Higher-Dimensional Automata for Petri Net Analysis

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Higher-Dimensional Automata?

PN'23 UF, K. Ziemiański: *A Myhill-Nerode theorem for higher-dimensional automata*

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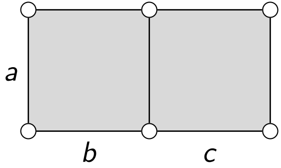
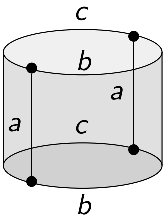
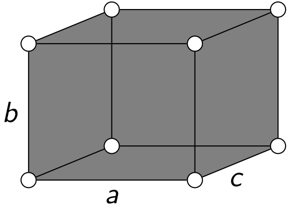
- Rob van Glabbeek 2006: *On the expressiveness of higher-dimensional automata*

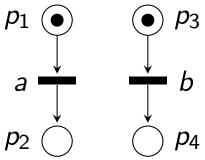






Semantics of Petri Nets

Petri net (S, T, F): places S; transitions T; weighted flows $F : S \times T \cup T \times S \to \mathbb{N}$



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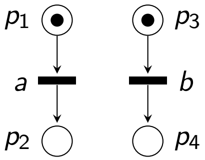

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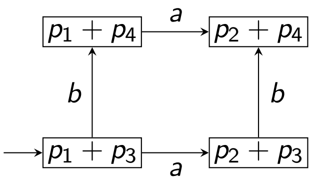
Interleaved semantics (reachability graph) (V, E):

- $V = \mathbb{N}^S$: all markings
- $E \subseteq V \times T \times V$: one transition at a time
- $E = \{(m, t, m') \mid {}^\bullet t \leq m, m' = m - {}^\bullet t + t^\bullet\}$
- initial marking $\implies$ initial state; take reachable part



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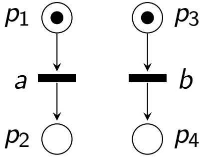

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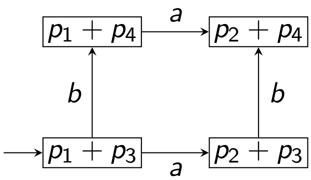
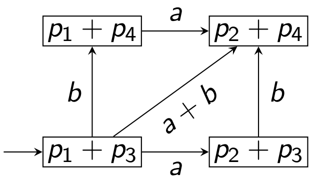
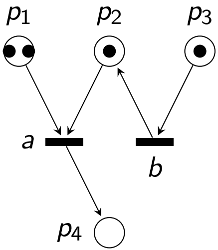


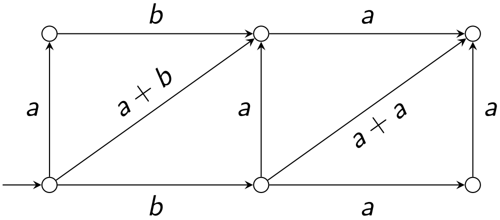
Concurrent step reachability graph (V, E'):

- $V = \mathbb{N}^S$
- $E' \subseteq V \times \mathbb{N}^T \times V$: multisets of transitions
- $E' = \{(m, U, m') \mid {}^\bullet U \le m, m' = m - {}^\bullet U + U^\bullet\}$



Another Example





- after firing b, a is auto-concurrent

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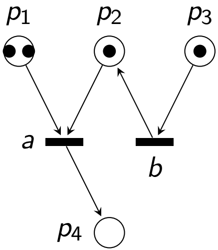

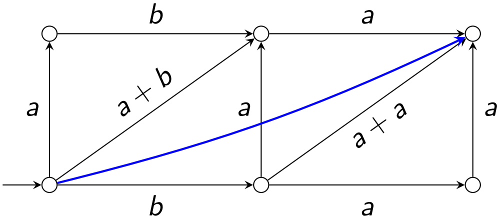

- after firing b, a is auto-concurrent
- (Concurrent step) semantics misses some behavior?
  - start a – start b – finish b – start another a – etc.

