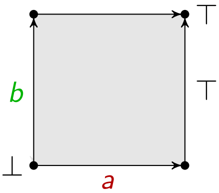
$X[a] = \{e, f\}$

$X[b] = \{g, h\}$

$X[ab] = \{q\}$

$\bot_X = \{v\}$

$\top_X = \{h, y\}$



Precubical sets as presheaves

A presheaf over a category $\mathcal{C}$ is a functor $\mathcal{C}^{\text{op}} \to \text{Set}$ (contravariant functor on $\mathcal{C}$)

The precube category $\square$ has (iso classes of) conclists as objects.
Morphisms are coface maps $d_{A,B} : U \to V$, where

- $A, B \subseteq V$ are disjoint subsets,
- $U \simeq V - (A \cup B)$ are isomorphic conclists,
- $d_{A,B} : U \to V$ is the unique label preserving monotonic map with image $V - (A \cup B)$.

Composition of coface maps $d_{A,B} : U \to V$ and $d_{C,D} : V \to W$ is

$$d_{\partial(A)\cup C,\partial(B)\cup D} : U \to W,$$

where $\partial : V \to W - (C \cup D)$ is the unique conclist isomorphism.

Intuitively, $d_{A,B}$ terminates events B and "unstarts" events A.

- precubical sets: presheaves over $\square$

Context

augmented presimplex category Δ	large augmented presimplex category $\blacktriangle$
objects $\{1 < \dots < n\}$ for $n \ge 0$	objects totally ordered sets
morphisms order injections	morphisms order injections
skeletal	isos are unique

$\Delta \hookrightarrow \blacktriangle$ equivalence with unique left inverse

(augmented) precube category $\square$	large (augmented) precube category $\boxdot$
objects $\{0, 1\}^n$ for $n \ge 0$	objects totally ordered sets
morphisms 0-1 injections	morphisms distinguished order injections
skeletal	isos are unique

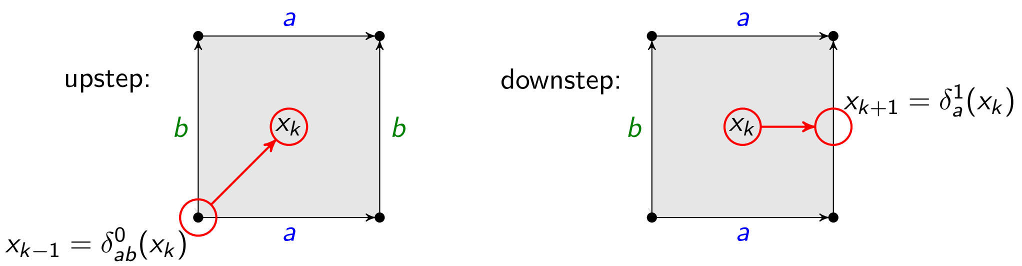
$\square \hookrightarrow \boxdot$ equivalence with unique left inverse

- presimplicial sets: $\mathsf{Set}^{\Delta^{op}}$ or $\mathsf{Set}^{\blacktriangle^{op}}$; makes no difference
- precubical sets: $\mathsf{Set}^{\square^{op}}$ or $\mathsf{Set}^{\boxdot^{op}}$; makes no difference

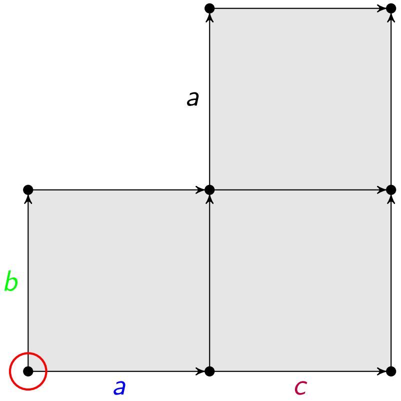
Computations of HDAs

A path on an HDA X is a sequence $(x_0, \phi_1, x_1, \dots, x_{n-1}, \phi_n, x_n)$ such that for every k, (x_{k-1}, ϕ_k, x_k) is either

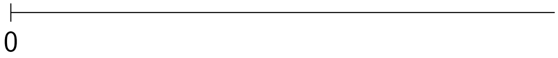
- $(\delta^0_A(x_k), \nearrow^A, x_k)$ for $A \subseteq \mathrm{ev}(x_k)$ or (upstep: start A)
- $(x_{k-1}, \searrow_B, \delta^1_B(x_{k-1}))$ for $B \subseteq \mathrm{ev}(x_{k-1})$ (downstep: terminate B)



