Generating Posets With and Without Interfaces

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(i)Po(m)set Project Online Seminar Februar 2024

Series-Parallel Posets

- a poset: *finite* set P plus strict partial order $<$: irreflexive, transitive, asymmetric
- parallel composition of posets $(P_1, <_1)$, $(P_2, <_2)$:

$$P_1 \otimes P_2 = (P_1 \sqcup P_2, <_1 \cup <_2)$$

$\upuparrows$ disjoint union

- serial composition:

$$P_1 * P_2 = (P_1 \sqcup P_2, (<_1 \cup <_2 \cup P_1 \times P_2)^+)$$

$\upuparrows$ P_1 before P_2 (transitive closure)

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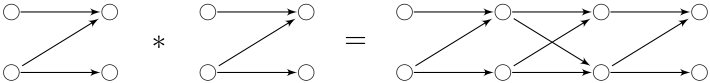
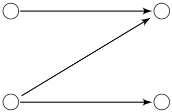


Series-Parallel Posets



Definition (Winkowski '77, Grabowski '81)

A poset is series-parallel (sp) if it is empty or can be obtained from the singleton poset by a finite number of serial and parallel compositions.

Theorem (Grabowski '81)

A poset is sp iff it does not contain **N** *as an induced subposet.*

The equational theory of sp-posets is well-understood: [Gischer 1988, TCS], [Bloom-Esik 1996, MSCS]

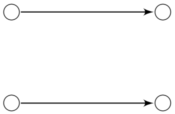
Interval Orders

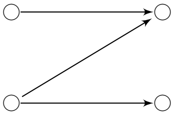
Definition (Fishburn '70)

A poset is an interval order if is has a representation as (real) intervals, ordered by $\max_1 \leq \min_2$

- posets which are good for concurrency?
- already in [Wiener 1914], then [Winkowski '77], [Lamport '86], [van Glabbeek '90], [Vogler '91], [Janicky '93], etc.
- Lemma (Fishburn '70): A poset is interval iff it does not contain $2{+}2 = \left(\begin{smallmatrix} \cdot \longrightarrow \cdot \\ \cdot \longrightarrow \cdot \end{smallmatrix}\right)$ as induced subposet.
- intuitively: if $a \longrightarrow b$ and $c \longrightarrow d$, then also $a \longrightarrow d$ or $c \longrightarrow b$

Interval Orders vs Series-Parallel Posets





- interval orders are used in concurrency theory and distributed computing
- but don't (*yet*) have a good algebraic theory
- sp-posets have nice algebraic theory and seem to be used in concurrency theory
- Concurrent Kleene algebra
- interval orders are 2+2-free; sp-posets are **N**-free
- incomparable: 2+2 is sp; **N** is interval

Interval Orders $\oplus$ Series-Parallel Posets

Goal (2018):

- develop common generalization of sp-posets and interval orders
- for use in concurrency theory etc.
- with good algebraic properties

$\Longrightarrow$ gluing-parallel (i)posets

Realization (2020):

- combinatorial properties of gluing-parallel iposets are complicated
- and interval orders seem to be enough for concurrency theory
- (languages of HDAs are sets of labeled interval orders)

This talk:

- algebra of gluing-parallel iposets (a bit)
- combinatorics of gluing-parallel iposets (a lot)

Nice people

- UF, Christian Johansen, Georg Struth, Ratan Bahadur Thapa: *Generating Posets Beyond N*. RAMiCS 2020
- Olavi Äikäs, Polytechnique intern, 2021
- UF, Christian Johansen, Georg Struth, Krzysztof Ziemiański: *Posets with Interfaces as a Model for Concurrency*. Information and Computation 2022
- Äikäs, UF, Christian Johansen, Krzysztof Ziemiański: *Generating Posets with Interfaces*. arxiv 2022
- Clarisse Blanco & Dorian Peron, EPITA interns, 2022

NEW Paul Fournillon & Quentin Hay-kergrohenn, EPITA interns, 2024

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Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A331156 Number of (weakly) connected gluing-parallel or GP-posets with n points. 4

1, 1, 1, 3, 10, 44, 23 (list; graph; refs; listen; history; text; internal format)

