Elements of Higher-Dimensional Automata Theory

Uli Fahrenberg

EPITA Research Laboratory (LRE), Rennes/Paris, France

Dagstuhl 25141, April 2025

Reading Material

- Languages of Higher-Dimensional Automata [MSCS 2021]
- Posets with Interfaces as a Model for Concurrency [I&C 2022]
- Kleene Theorem for Higher-Dimensional Automata [LMCS 2024]
- Myhill-Nerode Theorem for Higher-Dimensional Automata [FI 2024]
- Decision and Closure Properties for Higher-Dimensional Automata [TCS 2025]

- Languages of Higher-Dimensional Timed Automata [Petri Nets 2024]
- Presenting Interval Pomsets with Interfaces [RAMiCS 2024]
- Logic and Languages of Higher-Dimensional Automata [DLT 2024]
- Bisimulations and Logics for Higher-Dimensional Automata [ICTAC 2024]
- Petri Nets and Higher-Dimensional Automata [Petri Nets 2025]

Nice People

- Eric Goubault, Paris
- Christian Johansen, Oslo
- Georg Struth, Sheffield
- Krzysztof Ziemiański, Warsaw

- Amazigh Amrane, Hugo Bazille, Emily Clement, Jérémy Dubut, Marie Fortin, Loïc Hélouët, Jérémy Ledent, Philipp Schlehuber-Caissier, Safa Zouari, . . .

- See also `https://ulifahrenberg.github.io/pomsetproject/`

Higher-Dimensional Automata

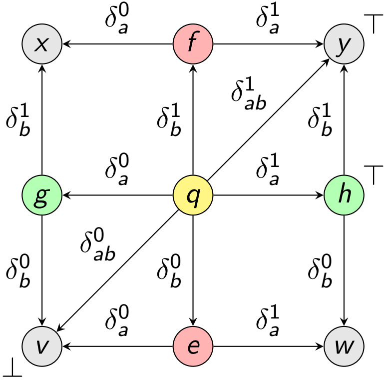
A conclist is a finite, totally ordered, Σ-labeled set. (a list of labeled events)

A precubical set X consists of:

- A set of cells X (cubes)
- Every cell $x \in X$ has a conclist $\mathsf{ev}(x)$ (list of events active in x)
- We write $X[U] = \{x \in X \mid \mathsf{ev}(x) = U\}$ for a conclist U (cells of type U)
- For every conclist U and $A \subseteq U$ there are:
  upper face map $\delta^1_A : X[U] \to X[U \setminus A]$ (terminating events A)
  lower face map $\delta^0_A : X[U] \to X[U \setminus A]$ ("unstarting" events A)
- Precube identities: $\delta^\mu_A \delta^\nu_B = \delta^\nu_B \delta^\mu_A$ for $A \cap B = \emptyset$ and $\mu, \nu \in \{0, 1\}$

A higher dimensional automaton (HDA) is a precubical set X with initial cells $\bot \subseteq X$ and accepting cells $\top \subseteq X$ (not necessarily vertices)