Event ipomset of a path



Lifetimes of events



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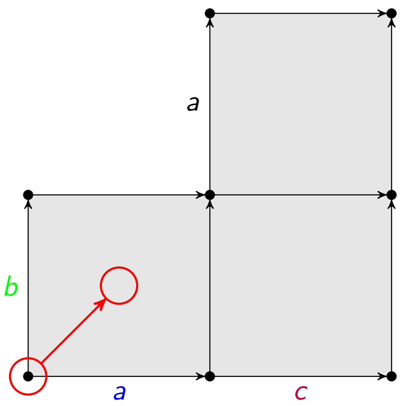


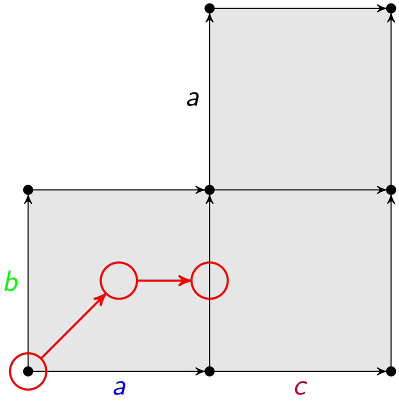
Lifetimes of events

$a\bullet$

$*$

$b\bullet$

Event ipomset of a path



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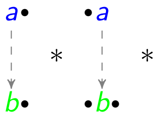
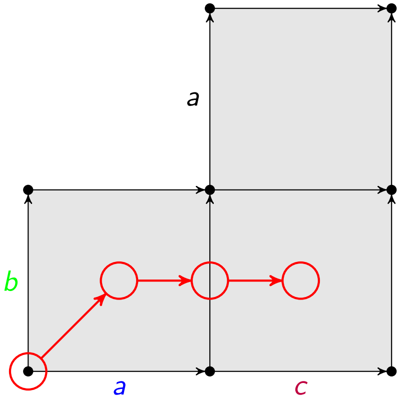
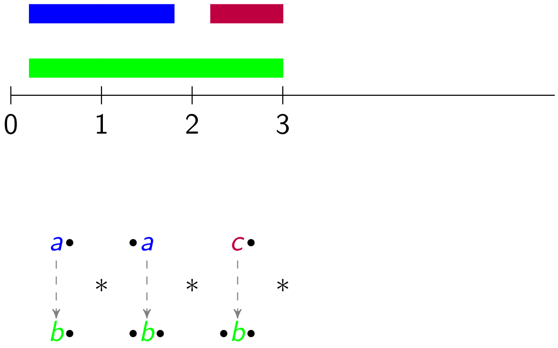


Event ipomset of a path



Lifetimes of events



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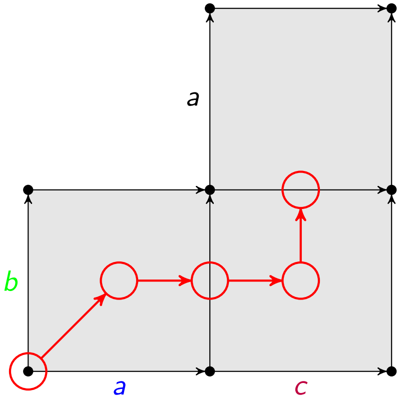
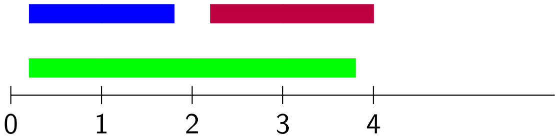
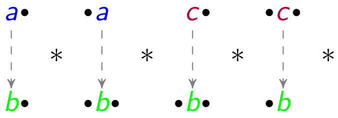


Lifetimes of events





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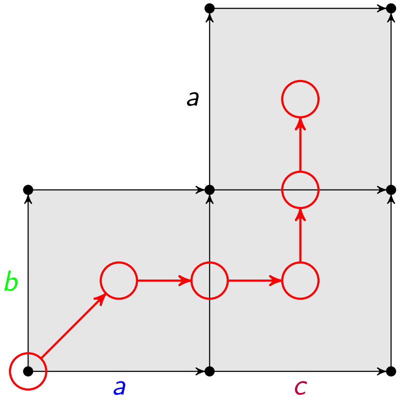
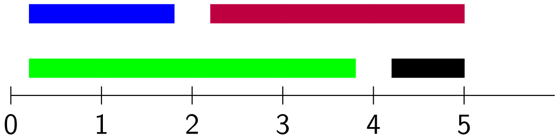
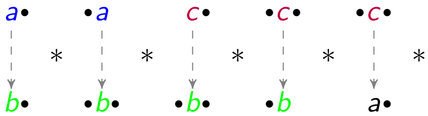