OFFSET	0,4
LINKS	Table of n, a(n) for n=0..6. Uli Fahrenberg, Christian Johansen, Georg Struth, Ratan Bahadur Thapa, Generating Posets Beyond N, arXiv:1910.06162 [cs.FL], 2019.
CROSSREFS	The seven sequences in the table of Uli Fahrenberg et al., 2019, are A000112, A003430, A079566, A331156, A331157, A331158, A331159. Sequence in context: A032269 A179501 A041737 * A279105 A246956 A026682 Adjacent sequences: A331153 A331154 A331155 * A331157 A331158 A331159
KEYWORD	nonn,more
AUTHOR	N. J. A. Sloane, Jan 16 2020, following a suggestion from Michael De Vlieger.

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EXTENSIONS Typo in a(6) corrected by Uli Fahrenberg, Feb 03 2024

Posets With Interfaces

Definition

A poset with interfaces (iposet) is a poset P plus two injections

$$[n] \xrightarrow{s} P \xleftarrow{t} [m]$$

such that $s[n]$ is minimal and $t[m]$ is maximal in P.

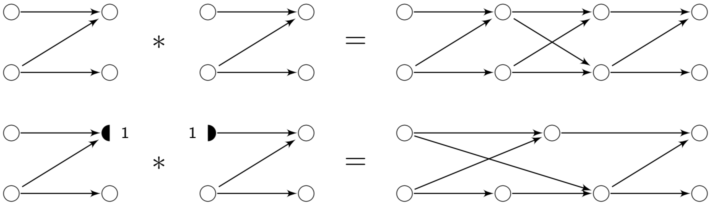
- $([n] = \{1, \dots, n\})$
- s: starting interface ; t: terminating interface
- events in $t[m]$ are *unfinished* ; events in $s[n]$ are "*unstarted*"

Gluing Composition

Definition

The gluing of iposets $s_1 : [n] \to (P_1, <_1) \leftarrow [m] : t_1$ and $s_2 : [m] \to (P_2, <_2) \leftarrow [k] : t_2$ is

$$P_1 * P_2 = \begin{cases} (P_1 \sqcup P_2)/t_1(i) = s_2(i) \\ (<_1 \cup <_2 \cup (P_1 \setminus t_1[m]) \times (P_2 \setminus s_2[m]))^+ \end{cases}$$



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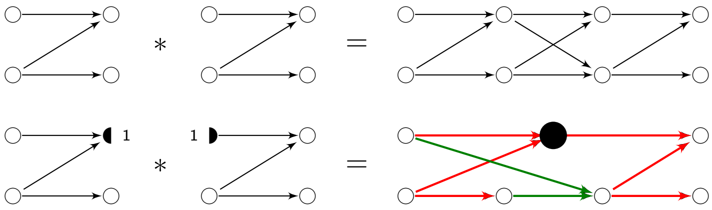

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Parallel Composition

- parallel composition of iposets: put posets in parallel and renumber interfaces
- for $[n_1] \to P_1 \leftarrow [m_1]$ and $[n_2] \to P_2 \leftarrow [m_2]$, have $[n_1 + n_2] \to P_1 \otimes P_2 \leftarrow [m_1 + m_2]$
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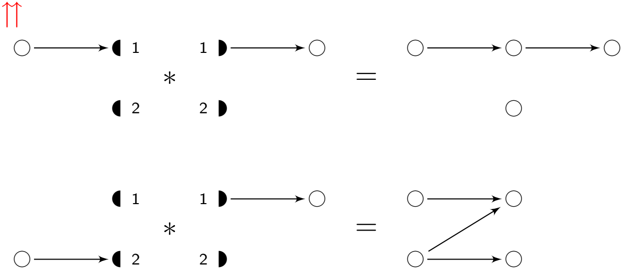

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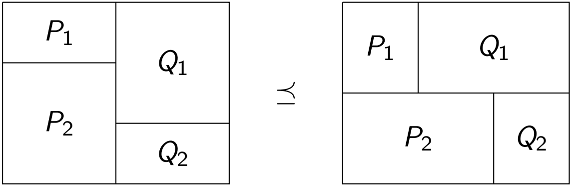
$\upuparrows$

$$(P_1 \otimes P_2) * (Q_1 \otimes Q_2) \preceq (P_1 * Q_1) \otimes (P_2 * Q_2)$$