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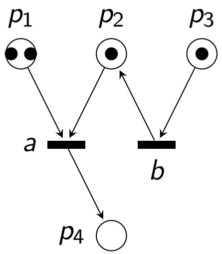
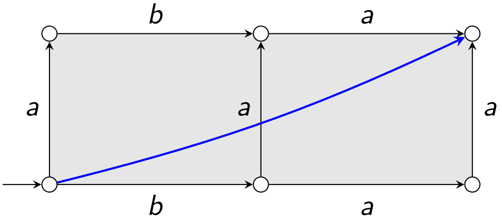




- after firing b, a is auto-concurrent
- (Concurrent step) semantics misses some behavior?
  - start a – start b – finish b – start another a – etc.
- enter higher-dimensional automata
  - replace multi-transitions by squares / cubes / etc.

Higher-Dimensional Automata

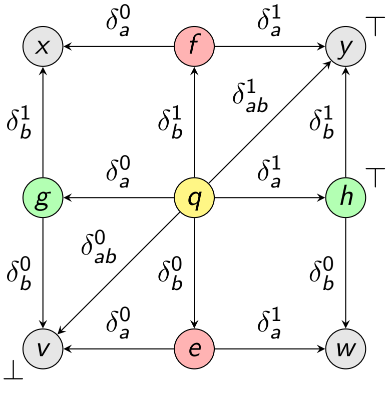
A conclist is a finite, totally ordered, Σ-labeled set. (a list of labeled events)

A precubical set X consists of:

- A set of cells X (cubes)
- Every cell $x \in X$ has a conclist $\mathrm{ev}(x)$ (list of events active in x)
- We write $X[U] = \{x \in X \mid \mathrm{ev}(x) = U\}$ for a conclist U (cells of type U)
- For every conclist U and $A \subseteq U$ there are:
  upper face map $\delta^1_A : X[U] \to X[U \setminus A]$ (terminating events A)
  lower face map $\delta^0_A : X[U] \to X[U \setminus A]$ ("unstarting" events A)
- Precube identities: $\delta^\mu_A \delta^\nu_B = \delta^\nu_B \delta^\mu_A$ for $A \cap B = \emptyset$ and $\mu, \nu \in \{0, 1\}$

A higher dimensional automaton (HDA) is a precubical set X with initial cells $\bot \subseteq X$ and accepting cells $\top \subseteq X$ (not necessarily vertices)

Example



$$X[\emptyset] = \{v, w, x, y\}$$

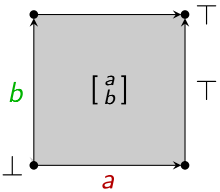
$$X[a] = \{e, f\}$$

$$X[b] = \{g, h\}$$

$$X[[\,{}^{a}_{b}\,]] = \{q\}$$

$$\perp_X = \{v\}$$

$$\top_X = \{h, y\}$$



Higher-Dimensional Automata & Concurrency Theory

HDAs as a model for concurrency:

- points: states
- edges: transitions
- squares, cubes etc.: independency relations / concurrently executing events
- two-dimensional automata $\cong$ asynchronous transition systems [Shields'85] [Bednarczyk'88]

- Introduced in [van Glabbeek'89]
- Generalize all main models of concurrency proposed in the literature
- (event structures; Petri nets; communicating automata; etc.)
- [van Glabbeek'06]: translations from Petri nets
    - individual vs. collective tokens; autoconcurrency or not

Our Contributions

- Update van Glabbeek's translation to our new event-based HDA formalism
- Implement translation in new tool `pn2HDA`
- Extend to inhibitor arcs
- Extend to generalized self-modifying nets
- Continuous effort to make `pn2HDA` less stupid and more efficient ⟵ Philipp

Concurrent Semantics of Petri Nets

Petri net (S, T, F): places S; transitions T;
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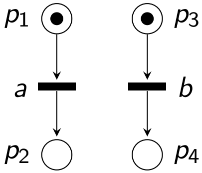