Lifetimes of events





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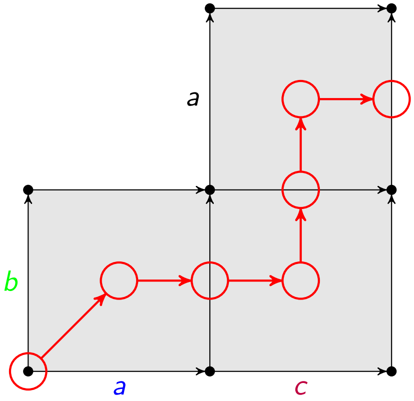
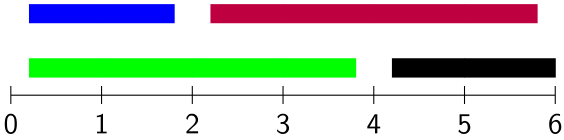
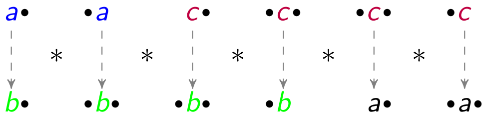
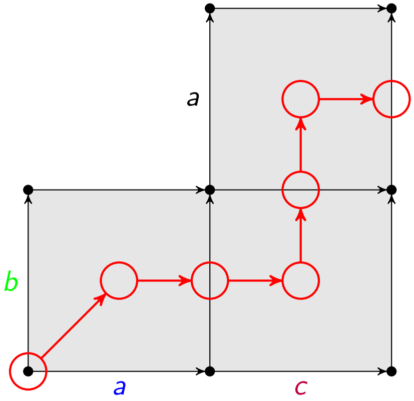


Lifetimes of events

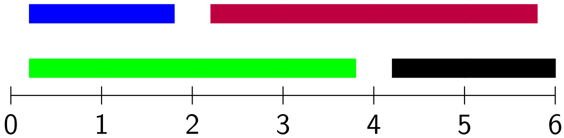




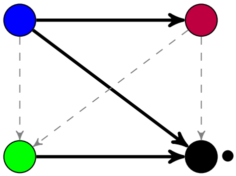
Event ipomset of a path



Lifetimes of events



Event ipomset

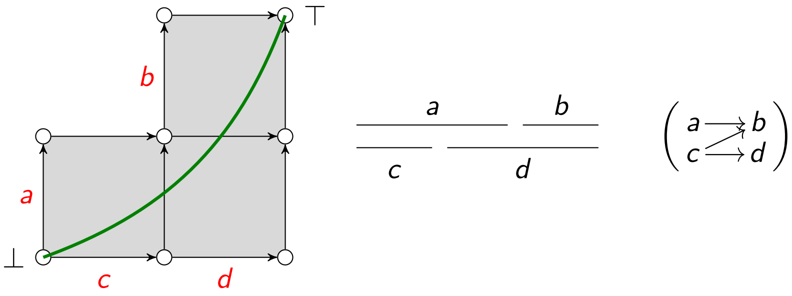


(not series-parallel!)

Are all pomsets generated by HDAs?

No, only (labeled) interval orders

- Poset $(P, \leq)$ is an interval order iff it has an interval representation:
  - a set $I = \{[l_i, r_i]\}$ of real intervals
  - with order $[l_i, r_i] \preceq [l_j, r_j]$ iff $r_i \leq l_j$
  - and an order isomorphism $(P, \leq) \leftrightarrow (I, \preceq)$
- [Fishburn 1970]