Gluing-Parallel Iposets

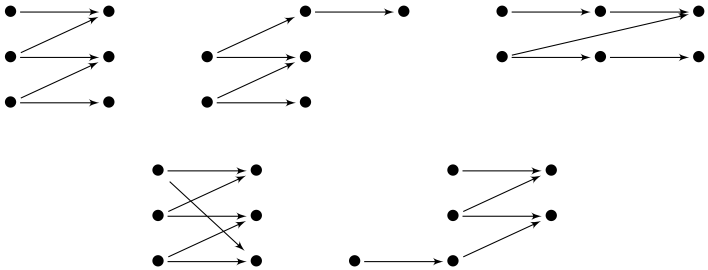
- recall *series-parallel posets*: generated from ○ using $*$ and $\otimes$
- the four singleton iposets:

  ○ 1◗ ◖1 1◗◖1

- gluing-parallel (gp) iposets: generated from ○ , 1◗ , ◖1 , 1◗◖1 using $*$ and $\otimes$
- gluing-parallel posets: gp-iposets without interfaces

Forbidden Substructures

- sp-posets $\hat{=}$ **N**-free
- interval orders $\hat{=}$ 2+2-free
- gluing-parallel posets $\Longrightarrow$ free of



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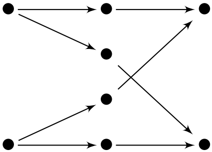

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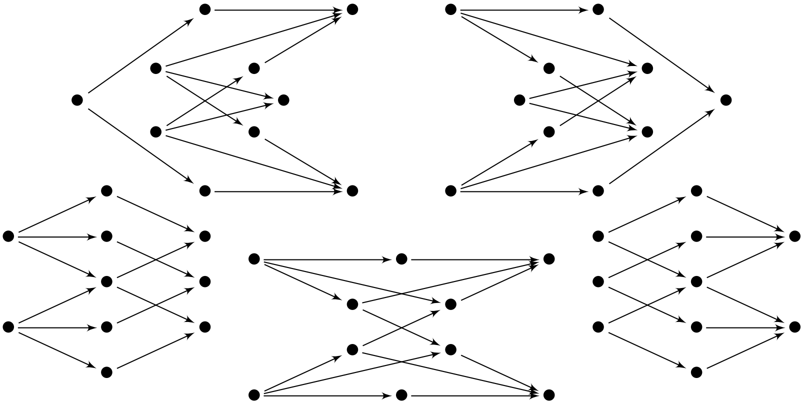
- sp-posets ≙ **N**-free
- interval orders ≙ 2+2-free
- gluing-parallel posets ⟹ free of



Forbidden Substructures

- sp-posets $\hat{=}$ **N**-free
- interval orders $\hat{=}$ 2+2-free
- gluing-parallel posets $\iff$ free of ???
  - it's complicated!

Gluing Decompositions

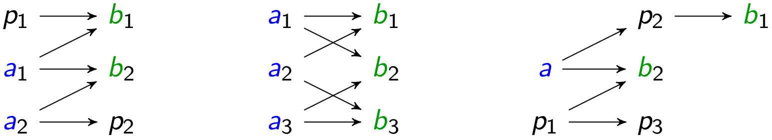
- an iposet P can be decomposed as $P = Q \otimes R$ iff P is disconnected as an undirected graph
- an iposet P can be decomposed as $P = Q * R$ iff ???

Definition

A gluing $P = Q * R$ is non-trivial if $P \neq Q$ and $P \neq R$ as posets.

Extremities

For a poset P, define $P_a = \{x \in P \mid \forall y \in P : |x\uparrow| \geq |y\uparrow|\}$ ("extreme left") and $P_b = \{x \in P \mid \forall y \in P : |x\downarrow| \geq |y\downarrow|\}$ ("extreme right").