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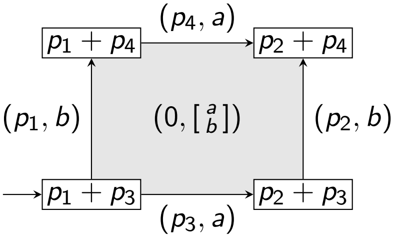
- $E = \{(m, t, m') \mid {}^\bullet t \le m, m' = m - {}^\bullet t + t^\bullet\}$

Concurrent semantics as HDA:
$\square = \square(T)$, $X = \mathbb{N}^S \times \square$, $\mathrm{ev}(m, \tau) = \tau$

- for $x = (m, \tau) \in X[\tau]$ with $\tau = (t_1, \dots, t_n)$:

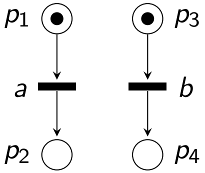
$$\delta^0_{t_i}(x) = (m + {}^\bullet t_i, (t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n))$$
$$\delta^1_{t_i}(x) = (m + t_i^\bullet, (t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n))$$

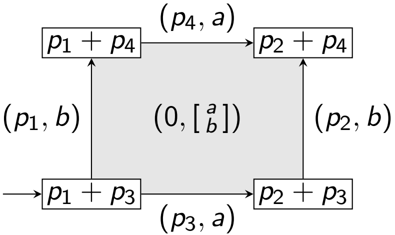
- initial marking $\implies$ initial cell; take reachable part
- (no accepting cells)



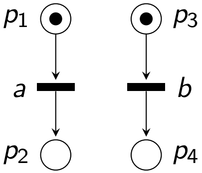
Event Order

- trouble with symmetry:
  have a cell $(0, \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right])$, but also $(0, \left[\begin{smallmatrix} b \\ a \end{smallmatrix}\right])$ (not shown)
- solution: fix an arbitrary order $\preccurlyeq$ on T
- and use $\square = \left\{ \left[\begin{smallmatrix} t_1 \\ \vdots \\ t_n \end{smallmatrix}\right] \;\middle|\; \forall i = 1, \dots, n-1 : t_i \preccurlyeq t_{i+1} \right\}$
  instead of $\square(T)$
- order $\preccurlyeq$ may be chosen (and re-chosen) at will
- here: lexicographic $a \prec b \prec \dots$





Example, Complete



$$
\begin{aligned}
X[\emptyset] &= \{p_1 + p_3, p_2 + p_3, p_1 + p_4, p_2 + p_4\}\\
X[a] &= \{(p_3, a), (p_4, a)\}\\
X[b] &= \{(p_1, b), (p_2, b)\}\\
X[[\begin{smallmatrix}a\\b\end{smallmatrix}]] &= \{(0, [\begin{smallmatrix}a\\b\end{smallmatrix}])\}
\end{aligned}
$$

$$
\begin{tikzcd}
 & p_1+p_4 \arrow[r, "(p_4,a)"] & p_2+p_4 \\
 \arrow[r] & p_1+p_3 \arrow[u, "(p_1,b)"] \arrow[r, "(p_3,a)"'] \arrow[ur, phantom, "(0,[\begin{smallmatrix}a\\b\end{smallmatrix}])"] & p_2+p_3 \arrow[u, "(p_2,b)"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & p_1+p_4 & (p_4,a) \arrow[l, "\delta_a^0"'] \arrow[r, "\delta_a^1"] & p_2+p_4 \\
 & (p_1,b) \arrow[u, "\delta_b^1"] \arrow[d, "\delta_b^0"'] & (0,[\begin{smallmatrix}a\\b\end{smallmatrix}]) \arrow[l, "\delta_a^0"'] \arrow[r, "\delta_a^1"] \arrow[u, "\delta_b^1"] \arrow[d, "\delta_b^0"'] \arrow[ur, "\delta_{ab}^1"] \arrow[dl, "\delta_{ab}^0"'] & (p_2,b) \arrow[u, "\delta_b^1"'] \arrow[d, "\delta_b^0"] \\
 \arrow[r] & p_1+p_3 & (p_3,a) \arrow[l, "\delta_a^0"] \arrow[r, "\delta_a^1"'] & p_2+p_3
\end{tikzcd}
$$