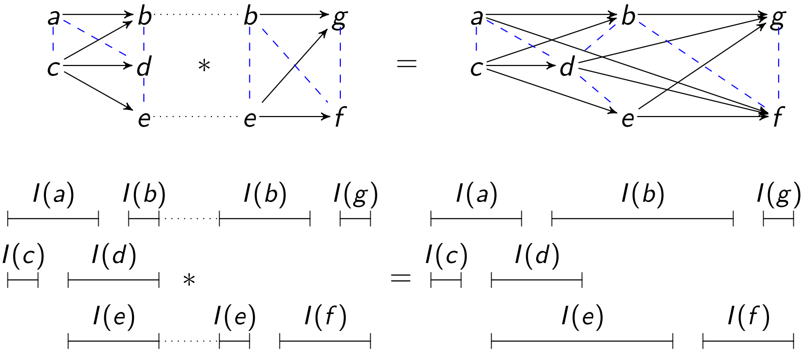
Pomsets with interfaces

Definition (Ipomset)

A pomset with interfaces (and event order): $(P, <, \dashrightarrow, S, T, \lambda)$:

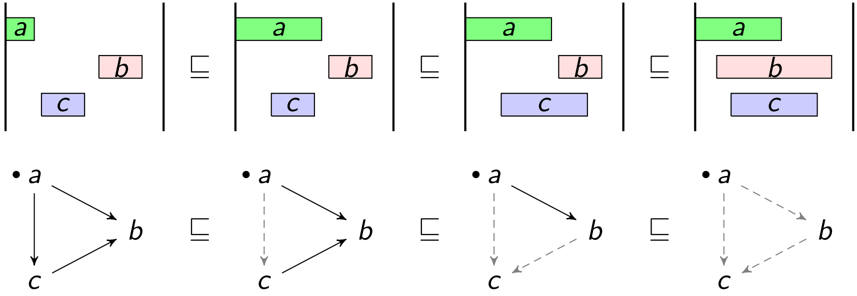
- finite set P;
- two partial orders $<$ (precedence order), $\dashrightarrow$ (event order)
  - s.t. $< \cup \dashrightarrow$ is a *total relation*;
- $S, T \subseteq P$ source and target interfaces
  - s.t. S is $<$-minimal, T is $<$-maximal.

Composition of ipomsets



- Gluing $P * Q$: P before Q, except for interfaces (which are identified)
- Parallel composition $P \parallel Q$: P above Q (disjoint union)

Subsumption



P refines Q / Q subsumes P / $P \sqsubseteq Q$ iff

- P and Q have same interfaces
- P has more $<$ than Q
- Q has more $\dashrightarrow$ than P

Languages of HDAs

Definition

The language of an HDA X is the set of event ipomsets of all accepting paths:

$$L(X) = \{\mathrm{ev}(\pi) \mid \pi \in \mathrm{Paths}(X), src(\pi) \in \bot_X, tgt(\pi) \in \top_X\}$$

- $L(X)$ contains only interval-order ipomsets
- and is closed under subsumption

Path objects

Important tool:

Proposition

For any interval-order ipomset P there exists an HDA $\square^P$ for which $L(\square^P) = \{P\}\downarrow$.

Lemma

For any HDA X and ipomset P, $P \in L(X)$ iff $\exists f : \square^P \to X$.

Path objects

Important tool:

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For any interval-order ipomset P there exists an HDA $\square^P$ for which $L(\square^P) = \{P\}{\downarrow}$.

Lemma

For any HDA X and ipomset P, $P \in L(X)$ iff $\exists f : \square^P \to X$.

Theorems

Definition (Rational Languages over Σ)

- Generated by $\emptyset$, $\{\epsilon\}$, and all $\{[a]\}$, $\{[\bullet a]\}$, $\{[a\bullet]\}$, $\{[\bullet a\bullet]\}$ for $a \in \Sigma$
- under operations $\cup$, $*$, $\|$ and (Kleene plus) $^+$

- $L^+ = \bigcup_{n\geq 1} L^n$
- no Kleene star; no parallel star

Theorem (à la Kleene)

A language is rational iff it is recognized by an HDA. CONCUR'22

Theorem (à la Myhill-Nerode)

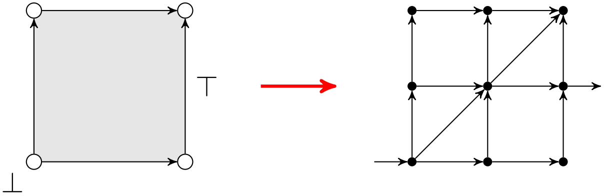
A language is rational iff it has finite prefix quotient. Petri Nets'23

Theorem (à la Büchi-Elgot-Trakhtenbrot)