Example



$$X[\emptyset] = \{v, w, x, y\}$$

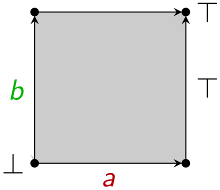
$$X[a] = \{e, f\}$$

$$X[b] = \{g, h\}$$

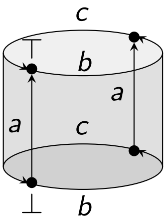
$$X[[\,{}^{a}_{b}\,]] = \{q\}$$

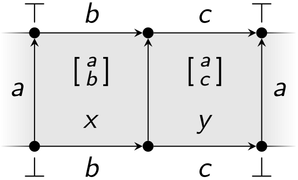
$$\bot_X = \{v\}$$

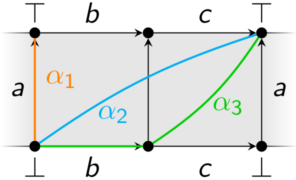
$$\top_X = \{h, y\}$$



Another One

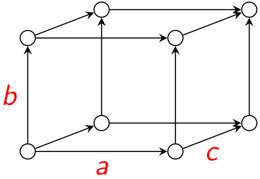




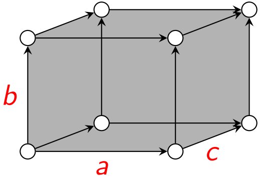


$$a \parallel (bc)^*$$

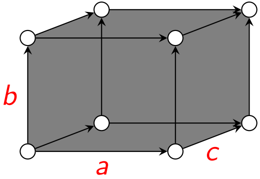
More Examples



no concurrency



two out of three



full concurrency

Higher-Dimensional Automata

A conclist is a finite, totally ordered, Σ-labeled set. (a list of labeled events)

A precubical set X consists of:

- A set of cells X (cubes)
- Every cell $x \in X$ has a conclist $\text{ev}(x)$ (list of events active in x)
- We write $X[U] = \{x \in X \mid \text{ev}(x) = U\}$ for a conclist U (cells of type U)
- For every conclist U and $A \subseteq U$ there are:
  upper face map $\delta^1_A : X[U] \to X[U \setminus A]$ (terminating events A)
  lower face map $\delta^0_A : X[U] \to X[U \setminus A]$ ("unstarting" events A)
- Precube identities: $\delta^\mu_A \delta^\nu_B = \delta^\nu_B \delta^\mu_A$ for $A \cap B = \emptyset$ and $\mu, \nu \in \{0, 1\}$

A higher dimensional automaton (HDA) is a precubical set X with initial cells $\bot \subseteq X$ and accepting cells $\top \subseteq X$ (not necessarily vertices)

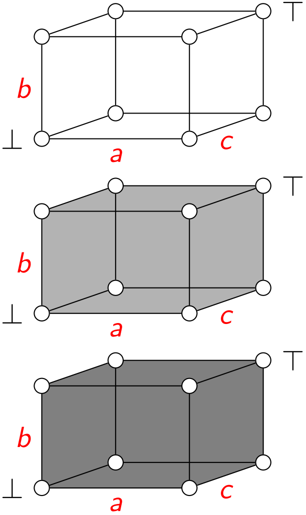
Higher-Dimensional Automata & Concurrency Theory

HDAs as a model for concurrency:

- points: states
- edges: transitions
- squares, cubes etc.: independency relations / concurrently executing events
- two-dimensional automata $\cong$ asynchronous transition systems

- Introduced in 1990
- Generalize all main models of concurrency proposed in the literature
- (event structures; Petri nets; communicating automata; etc.)

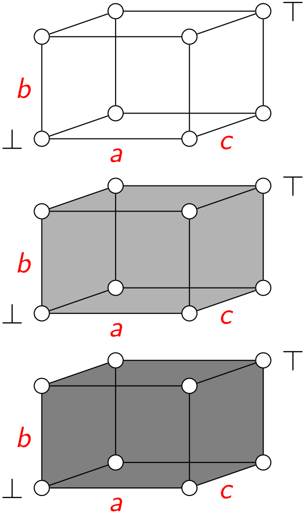
Languages of HDAs: Examples



$$L_1 = \{abc, acb, bac, bca, cab, cba\}$$

$$L_3 = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \dots \right\}$$

Languages of HDAs: Examples



$$L_1 = \{abc, acb, bac, bca, cab, cba\}$$

$$L_2 = \left\{ \begin{bmatrix} a \\ b \to c \end{bmatrix}, \begin{bmatrix} a \\ c \to b \end{bmatrix}, \begin{bmatrix} b \\ a \to c \end{bmatrix}, \begin{bmatrix} b \\ c \to a \end{bmatrix}, \begin{bmatrix} c \\ a \to b \end{bmatrix}, \begin{bmatrix} c \\ b \to a \end{bmatrix} \right\} \cup L_1 \cup \dots$$

