Developments in Higher-Dimensional Automata Theory

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RAMiCS 2024

Languages of higher-dimensional automata

- Generating Posets Beyond N [RAMiCS 2020]
- Languages of Higher-Dimensional Automata [MSCS 2021]
- Posets with Interfaces as a Model for Concurrency [I&C 2022]
- A Kleene Theorem for Higher-Dimensional Automata [CONCUR 2022]
- A Myhill-Nerode Theorem for Higher-Dimensional Automata [Petri Nets 2023]
- Decision and Closure Properties for Higher-Dimensional Automata [ICTAC 2023]
- Logic and Languages of Higher-Dimensional Automata [DLT 2024]
- ...

Today:

1. What are HDAs (and why should I be interested)?
2. What are languages of HDAs (and why should I be interested)?
3. What can I do with languages of HDAs (that I cannot do with other models)?

Nice people

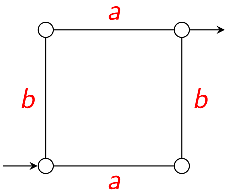
- Eric Goubault, Paris
- Christian Johansen, Oslo
- Georg Struth, Sheffield
- Krzysztof Ziemiański, Warsaw

- Amazigh Amrane, Hugo Bazille, Emily Clement, Jérémy Dubut, Marie Fortin, Roman Kniazev, Jérémy Ledent, Safa Zouari, ...

- See also https://ulifahrenberg.github.io/pomsetproject/

Higher-dimensional automata

semantics of "*a* parallel *b*":



$a.b + b.a$

Higher-dimensional automata

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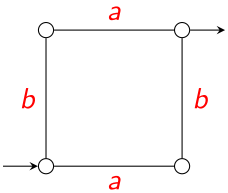
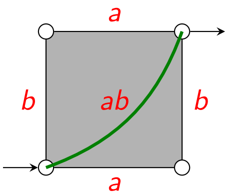


$a.b + b.a$



a and *b* are independent

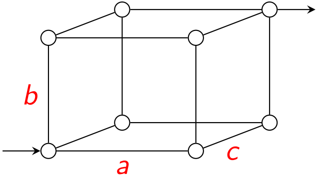
Higher-dimensional automata & concurrency

HDAs as a model for concurrency:

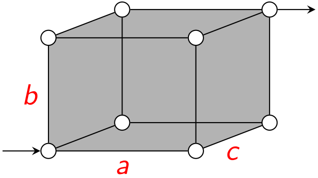
- points: states
- edges: transitions
- squares, cubes etc.: independency relations (concurrently executing events)
- two-dimensional automata $\cong$ asynchronous transition systems [Bednarczyk]

[van Glabbeek 2006, TCS]: Up to history-preserving bisimilarity, HDAs "generalize the main models of concurrency proposed in the literature" (notably, event structures and Petri nets)

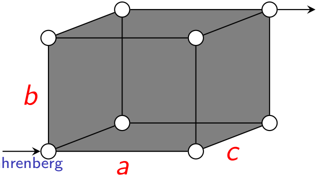
Examples



no concurrency



two out of three



full concurrency

Higher-dimensional automata

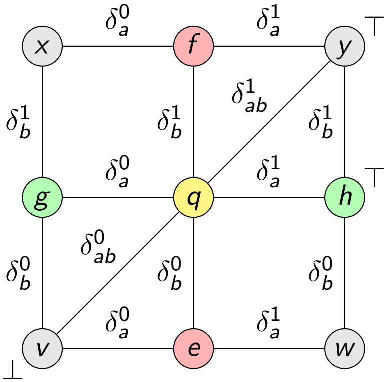
A conclist is a finite, ordered and Σ-labelled set. (a list of events)

A precubical set X consists of:

- A set of cells X (cubes)
- Every cell $x \in X$ has a conclist $\text{ev}(x)$ (list of events active in x)
- We write $X[U] = \{x \in X \mid \text{ev}(x) = U\}$ for a conclist U (cells of type U)
- For every conclist U and $A \subseteq U$ there are:
  upper face map $\delta_A^1 : X[U] \to X[U \setminus A]$ (terminating events A)
  lower face map $\delta_A^0 : X[U] \to X[U \setminus A]$ (unstarting events A)
- Precube identities: $\delta_A^\mu \delta_B^\nu = \delta_B^\nu \delta_A^\mu$ for $A \cap B = \emptyset$ and $\mu, \nu \in \{0, 1\}$