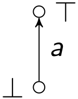
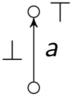
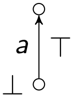
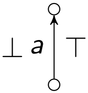
A language is rational iff it is MSO-definable. arxiv'24

Kleene theorem: easy parts

- regular $\Rightarrow$ rational: by reduction to ST-automata



- rational $\Rightarrow$ regular: generators:

$L(X)$	$\emptyset$	$\{\epsilon\}$	$\{[a]\}$	$\{[\bullet\, a]\}$	$\{[a\, \bullet]\}$	$\{[\bullet\, a\, \bullet]\}$
X	$\emptyset$	$\bot \circ \top$				

- rational $\Rightarrow$ regular: $\cup$ and $\|$

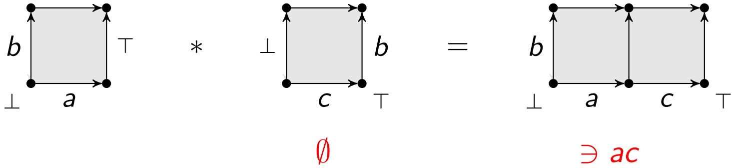
$$L(X) \cup L(Y) = L(X \sqcup Y) \qquad L(X) \parallel L(Y) = L(X \otimes Y)$$

Kleene theorem: difficult parts

- miss to see: gluings and iterations of regular languages are regular:

$$L(X) * L(Y) = L(X * Y) \qquad\qquad L(X)^+ = L(X^+)$$

- much more difficult: higher-dimensional gluings identify too much
- for example:



Kleene theorem: difficult parts

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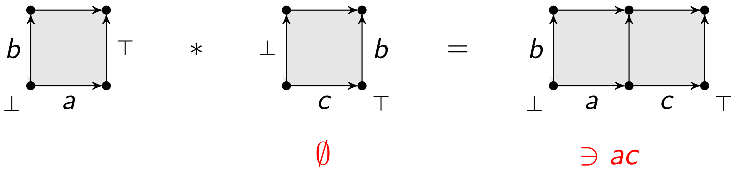


- use HDAs with interfaces and cylinder objects

HDAs with interfaces

A conclist with interfaces (iconclist) is a conclist U with subsets $S \subseteq U \supseteq T$ (notation: ${}_S U_T$).
(events in T cannot be terminated; events in S cannot be "unstarted")

A precubical set with interfaces (ipc-set) X consists of a set of cells X such that:

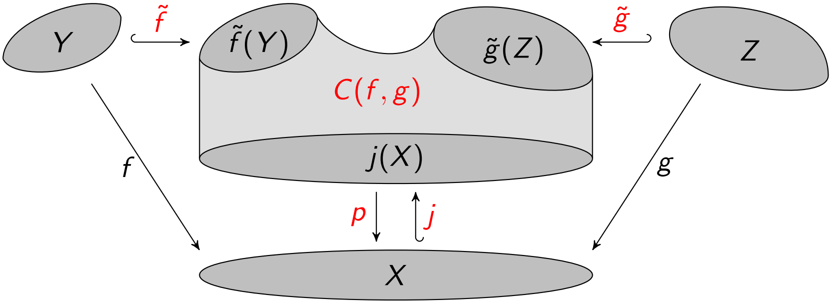
- Every cell $x \in X$ has an iconclist $\text{ev}(x)$
- We write $X[{}_S U_T] = \{x \in X \mid \text{ev}(x) = {}_S U_T\}$.
- For every $A \subseteq U - S$ there is a lower face map $\delta_A^0 : X[U] \to X[{}_S U_T - A]$.
- For every $B \subseteq U - T$ there is an upper face map $\delta_B^1 : X[U] \to X[{}_S U_T - b]$.
- Precubical identities: $\delta_A^\mu \delta_B^\nu = \delta_B^\nu \delta_A^\mu$ for $A \cap B = \emptyset$ and $\mu, \nu \in \{0, 1\}$
- presheaves over $\mathrm{I}\square$

An HDA with interfaces (iHDA) is a finite ipc-set with start and accept cells.

Extra conditions:
If $x \in X[{}_S U_T]$ is a start cell, then $S = U$.
If $x \in X[{}_S U_T]$ is an accept cell, then $T = U$.

Cylinders

Let X, Y, Z be ipc-sets and $f : Y \to X$, $g : Z \to X$ ipc-maps such that $f(Y) \cap g(Z) = \emptyset$. There is a diagram of ipc-sets



such that

- $\tilde{f}$ is an initial inclusion;
- $\tilde{g}$ is a final inclusion;
- all paths in X from $f(Y)$ to $g(Z)$ lift to paths in $C(f, g)$.