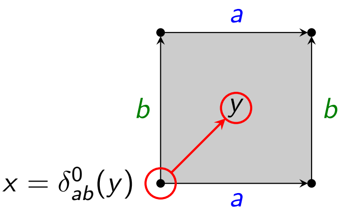
sets of pomsets

$$L_3 = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right\} \cup L_2$$

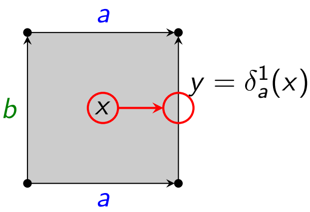
Computations of HDAs

An HDA computes by starting and terminating events in sequence:

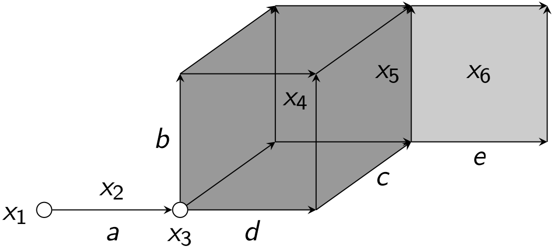
upstep $x \nearrow y$, starting $[\,{}^{a}_{b}\,]$:



downstep $x \searrow y$, terminating a:

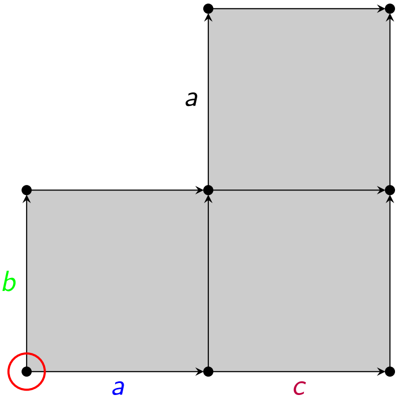


Example

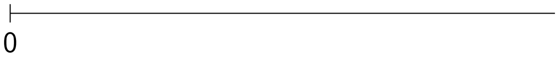


$$\big(x_1 \nearrow^{a} x_2 \searrow_{a} x_3 \nearrow^{\{b,c,d\}} x_4 \searrow_{\{c,d\}} x_5 \nearrow^{e} x_6\big)$$