A higher dimensional automaton (HDA) is a precubical set X with start cells $\bot \subseteq X$ and accept cells $\top \subseteq X$ (not necessarily vertices)

Example



$$X[\emptyset] = \{v, w, x, y\}$$

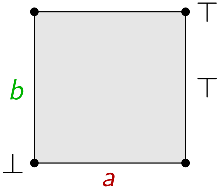
$$X[a] = \{e, f\}$$

$$X[b] = \{g, h\}$$

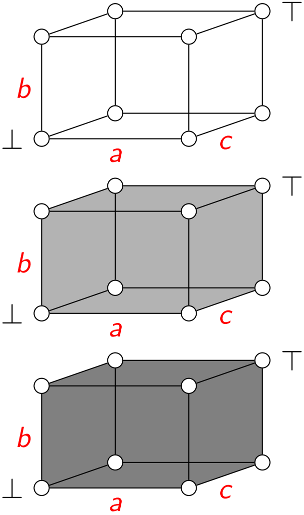
$$X[ab] = \{q\}$$

$$\bot_X = \{v\}$$

$$\top_X = \{h, y\}$$



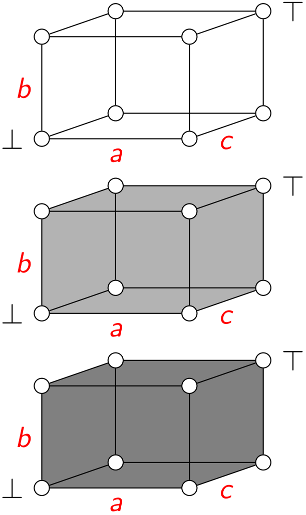
Languages of HDAs: Examples



$$L_1 = \{abc, acb, bac, bca, cab, cba\}$$

$$L_3 = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \dots \right\}$$

Languages of HDAs: Examples



$$L_1 = \{abc, acb, bac, bca, cab, cba\}$$

$$L_2 = \left\{ \begin{pmatrix} a \\ b \to c \end{pmatrix}, \begin{pmatrix} a \\ c \to b \end{pmatrix}, \begin{pmatrix} b \\ a \to c \end{pmatrix}, \begin{pmatrix} b \\ c \to a \end{pmatrix}, \begin{pmatrix} c \\ a \to b \end{pmatrix}, \begin{pmatrix} c \\ b \to a \end{pmatrix} \right\} \cup L_1 \cup \dots$$

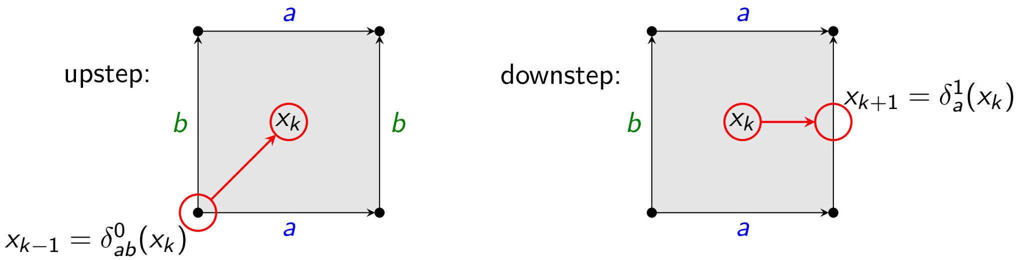
$$L_3 = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\} \cup L_2$$

sets of pomsets

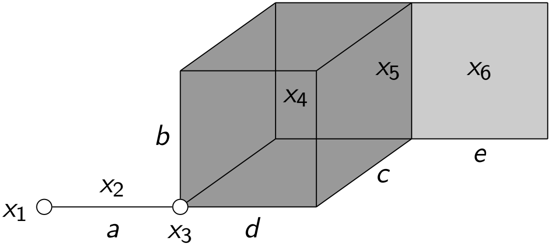
Computations of HDAs

A path on an HDA X is a sequence $(x_0, \varphi_1, x_1, \dots, x_{n-1}, \varphi_n, x_n)$ such that for every k, $(x_{k-1}, \varphi_k, x_k)$ is either