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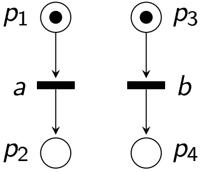
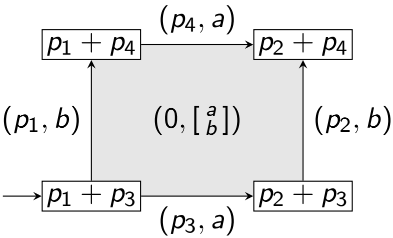
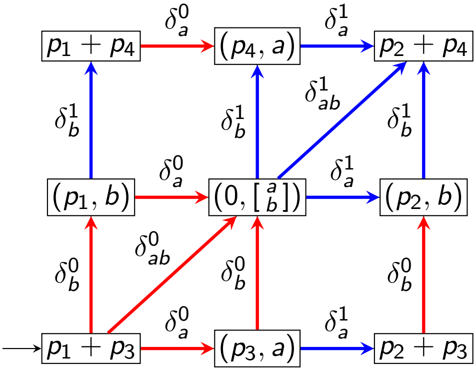




– for computations, *invert* lower face maps



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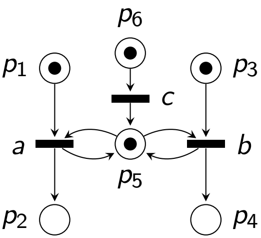

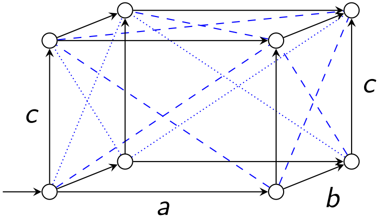

- initially, p_5 is a mutex place: it disables concurrency of a and b
- after c fires, p_5 holds two tokens, so a and b become concurrent
- semantically, a hollow cube without bottom face
- the five faces: front: $(p_3, \left[\begin{smallmatrix} a \\ c \end{smallmatrix}\right])$, back: $(p_4, \left[\begin{smallmatrix} a \\ c \end{smallmatrix}\right])$
  left: $(p_1, \left[\begin{smallmatrix} b \\ c \end{smallmatrix}\right])$, right: $(p_2, \left[\begin{smallmatrix} b \\ c \end{smallmatrix}\right])$
  top: $(0, \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right])$

Working with Petri nets

We wish to build a tool for analyzing and verifying Petri nets based on HDAs.

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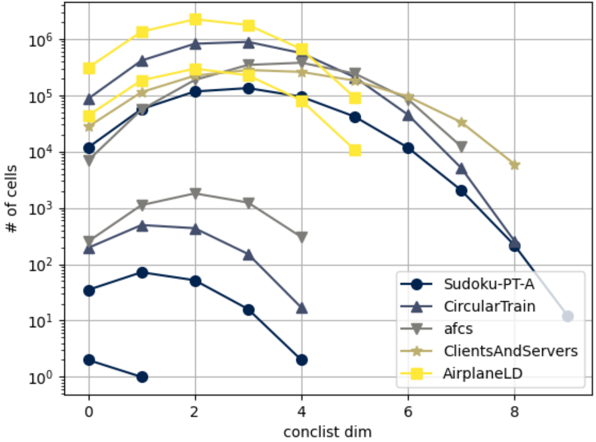
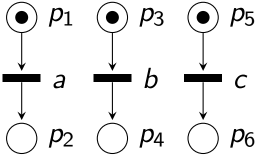