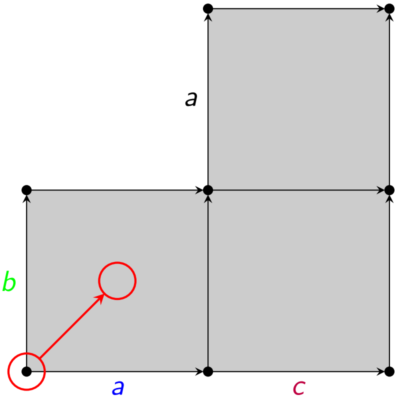
Event Ipomset of a Path



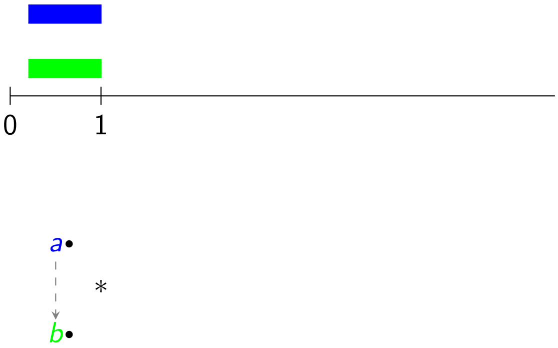
Lifetimes of events



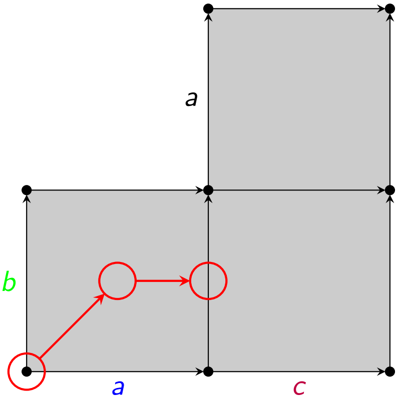
Event Ipomset of a Path



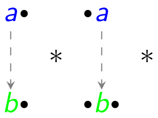
Lifetimes of events



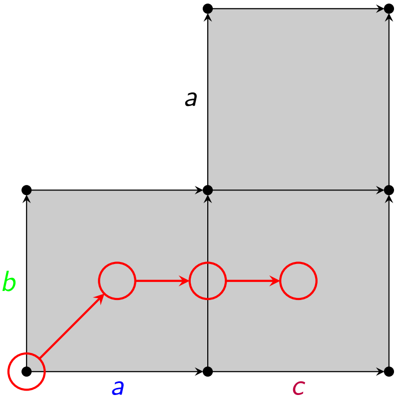
Event Ipomset of a Path



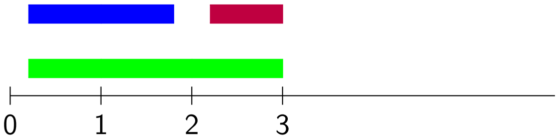
Lifetimes of events

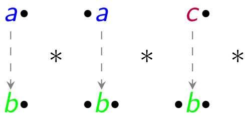


Event Ipomset of a Path

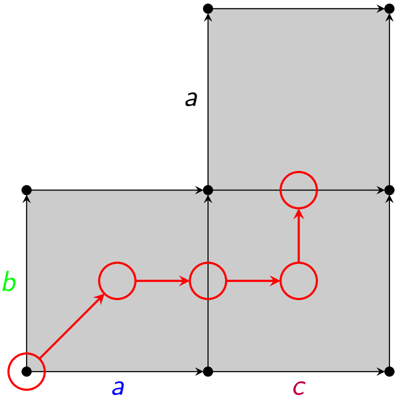


Lifetimes of events

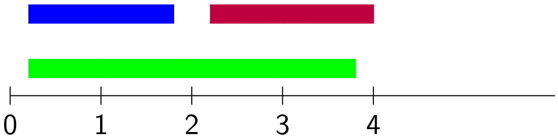


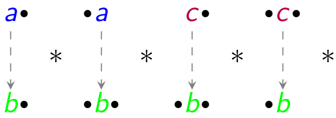


Event Ipomset of a Path

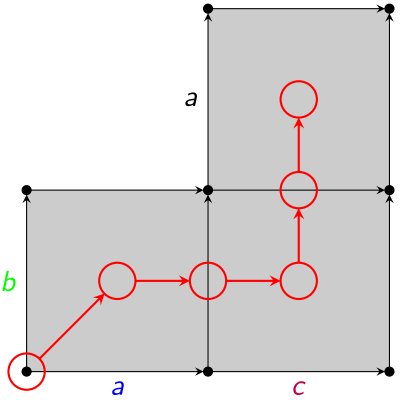


Lifetimes of events

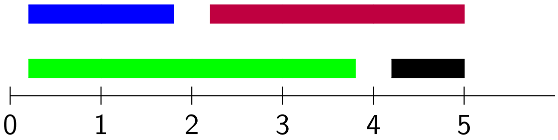


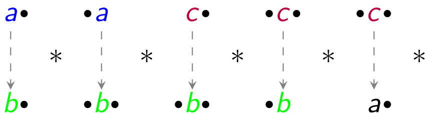


Event Ipomset of a Path

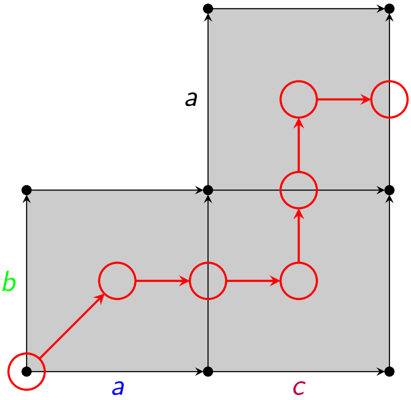


Lifetimes of events

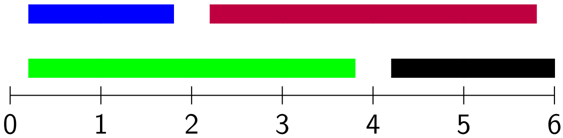


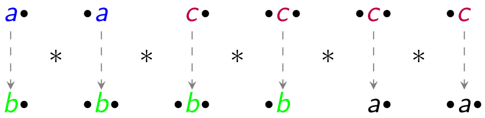


Event Ipomset of a Path