- $(\delta^0_A(x_k), \nearrow^A, x_k)$ for $A \subseteq \mathrm{ev}(x_k)$ or (upstep: start A)
- $(x_{k-1}, \searrow_B, \delta^1_B(x_{k-1}))$ for $B \subseteq \mathrm{ev}(x_{k-1})$ (downstep: terminate B)

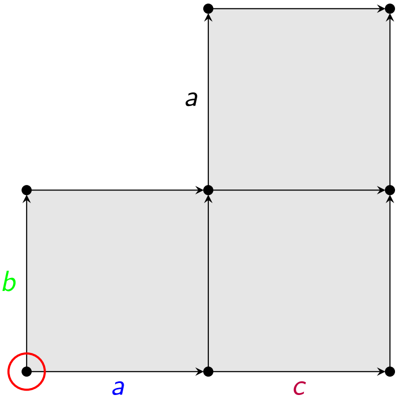


Example

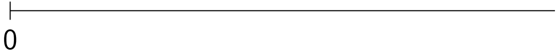


$$(x_1 \nearrow^{a} x_2 \searrow_{a} x_3 \nearrow^{\{b,c,d\}} x_4 \searrow_{\{c,d\}} x_5 \nearrow^{e} x_6)$$

Event ipomset of a path



Lifetimes of events



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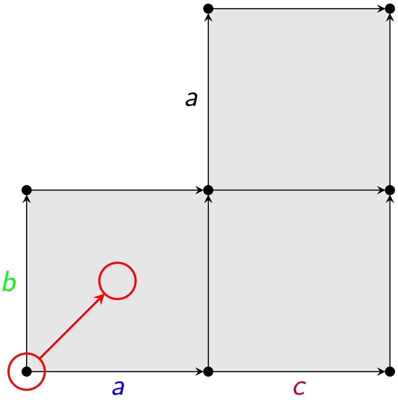

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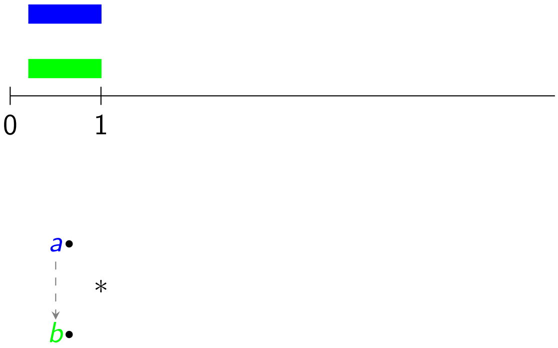
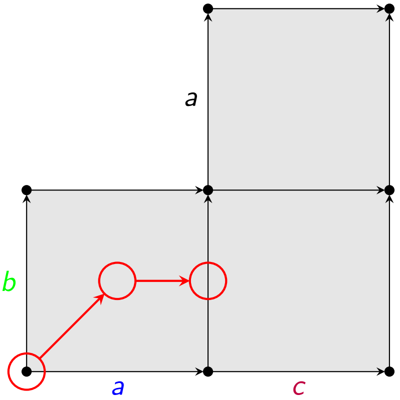


Event ipomset of a path



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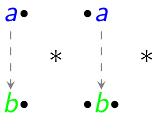

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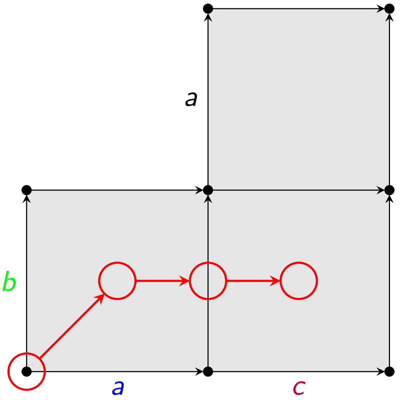