Decomposition Lemma

Lemma

Suppose P admits a non-trivial gluing decomposition. Then there is $\varphi : P \to \{0, *, 1\}$ such that

1. *if* $x < y$, *then* $(\varphi(x), \varphi(y)) \in \{(1,0), (1,*), (1,1), (*,0), (0,0)\}$;
2. *if* $(\varphi(x), \varphi(y)) = (1,0)$, *then* $x < y$;
3. *if* $(\varphi(x), \varphi(y)) \in \{(1,*), (*,0), (1,0)\}$, *then* $y \not< x$;
4. *if* $(\varphi(x), \varphi(y)) = (*,*)$, *then* $x \not< y$ *and* $y \not< x$;
5. $\varphi(x) = 1$ *for* $x \in P_a$ *and* $\varphi(y) = 0$ *for* $y \in P_b$.

- $0 \triangleq$ not started yet
- $* \triangleq$ running
- $1 \triangleq$ terminated

Decomposition Lemma: Proof

Lemma

P has non-trivial gluing decomposition $\implies \exists \varphi : P \to \{0, *, 1\}$:

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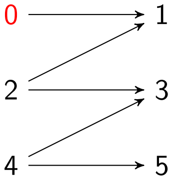
Proof: (or maybe not!?)

Small Forbidden Substructures

Lemma

N *does not admit a non-trivial gluing decomposition.*

First proof: Assume N $= P \ast Q$.



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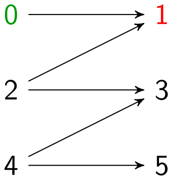

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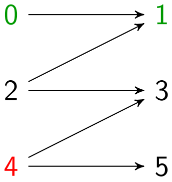

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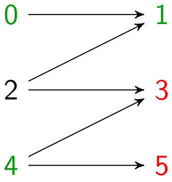

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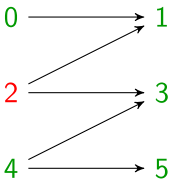

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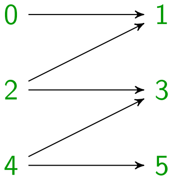

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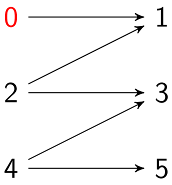

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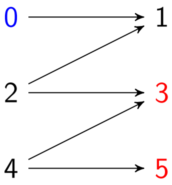

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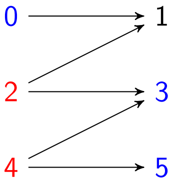

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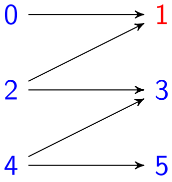

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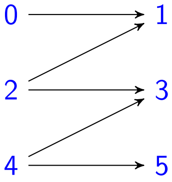

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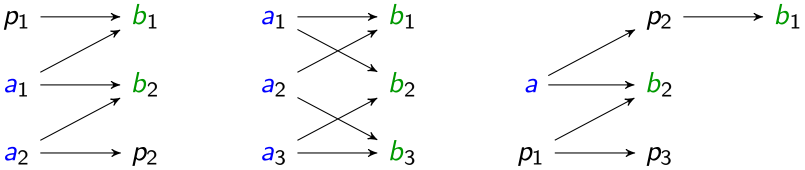

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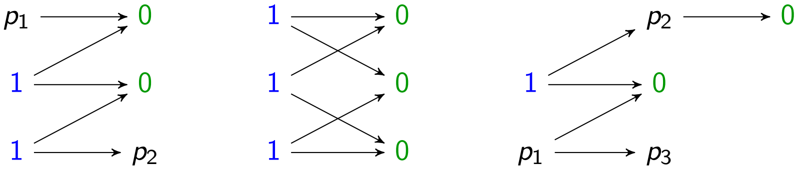

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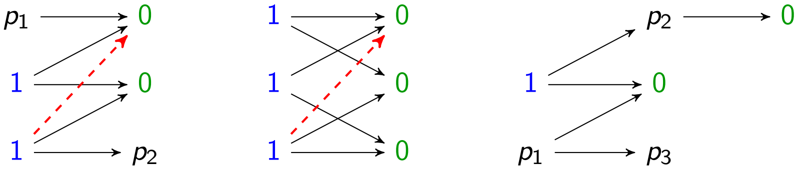