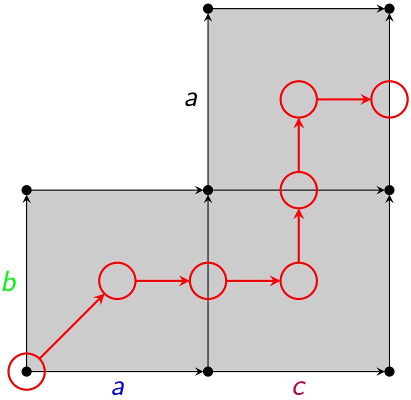


Lifetimes of events

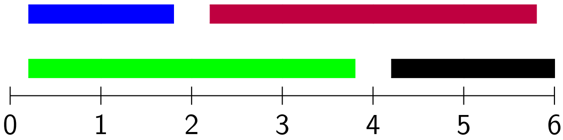




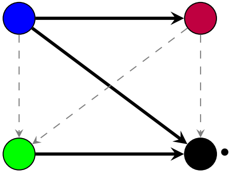
Event Ipomset of a Path



Lifetimes of events



Event ipomset



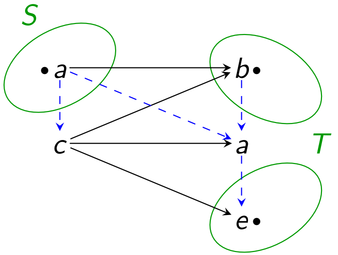
(not series-parallel!)

Pomsets with Interfaces

Definition

A pomset with interfaces (ipomset): $(P, <, \dashrightarrow, S, T, \lambda)$:

- P finite set of events, $\lambda : P \to \Sigma$
- two partial orders $<$ (precedence order), $\dashrightarrow$ (event order)
  - s.t. $< \cup \dashrightarrow$ is a total relation;
- $S, T \subseteq P$ source and target interfaces
  - s.t. S is $<$-minimal and T is $<$-maximal.

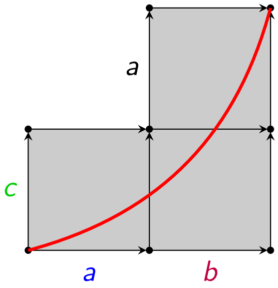


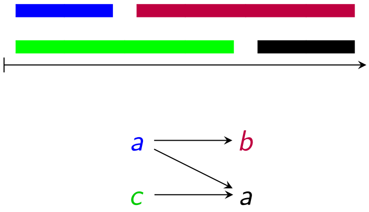
Interval Orders

Definition

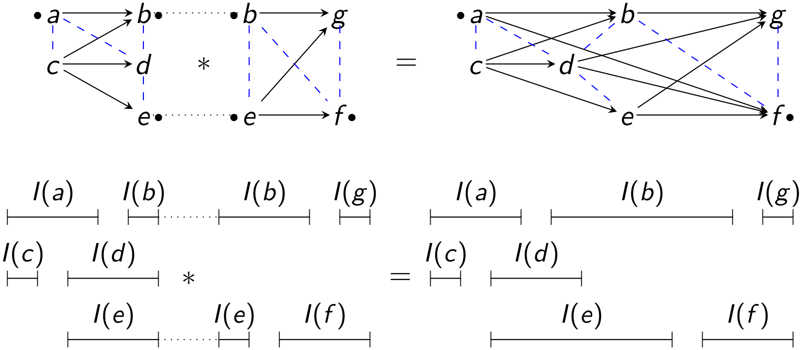
An ipomset $(P, <_P, \dashrightarrow, S, T, \lambda)$ is interval if $(P, <_P)$ has an interval representation: functions $b, e : P \to \mathbb{R}$ s.t.