- ⊕ Implementation very close to mathematical definition
- ⊖ Very memory and cpu intensive
- ⊖ Only able to treat small instances from the MCC

Taking a closer look at HDA cells



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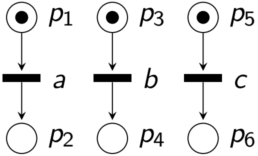
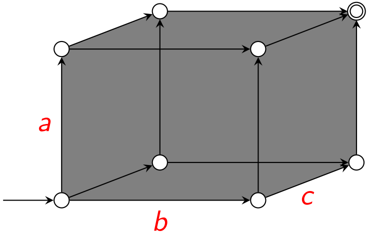
Three independent transitions form a full, three-dimensional cell: $(0, \begin{bmatrix} a \\ b \\ c \end{bmatrix})$.





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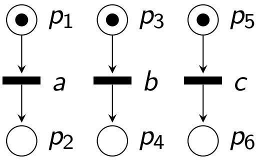
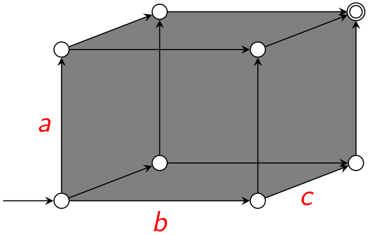
But, in total we have
1 3-cell, 6 2-cells, 12 1-cells, and 8 0-cells.
Plus all of the face-map entries!
($2n$ for n-dim cells)





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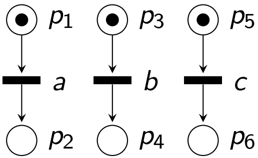
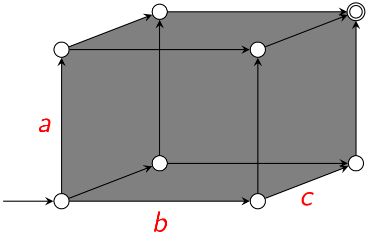
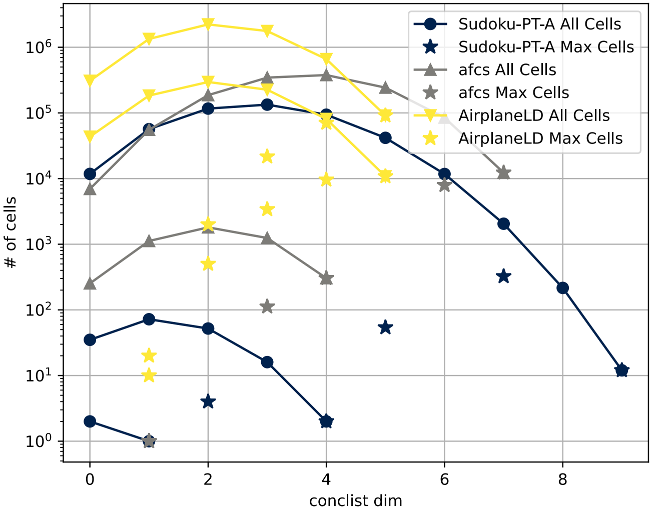
In fact, for a full HDA, any n-dimensional cell has $3^n - 1$ faces, which are guaranteed to exist.

An “exponential” waste!





Maybe it is not that bad in practice?



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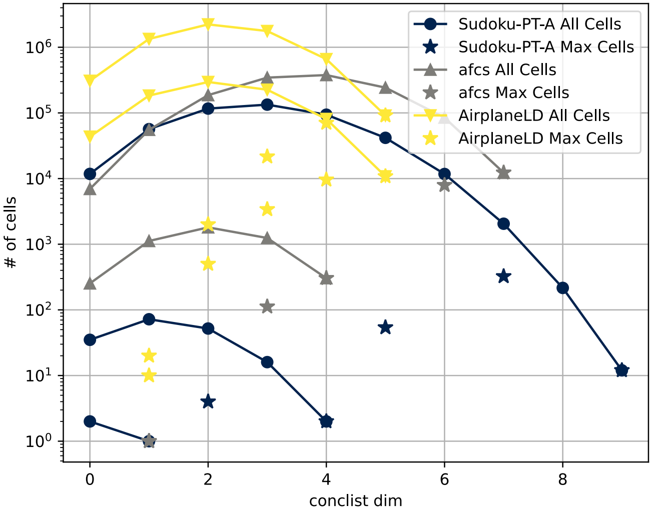