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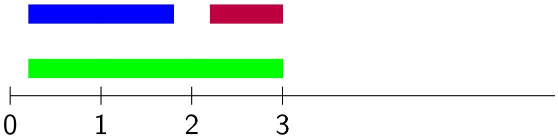

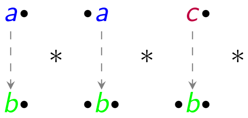

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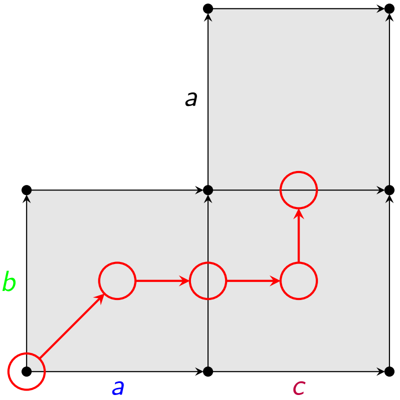

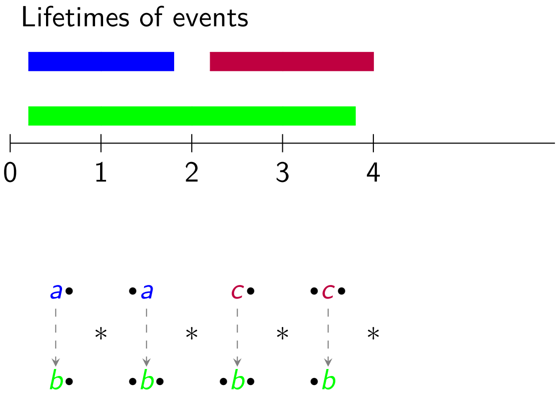

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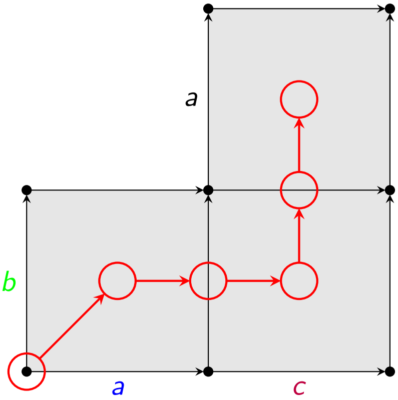

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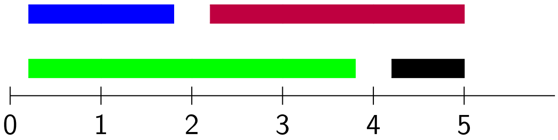

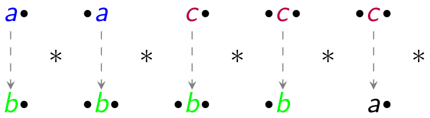

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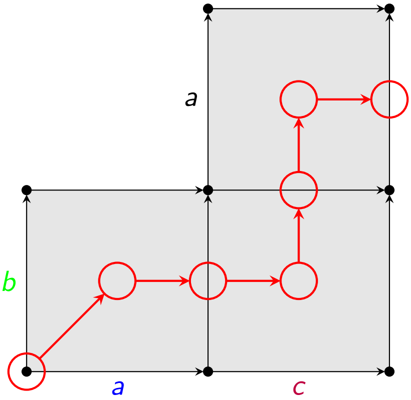

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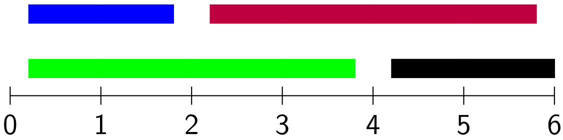

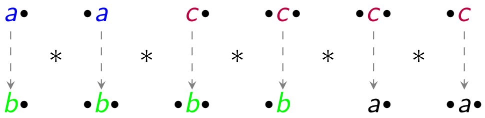

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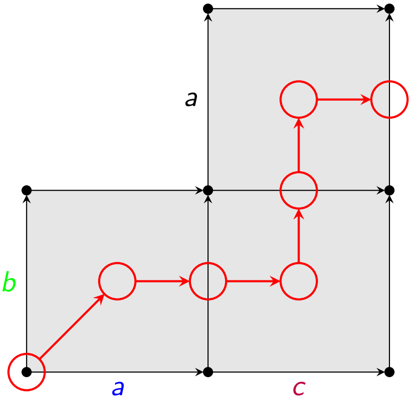