- $\forall x \in P : b(x) \leq_{\mathbb{R}} e(x)$;
- $\forall x, y \in P : e(x) <_{\mathbb{R}} b(y) \iff x <_P y$



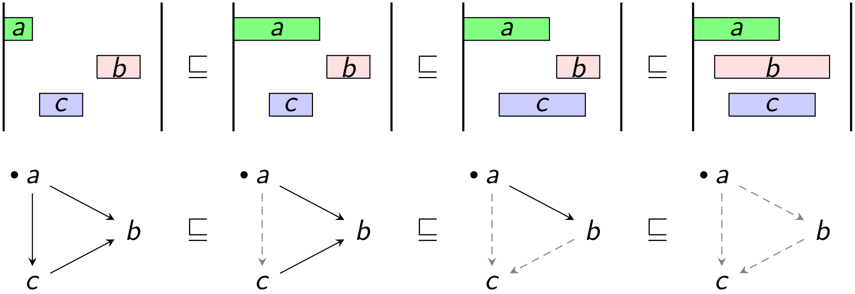


Gluing Composition



- Gluing $P * Q$: P before Q, except for interfaces (which are identified)
- (also have parallel composition $P \parallel Q$: disjoint union)

Subsumption



P refines Q / Q subsumes P / $P \sqsubseteq Q$ iff

- P and Q have same interfaces
- P has more $<$ than Q
- Q has more $\dashrightarrow$ than P

Languages of HDAs

Definition

The language of an HDA X is the set of event ipomsets of all accepting paths:

$$L(X) = \{\mathsf{ev}(\pi) \mid \pi \in \mathsf{Paths}(X), \mathsf{src}(\pi) \in \bot_X, \mathsf{tgt}(\pi) \in \top_X\}$$

- $L(X)$ contains only interval ipomsets,
- is closed under subsumption,
- and has finite width

Definition

A language $L \subseteq \mathsf{iiPoms}$ is regular if there is an HDA X with $L = L(X)$.

Theorems

Definition (Rational Languages over Σ)

- Generated by $\emptyset$, $\{\epsilon\}$, and all $\{[a]\}$, $\{[\bullet\, a]\}$, $\{[a\, \bullet]\}$, $\{[\bullet\, a\, \bullet]\}$ for $a \in \Sigma$
- under operations $\cup$, $*$, $\parallel$ and (Kleene plus) $^+$
- (these need to take subsumption closure into account)

Definition (Monadic Second-Order Logics over Ipomsets)

$$\psi ::= a(x) \mid \mathsf{s}(x) \mid \mathsf{t}(x) \mid x < y \mid x \dashrightarrow y \mid x \in X \mid \\ \exists x.\, \psi \mid \forall x.\, \psi \mid \exists X.\, \psi \mid \forall X.\, \psi \mid \psi_1 \wedge \psi_2 \mid \psi_1 \vee \psi_2 \mid \neg \psi$$