instance	compr.
AirplaneLD-pt-0010	97.1%
AirplaneLD-pt-0020	97.1%
afcs_01_a	91.2%
afcs_02_a	98.5%
Sudoku-PT-A-N01	66.7%
Sudoku-PT-A-N02	96.6%
Sudoku-PT-A-N03	99.2%

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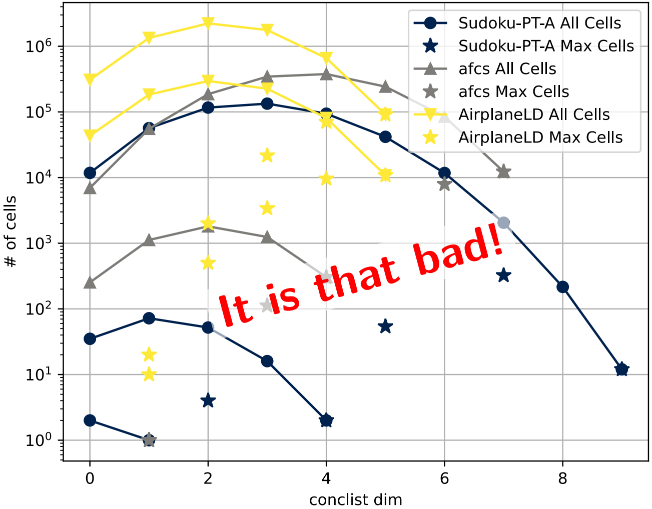

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Extending pn2HDA

We are currently working on extending pn2HDA to support max-cells.

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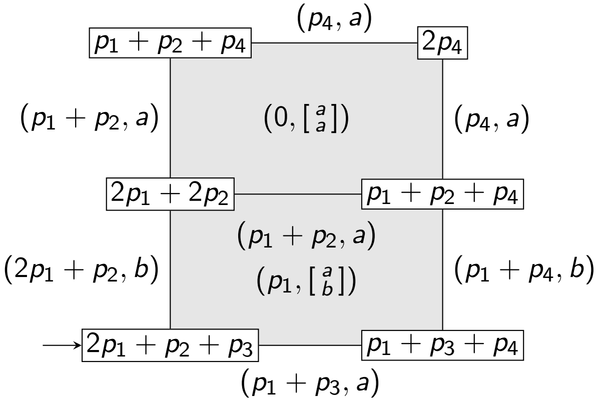
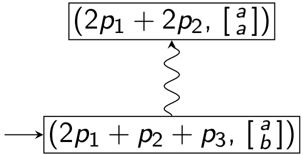
We strive for a model with minimal memory consumption, possibly at the cost of speed.



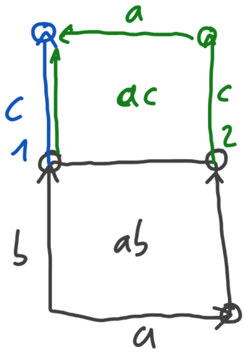


- Keep "lower-left"-marking and maximal conclist
- Keep which cells are connected (but not how?)