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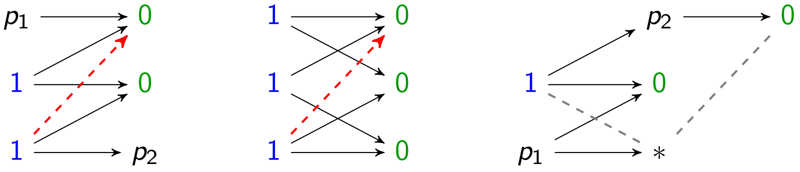

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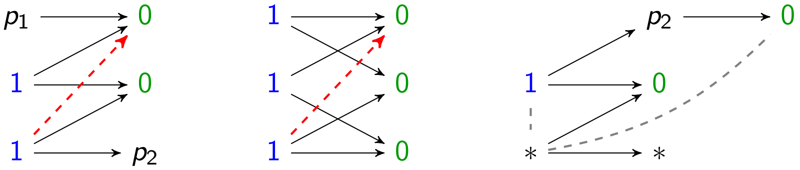

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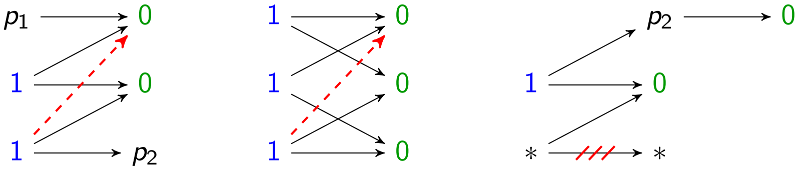
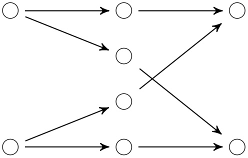


8-Point Forbidden Substructure



- not gluing-parallel

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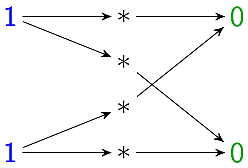
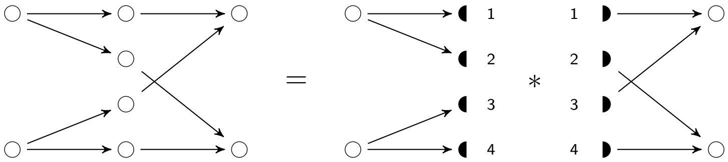


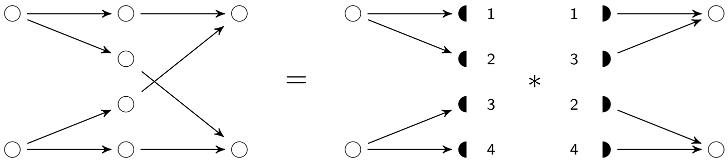
- not gluing-parallel
- but satisfies decomposition lemma

8-Point Forbidden Substructure



- not gluing-parallel
- but satisfies decomposition lemma
- and can be decomposed?!?

8-Point Forbidden Substructure



- not gluing-parallel
- but satisfies decomposition lemma
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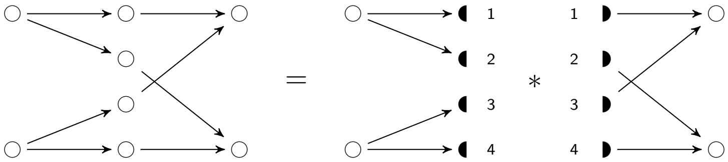
8-Point Forbidden Substructure



- not gluing-parallel
- but satisfies decomposition lemma
- and can be decomposed?!?
- interfaces "permuted wrong"
- same for all 10-point forbidden substructures: all "decomposable up to interface permutation"

Gluing-Parallel-Symmetric Iposets

- recall gp-iposets: generated from ○ , 1◗ , ◖1 , and 1⋈1 (using $*$ and $\otimes$)
- let $\begin{smallmatrix} 1 \bowtie 2 \\ 2 \bowtie 1 \end{smallmatrix} = (s, [2], t) : 2 \to 2$ be the non-trivial symmetry on 2
- gps-iposets: generated from ○ , 1◗ , ◖1 , 1⋈1 , and $\begin{smallmatrix} 1 \bowtie 2 \\ 2 \bowtie 1 \end{smallmatrix}$ (using $*$ and $\otimes$)

Lemma

An iposet is gps iff its underlying poset is.