Cylinders: construction

X, Y, Z: ipc-sets, $f : Y \to X$, $g : Z \to X$: ipc-maps with $f(Y) \cap g(Z) = \emptyset$.
For ${}_S U_T \in \mathrm{I}\square$ let

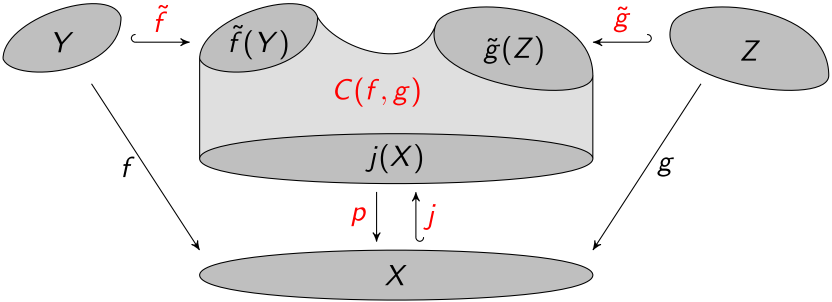
$$C(f, g)[{}_S U_T] = \{(x, K, L, \phi, \psi)\}$$

such that

- $x \in X[{}_S U_T]$;
- $K \subseteq \mathrm{I}\square^U$ is an initial subset;
- $L \subseteq \mathrm{I}\square^U$ is a final subset;
- $\phi : K \to Y$, $\psi : L \to Z$ are ipc-maps satisfying $f \circ \phi = \iota_x|_K$ and $g \circ \psi = \iota_x|_L$:

$$\begin{array}{ccccc} K & \hookrightarrow & \mathrm{I}\square^U & \hookleftarrow & L \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\iota_x} & & \downarrow{\scriptstyle\psi} \\ Y & \xrightarrow{f} & X & \xleftarrow{g} & Z \end{array}$$

Cylinders?!



- a factorization system?
- directed model categories?
- (note: no homotopy has been used)

Conclusion

- Programs are directed topological spaces
  - po-spaces $\hookrightarrow$ lpo-spaces $\hookrightarrow$ d-spaces
  - (lpo-spaces $\to$ po-spaces: delooping / universal dicover)
  - executions are dipaths; equivalence of executions is dihomotopy
  - $\rightsquigarrow$ dihomotopy invariants; dihomology; homotopy vs reversibility; etc.
- Programs are precubical sets
  - higher-dimensional automata
  - executions are combinatorial paths; equivalence of such is combinatorial homotopy
  - strong link to spaces via geometric realization
- Language theory of higher-dimensional automata
  - languages are sets of interval pomsets with interfaces
  - partial order semantics, trace theory etc.
  - Kleene, Myhill-Nerode, Büchi-Elgot-Trakhtenbrot ✓
  - $\rightsquigarrow$ timed HDAs; hybrid HDAs; ω-HDAs; weighted HDAs; active learning; etc.
  - no homotopy has been used!?

Contents

Program Committee

Henning Basold
Uli Fahrenberg
Christian Johansen
Tobias Kappé
Sergio Rajsbaum
Georg Struth
Rob van Glabbeek
Krzysztof Ziemiański

Organizing Committee

Amazigh Amrane
Uli Fahrenberg
Loïc Hélouët

Workshop on Pomsets and Related Structures RaPS

The Workshop on Pomsets and Related Structures will take place at EPITA Rennes on **24 April 2024**.

The workshop is associated with the (i)Po(m)set Project, a research project at the crossroads of concurrency theory, automata theory, algebra, and geometry. The project also has an online seminar, and the workshop is an offshoot of that seminar.

The RaPS workshop is collocated with ATLAS'24 and will be followed by the Journées GT DAAL.

Venue

The workshop will take place at EPITA Rennes in downtown Rennes, close to the train station.

Participants

Preliminary list of participants (updated 20 March):

- Samy Abbes, Université Paris Cité, France
- Amazigh Amrane, EPITA Paris, France
- Henning Basold, Universiteit Leiden, The Netherlands
- Hugo Bazille, EPITA Rennes, France
- Christian Choffrut, Université Paris Cité, France
- Emily Clement, Université Paris Cité, France