Theorem (à la Kleene): regular $\iff$ rational

Theorem (à la Myhill-Nerode): regular $\iff$ finite prefix quotient

Theorem (à la Büchi-Elgot-Trakhtenbrot): [DLT 2024]

regular $\iff$ MSO-definable, of finite width, and subsumption-closed

Special ipomsets

Definition

An ipomset $(P, <, \dashrightarrow, S, T, \lambda)$ is

- discrete if $<$ is empty (hence $\dashrightarrow$ is total) $\begin{bmatrix}\bullet a\\ \bullet b\bullet\\ a\bullet\end{bmatrix}$
  - also written ${}_S P_T$
- a conclist ("concurrency list") if it is discrete and $S = T = \emptyset$ $\begin{bmatrix}a\\ b\\ a\end{bmatrix}$
- a starter if it is discrete and $T = P$ $\begin{bmatrix}\bullet a\bullet\\ \bullet b\bullet\\ a\bullet\end{bmatrix}$
- a terminator if it is discrete and $S = P$ $\begin{bmatrix}\bullet a\\ \bullet b\bullet\\ \bullet a\bullet\end{bmatrix}$
- an identity if it is both a starter and a terminator $\begin{bmatrix}\bullet a\bullet\\ \bullet b\bullet\\ \bullet a\bullet\end{bmatrix}$

Lemma (Janicki-Koutny 93; reformulated)

An ipomset is interval iff it has a decomposition into discrete ipomsets.

Decompositions

Lemma (Janicki-Koutny 93)

A poset $(P, <)$ is an interval order iff the order defined on its maximal antichains defined by $A \preceq B \iff \forall a \in A, b \in B : b \not< a$ is total.

Corollary

An ipomset is interval iff it has a decomposition into discrete ipomsets.

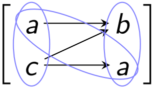
Lemma

Any discrete ipomset is a gluing of a starter and a terminator.

$$\begin{bmatrix} \bullet a \\ \bullet b \bullet \\ a \bullet \end{bmatrix} = \begin{bmatrix} \bullet a \bullet \\ \bullet b \bullet \\ a \bullet \end{bmatrix} * \begin{bmatrix} \bullet a \\ \bullet b \bullet \\ \bullet a \bullet \end{bmatrix}$$

Corollary

Any interval ipomset has a decomposition as a sequence of starters and terminators.



$$= \begin{bmatrix} a \bullet \\ c \end{bmatrix} * \begin{bmatrix} \bullet a \\ a \bullet \end{bmatrix} * \begin{bmatrix} b \\ \bullet a \end{bmatrix} = \begin{bmatrix} a \bullet \\ c \bullet \end{bmatrix} * \begin{bmatrix} \bullet a \bullet \\ \bullet c \end{bmatrix} * \begin{bmatrix} \bullet a \bullet \\ a \bullet \end{bmatrix} * \begin{bmatrix} \bullet a \\ \bullet a \bullet \end{bmatrix} * \begin{bmatrix} b \bullet \\ \bullet a \bullet \end{bmatrix} * \begin{bmatrix} \bullet b \\ \bullet a \end{bmatrix}$$

Unique decompositions

Notation: St: set of starters ${}_S U_U$
Te: set of terminators ${}_U U_T$
$\text{Id} = \text{St} \cap \text{Te}$: set of identities ${}_U U_U$
$\Omega = \text{St} \cup \text{Te}$

Definition

A word $w = (S_1, U_1, T_1) \dots (S_n, U_n, T_n) \in \Omega^+$ is coherent if $T_i = S_{i+1}$ for all i.

Definition

A coherent word is sparse if proper starters and proper terminators are alternating.

- additionally, all $w \in \text{Id} \subseteq \Omega^+$ are sparse

Lemma

Any interval ipomset P has a unique decomposition $P = P_1 * \dots * P_n$ such that $P_1 \dots P_n \in \Omega^+$ is sparse.

Step sequences

Let $\sim$ be the congruence on Ω^+ generated by the relation

$$ {}_S U_U \cdot {}_U T_T \sim {}_S T_T \qquad {}_S S_U \cdot {}_U U_T \sim {}_S S_T $$

- compose subsequent starters and subsequent terminators

Definition

A step sequence is a $\sim$-equivalence class of coherent words in Ω^+.

Lemma

Any step sequence has a unique sparse representant.

Theorem

The category of interval ipomsets is isomorphic to the category of step sequences.

Categories?