Proof.

The symmetric groups are generated by transpositions. □

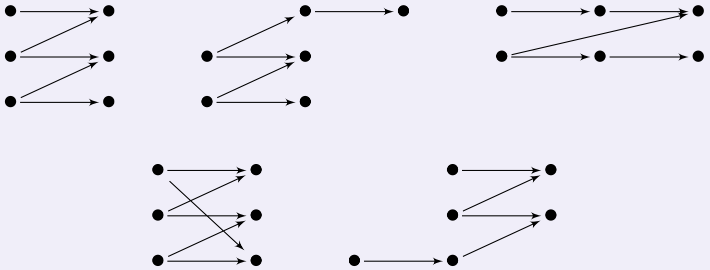
- so all interface permutations included

$\Longrightarrow$ all "big" forbidden substructures are gps

Conjecture

Conjecture

A poset is gps iff it is free of



Gps = Sp-Intervals

Definition (generalized interval orders)

Let P and V be posets. An interval representation of P in V is a pair of functions $f, g : P \to V$ such that:

1. $f(p) \leq_V g(p)$ for all $p \in P$,
2. $p <_P q$ iff $g(p) <_V f(q)$ for all $p, q \in P$.

P is a V-interval order if it admits an interval representation in V.

Lemma

If P is a V-interval order and V is a W-interval order, then P is a W-interval order.

Theorem (Ziemiański)

A poset is gps iff it is an sp-interval order.

Gps = Sp-Intervals, 2.

Lemma

If P is gps, then P is sp-interval.

Proof.

Induction. $P \in \{ \circ, {\bullet}_1, {}_1{\bullet}, {}_1{\bowtie}_1 \}$: fine; all are $\circ$-interval.

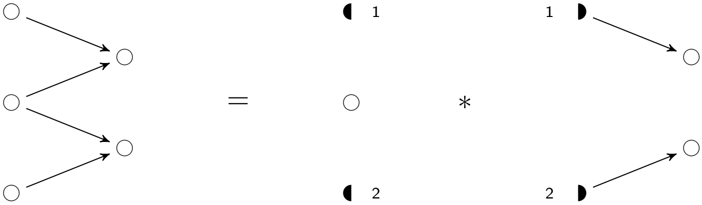
- if $P = Q \otimes R$: when Q is V-interval and R is W-interval, then $Q \otimes R$ is $V \otimes W$-interval.
- if $P = Q * R$: when Q is V-interval and R is W-interval, then $Q * R$ is $V * W$-interval. $\longleftarrow$ requires proof

In both cases:

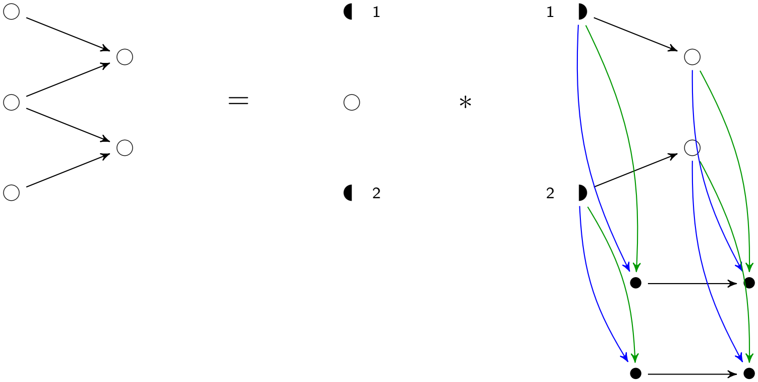
$$f(p) = \begin{cases} f_Q(p) & \text{for } p \in Q \\ f_R(p) & \text{for } p \notin Q \end{cases} \qquad g(p) = \begin{cases} g_R(p) & \text{for } p \in R \\ g_Q(p) & \text{for } p \notin R \end{cases}$$