Direct Translation of Petri Nets to MHDA - 1

Given a partially constructed MHDA A and a max cell (m, c) to explore

- Compute the marking for each 0-cell
- Compute the set of maximal concurrent steps
- For each of them form the candidate cell (m', c')
- Check inclusion with all existing cells (m_e, c_e)
  - $(m', c') \sqsubseteq (m_e, c_e)$
  - $(m_e, c_e) \sqsubseteq (m', c')$



Direct Translation of Petri Nets to MHDA - 2

Inclusion checking is translated to an integer program, solved via z3.
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	HDA			MHDA			time (s)	
Name	cells	conclists	markings	cells	conclists	markings	PN $\to$ HDA	PN $\to$ MHDA
abx_3	272	50	146	10	10	4	0.003	0.7
abx_4	846	105	371	15	15	5	0.005	4
abx_5	2232	196	812	21	21	6	0.008	13
abx_6	5214	336	1596	28	28	7	0.01	44
Sudoku-PT-A-N02	177	35	176	6	23	5	0.4	19

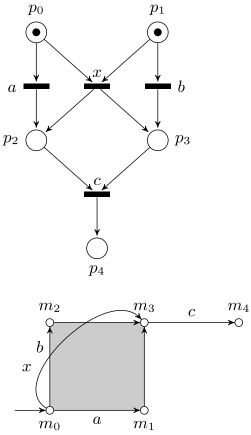
Problem Analysis

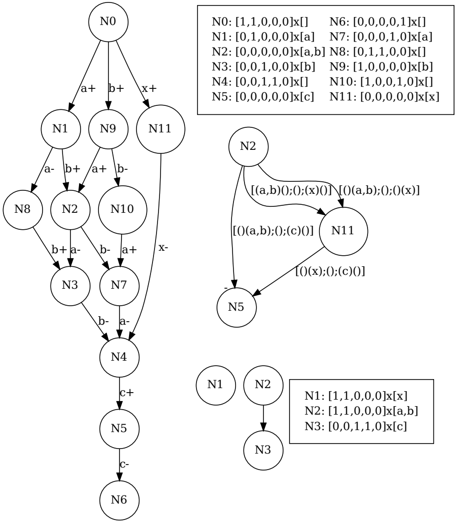
Why is it sooo slow??

What are the expensive parts?

- Investigate from all 0-cells ($2^{|C|}$)
- Finding (maximal) concurrent steps from a 0-cell (NP)
- Inclusion checks (linear in current size, but costly)

Intermezzo: Presenting abx





Simplifying Inclusion Checks

Integer programming is generally expensive

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Luckily our problem for $(m, c) \sqsubseteq (m', c')$ looks like this

$$m_i = m'_i - \sum_t \mathtt{pre}(i, t) n_t + \sum_t \mathtt{post}(i, t) n_t$$
$$0 \leq n_t \leq |c' \setminus c|$$

with $\mathtt{pre}(i, t)/\mathtt{post}(i, t)$ being 1 if $i \in {}^\bullet t / t^\bullet$ else 0

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What about non-integer solutions?
Claim:
They only arise for parallel transitions and imply the existence of an integer solution.

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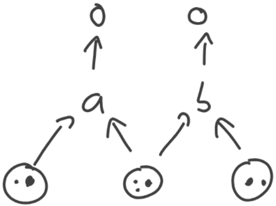
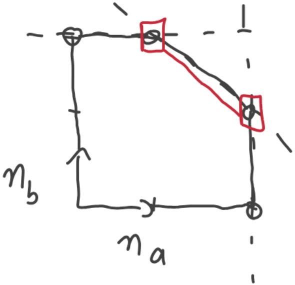
Even better, the polytope is bounded and all the coefficients are integer (even even better they are 0 or 1)!

Why does it matter? # 2





We can find the V-representation!
The pareto optimal vertices are those of interest for max-cell successors!