Definition (Category iiPoms)

objects: conclists U (discrete ipomsets without interfaces)

morphisms in iiPoms(U, V): interval ipomsets P with sources U and targets V

composition: gluing

identities $\mathrm{id}_U = {}_U U_U$

Definition (Category SSeq)

objects: conclists U (discrete ipomsets without interfaces)

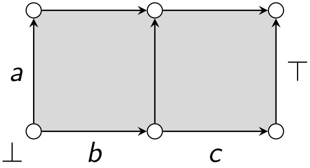
morphisms in SSeq(U, V): step sequences $[(S_1, U_1, T_1) \dots (S_n, U_n, T_n)]_\sim$ with $S_1 = U$ and $T_n = V$

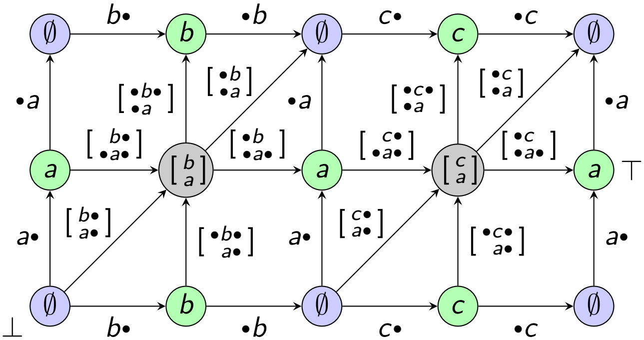
composition: concatenation

identities $\mathrm{id}_U = {}_U U_U$

- SSeq is category generated from (directed multi)graph Ω under relations $\sim$
- isomorphisms Φ : iiPoms $\leftrightarrow$ SSeq : Ψ provided by
  - $\Phi(P) = [w]_\sim$, where w is any step decomposition of P;
  - $\Psi([P_1 \dots P_n]_\sim) = P_1 * \dots * P_n$

ST-automata





- The operational semantics of an HDA $(X, \bot, \top, \Sigma)$ is the "ST-automaton" with states X, alphabet Ω, state labeling $\mathsf{ev} : X \to \square$, and transitions

$$E = \{\delta^0_A(\ell) \xrightarrow{{}_A\!\uparrow\mathsf{ev}(\ell)} \ell \mid A \subseteq \mathsf{ev}(\ell)\} \cup \{\ell \xrightarrow{\mathsf{ev}(\ell)\downarrow_A} \delta^1_A(\ell) \mid A \subseteq \mathsf{ev}(\ell)\}.$$

- An automaton on graph alphabet Ω

Properties

- alphabet Σ
- St: set of starters ${}_S U_U$ on Σ; Te: set of terminators ${}_U U_T$ on Σ; $\Omega = \text{St} \cup \text{Te}$
- (equivalently: St and Te are marked inclusions $\Sigma^* \hookrightarrow \Sigma^*$)
- ST-automaton: automaton on (infinite) graph alphabet Ω
- language of ST-automaton: subset of (morphisms of) SSeq
- SSeq: category generated from Ω under relation $\sim$ (not free)
- Kleene theorem? regular $\iff$ generated from Ω using $.$, $\cup$ and *
- BET-König theorem? "localization" of MSO on SSeq to Ω
- generalization?

Bibliography

Currently best intro to HDAs:

- UF, K.Ziemiański: *Myhill-Nerode Theorem for Higher-Dimensional Automata* [FI 2024]

HDAs and Petri nets:

- A.Amrane, H.Bazille, UF, L.Hélouët, P.Schlehuber-Caissier: *Petri Nets and Higher-Dimensional Automata* [Petri Nets 2025]

MSO for HDAs:

- A.Amrane, H.Bazille, UF, M.Fortin: *Logic and Languages of Higher-Dimensional Automata* [DLT 2024]

This talk mostly based on:

- A.Amrane, H.Bazille, E.Clement, UF, K.Ziemiański: *Presenting Interval Pomsets with Interfaces* [RAMiCS 2024]