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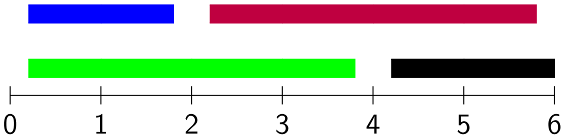
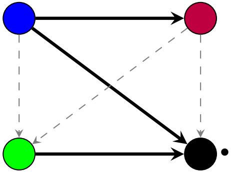


Event ipomset



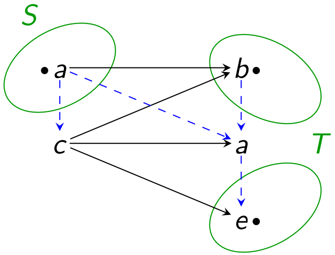
(not series-parallel!)

Pomsets with interfaces

Definition

A pomset with interfaces (ipomset): $(P, <, \dashrightarrow, S, T, \lambda)$:

- finite set P;
- two partial orders $<$ (precedence order), $\dashrightarrow$ (event order)
  - s.t. $< \cup \dashrightarrow$ is a *total relation*;
- $S, T \subseteq P$ source and target interfaces
  - s.t. S is $<$-minimal and T is $<$-maximal.

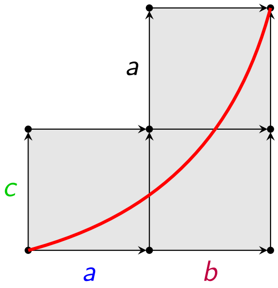


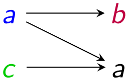
Interval orders

Definition

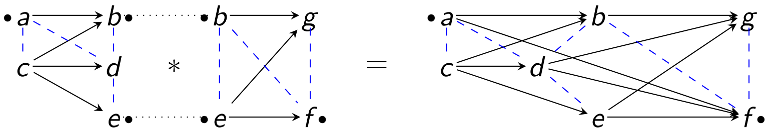
An ipomset $(P, <_P, \dashrightarrow, S, T, \lambda)$ is interval if $(P, <_P)$ has an interval representation: functions $b, e : P \to \mathbb{R}$ s.t.

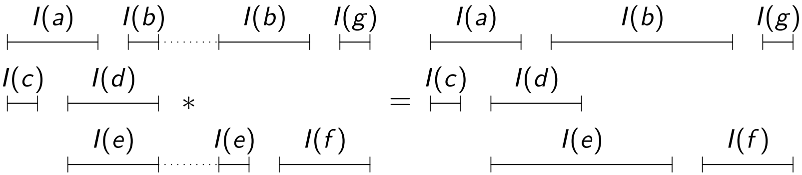
- $\forall x \in P : b(x) \le_{\mathbb{R}} e(x)$;
- $\forall x, y \in P : e(x) <_{\mathbb{R}} b(y) \iff x <_P y$





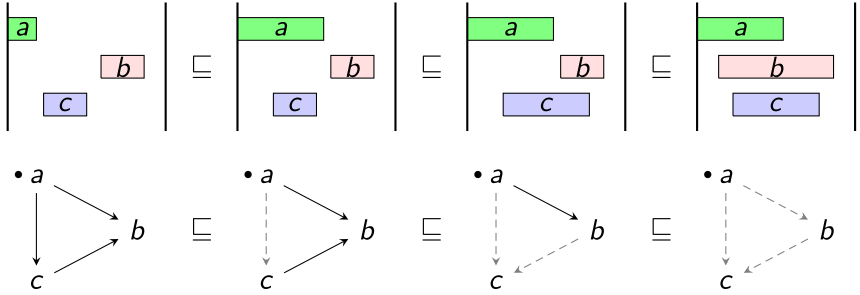
Gluing composition





- Gluing $P * Q$: P before Q, except for interfaces (which are identified)
- (also have parallel composition $P \parallel Q$: disjoint union)