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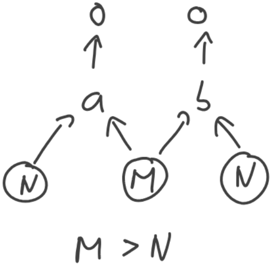

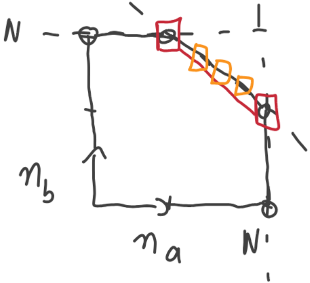

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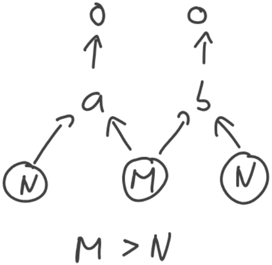

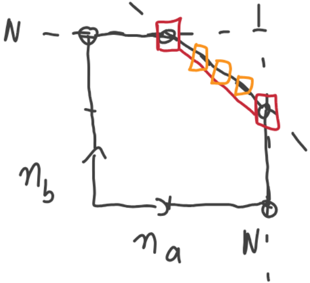

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Going from H- to V-representation is independent of N and M.
Unfortunately. Problems enumerating the integer solutions in higher dimensions.

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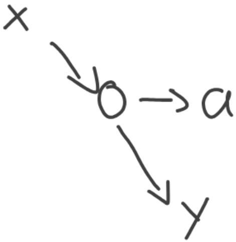

- $x^{-}a^{+}$
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Directions only *demand* at most one token per place.
We only construct maximal directions.

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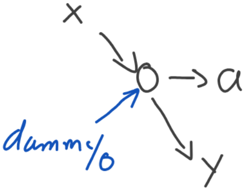

- $x^- a^+$
- $y^{-1} a^+$
- $dummy_0^- a^+$

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Why does it matter? # 3

This is once again a (bounded,...) polytope.

$$|c|_t \geq \sum_d \texttt{req}(d,t)n_d$$

with $\texttt{req}(d,t)$ being 1 if the direction d unstarts or terminates transition t, 2 if it unstarts AND terminates t and is 0 otherwise.

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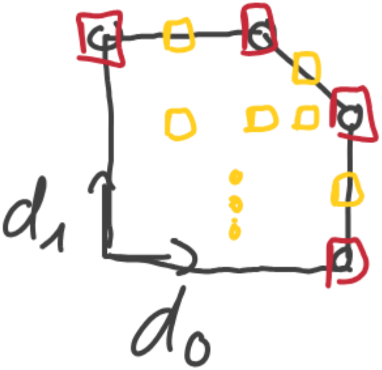

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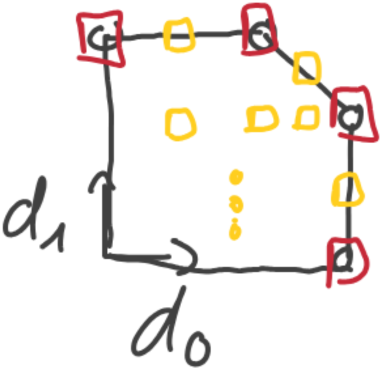
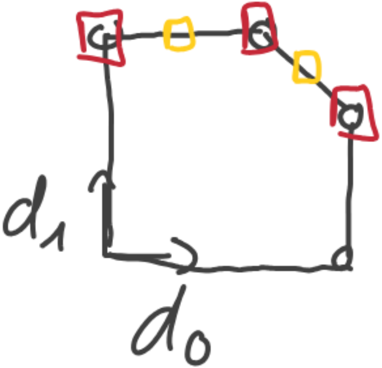


Here, all integer solutions are of interest. - Not only the pareto optimal ones

Why does it matter? # 3



Well except those who only terminate dummy transitions. $dummy_0^- \, dummy_1^- \, a^+$

Conclusion and Future Work

- HDAs provide detailed semantics for concurrent systems and Petri Nets.
- Fixing the event-order only solves half of the problem
  What is actually needed is to switch to anonymous HDAs.
  As for Petri Nets, the different events have no identity and can be interchanged.

- Improve data structures used for max-cell HDAs
- Improve algorithms
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Thank you for your attention!