□

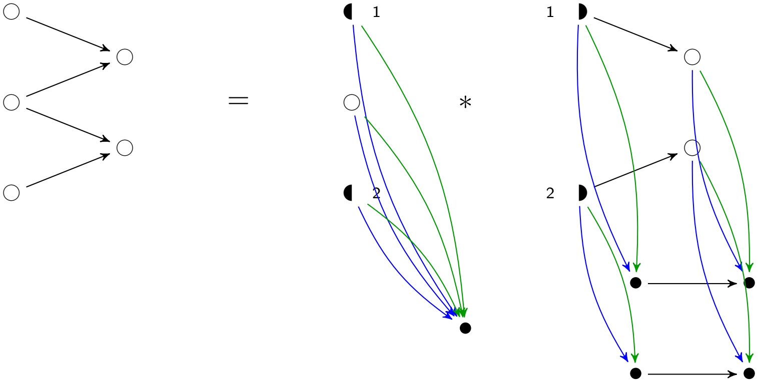
Example



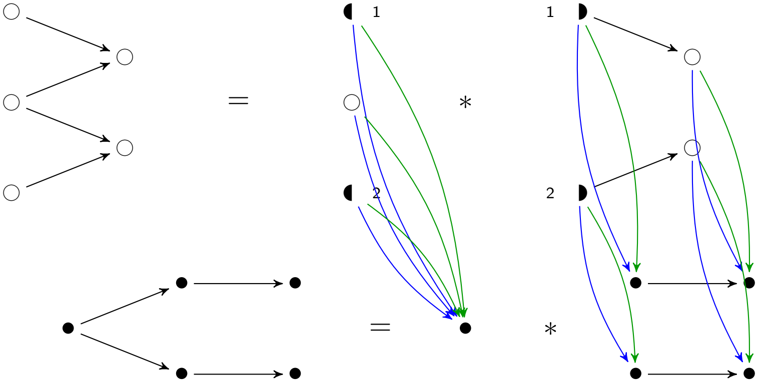
Example



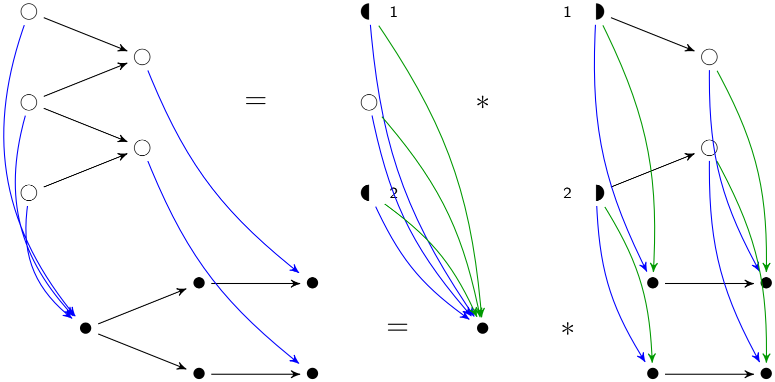
Example



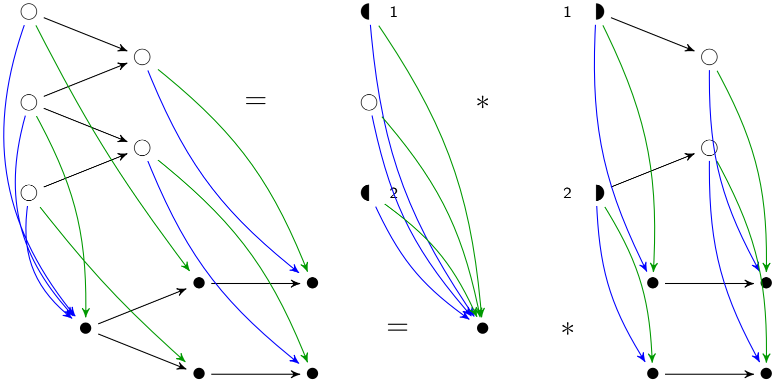
Example



Example



Example



Gps = Sp-Intervals, 3.

Lemma

If P is sp-interval, then P is gps.