Subsumption



P refines Q / Q subsumes P / $P \sqsubseteq Q$ iff

- P and Q have same interfaces
- P has more $<$ than Q
- Q has more $\dashrightarrow$ than P

Languages of HDAs

Definition

The language of an HDA X is the set of event ipomsets of all accepting paths:

$$L(X) = \{\mathrm{ev}(\pi) \mid \pi \in \mathrm{Paths}(X), \mathrm{src}(\pi) \in \bot_X, \mathrm{tgt}(\pi) \in \top_X\}$$

- $L(X)$ contains only interval ipomsets,
- is closed under subsumption,
- and has finite width

Definition

A language $L \subseteq \mathrm{iiPoms}$ is regular if there is an HDA X with $L = L(X)$.

Theorems

Definition (Rational Languages over Σ)

- Generated by $\emptyset$, $\{\epsilon\}$, and all $\{[a]\}$, $\{[\bullet\, a]\}$, $\{[a\, \bullet]\}$, $\{[\bullet\, a\, \bullet]\}$ for $a \in \Sigma$
- under operations $\cup$, $*$, $\parallel$ and (Kleene plus) $^+$
- (these need to take subsumption closure into account)

Definition (Monadic Second-Order Logics over Ipomsets)

$$\psi ::= a(x) \mid \mathsf{s}(x) \mid \mathsf{t}(x) \mid x < y \mid x \dashrightarrow y \mid x \in X \mid \exists x.\,\psi \mid \forall x.\,\psi \mid \exists X.\,\psi \mid \forall X.\,\psi \mid \psi_1 \wedge \psi_2 \mid \psi_1 \vee \psi_2 \mid \neg\psi$$

Theorem (à la Kleene [CONCUR 2022])

A language is rational iff it is regular.

Theorem (à la Büchi-Elgot-Trakhtenbrot [DLT 2024])

A language is rational iff it is MSO-definable, of finite width, and subsumption-closed.

More theorems

Theorem (à la Myhill-Nerode [Petri Nets 2023])

A language is rational iff it has finite prefix quotient.

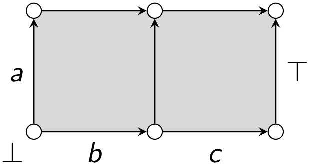
Theorem (Closure properties [ICTAC 2023])

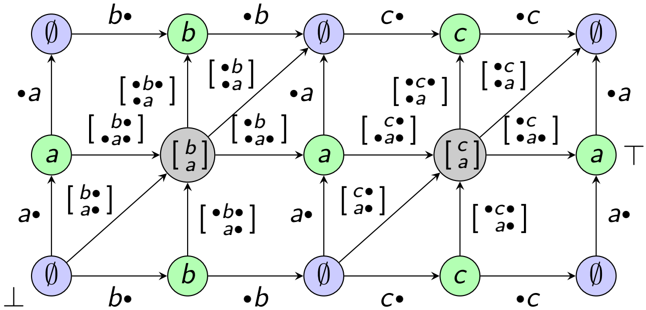
Rational languages are closed under intersection but not under complement.

Theorem (Determinizability & ambiguity [Petri Nets 2023])

Not all HDAs are determinizable.
There is a rational language which is inherently infinitely ambiguous.

Important tool: ST-automata





- The operational semantics of an HDA $(X, \bot, \top, \Sigma)$ is the "ST-automaton" with states X, (infinite) alphabet Ω, state labeling $\mathsf{ev} : X \to \square$, and transitions

$$E = \{\delta^0_A(\ell) \xrightarrow{A\uparrow \mathsf{ev}(\ell)} \ell \mid A \subseteq \mathsf{ev}(\ell)\} \cup \{\ell \xrightarrow{\mathsf{ev}(\ell)\downarrow A} \delta^1_A(\ell) \mid A \subseteq \mathsf{ev}(\ell)\}.$$

- Takes care of half of all theorems: regular ⇒ rational; MSO-definable ⇒ regular; regular ⇒ finite prefix quotient; decidability of inclusion