Proof.

Induction. $P \in \{ \bigcirc , ◖_1 , {}_1◗ , {}_1⧓_1 \}$: fine.

- if P is $V \otimes W$-interval: let $Q = f^{-1}(V) = g^{-1}(V)$ and $R = f^{-1}(W) = g^{-1}(W)$ (true because $V \otimes W$ is disconnected), then $P = Q \otimes R$. Using $f_Q = f_{\restriction Q}$ etc. (the restrictions), Q is V-interval and R is W-interval.
- if P is $V * W$-interval: let $Q = f^{-1}(V)$ and $R = g^{-1}(W)$, then $P = Q * R$ (interfaces are $Q \cap R$). Define $f_Q = f_{\restriction Q}$ and
$$g_Q(p) = \begin{cases} g(p) \text{ if } p \in V, \\ \text{some } x \in V^{\max} \text{ with } f(p) \leq x \text{ otherwise.} \end{cases}$$
Definitions of f_R and g_R are symmetric, and then Q is V-interval and R is W-interval. $\longleftarrow$ requires proof □

Conclusion

- gluing-parallel iposets: generated from ○, $_1$▶, ◀$_1$, $_1$⋈$_1$ using $*$ and $\otimes$
- complicated combinatorics; so far 11 forbidden substructures; unknown whether set of forbidden substructures is finite

- gluing-parallel-symmetric iposets: generated from ○, $_1$▶, ◀$_1$, $_1$⋈$_1$, $^1_2$⋈$^2_1$ using $*$ and $\otimes$
- less complicated: iposet is gps iff underlying poset is
- (and can be generated from ○ without using interfaces)
- Theorem: gps-posets = interval orders in sp-posets
- Conjecture: precisely five forbidden substructures

Also interesting:

- Relational and Algebraic Methods in CS (RAMiCS), Prague 19-23 Aug.
- Geometric and Topological Methods in CS (GETCO), Tallinn 6-7 July
- Pomsets and Related Structures (RaPS), Rennes 24 April

Gps-Posets Without Interfaces

Definition

Let P_1 and P_2 be posets.

- The right-interior gluing composition $P_1 *^{\mathrm{i}} P_2$: carrier set $P_1 \sqcup P_2$, $(p,i) < (q,j) \Leftrightarrow (i = j \wedge p <_i q) \vee (i < j \wedge q \notin P_2^{\min})$
- The left-interior gluing composition $P_1 \, {}^{\mathrm{i}}{*} \, P_2$: carrier set $P_1 \sqcup P_2$, $(p,i) < (q,j) \Leftrightarrow (i = j \wedge p <_i q) \vee (i < j \wedge p \notin P_1^{\max})$
- The Winkowski multi-composition $P_1 \gtrdot P_2$: defined if $|P_1^{\max}| = |P_2^{\min}|$, and then $P_1 \gtrdot P_2 = \{P_1 \gtrdot_f P_2 \mid f \text{ bijection } P_1^{\max} \to P_2^{\min}\}$, where $P_1 \gtrdot_f P_2$ is the poset with carrier set $(P_1 \sqcup P_2)_{/x=f(x)}$ and order $(p,i) < (q,j) \Leftrightarrow (i = j \wedge p <_i q) \vee (i < j \wedge p \notin P_1^{\max} \wedge q \notin P_2^{\min})$

Lemma

Gps-posets are generated from $\circ$ *using* $\otimes$, $*$, $*^{\mathrm{i}}$, ${}^{\mathrm{i}}{*}$, *and* $\gtrdot$.

Some numbers

n	P(n)	GP(n)	GPS(n)	IP(n)	GPI(n)	GPSI(n)
0	1	1	1	1	1	1
1	1	1	1	4	4	4
2	2	2	2	17	16	17
3	5	5	5	86	74	86
4	16	16	16	532	419	532
5	63	63	63	4068	2980	4068
6	318	313	313	38.933	26.566	38.447
7	2045	1903	1903	474.822	289.279	
8	16.999	13.943	13.944	7.558.620	3.726.311	
9	183.231	120.442	120.465			
10	2.567.284	1.206.459				