Higher-Dimensional Automata: Operational Semantics

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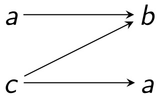
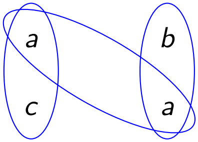
Motivation: Understand Algebraic Structure of Interval Orders

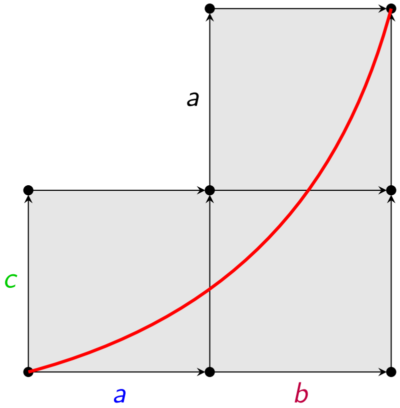
- non-interleaving concurrency:
- languages consist of pomsets instead of words
- (partially ordered multisets)
- but not *all* pomsets: only interval orders
- (elements can be represented as real intervals)
- Janicki-Koutny 1993 (TCS): represent interval orders as sequences of overlapping maximal antichains
- use that to understand algebra of interval pomsets
- with an application to higher-dimensional automata



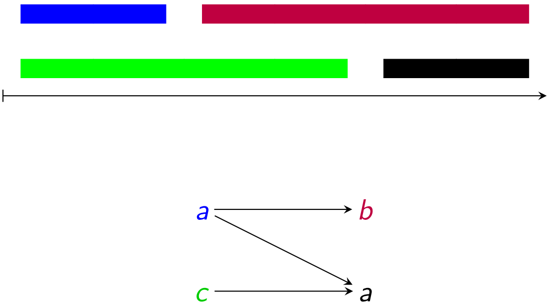


$$\begin{bmatrix} a\bullet \\ c \end{bmatrix} \begin{bmatrix} \bullet a \\ a\bullet \end{bmatrix} \begin{bmatrix} b \\ \bullet a \end{bmatrix}$$

Example



Lifetimes of events



1. Motivation: Understand Algebraic Structure of Interval Orders
2. Pomsets with Interfaces
3. Application: Higher-Dimensional Automata

Nice People

- Eric Goubault, Paris
- Christian Johansen, Oslo
- Georg Struth, Sheffield
- Krzysztof Ziemiański, Warsaw

- Amazigh Amrane, Hugo Bazille, Emily Clement, Jérémy Dubut, Marie Fortin, Loïc Hélouët, Jérémy Ledent, Philipp Schlehuber-Caissier, Safa Zouari, . . .

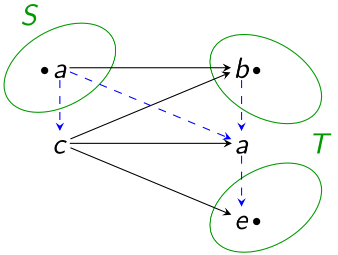
- See also https://ulifahrenberg.github.io/pomsetproject/

Pomsets with interfaces

Definition

A pomset with interfaces (ipomset): $(P, <, \dashrightarrow, S, T, \lambda)$:

- finite set P;
- two partial orders $<$ (precedence order), $\dashrightarrow$ (event order)
  - s.t. $< \cup \dashrightarrow$ is a *total relation*;
- $S, T \subseteq P$ source and target interfaces
  - s.t. S is $<$-minimal and T is $<$-maximal.

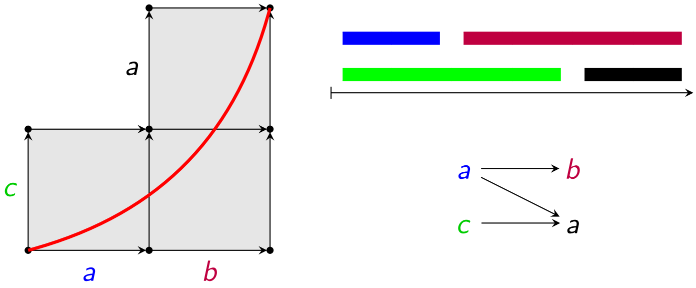


Interval orders

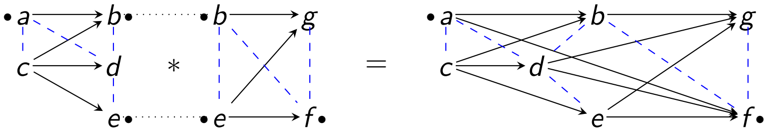
Definition

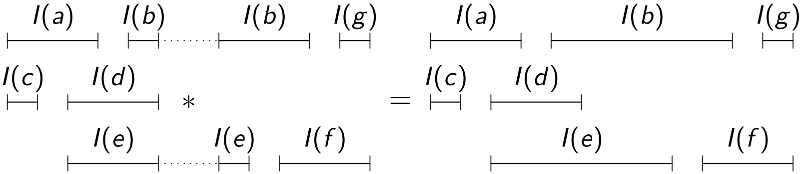
An ipomset $(P, <_P, \dashrightarrow, S, T, \lambda)$ is interval if $(P, <_P)$ has an interval representation: functions $b, e : P \to \mathbb{R}$ s.t.

- $\forall x \in P : b(x) \leq_{\mathbb{R}} e(x)$;
- $\forall x, y \in P : e(x) <_{\mathbb{R}} b(y) \iff x <_P y$



Gluing composition





- Gluing $P * Q$: P before Q, except for interfaces (which are identified)
- (also have parallel composition $P \parallel Q$: disjoint union)

Special ipomsets

Definition

An ipomset $(P, <, \dashrightarrow, S, T, \lambda)$ is

- discrete if $<$ is empty (hence $\dashrightarrow$ is total) $\begin{bmatrix} \bullet a \\ \bullet b \bullet \\ a \bullet \end{bmatrix}$
  - also written ${}_S P_T$
- a conclist ("concurrency list") if it is discrete and $S = T = \emptyset$ $\begin{bmatrix} a \\ b \\ a \end{bmatrix}$
- a starter if it is discrete and $T = P$ $\begin{bmatrix} \bullet a \bullet \\ \bullet b \bullet \\ a \bullet \end{bmatrix}$
- a terminator if it is discrete and $S = P$ $\begin{bmatrix} \bullet a \\ \bullet b \bullet \\ \bullet a \bullet \end{bmatrix}$
- an identity if it is both a starter and a terminator $\begin{bmatrix} \bullet a \bullet \\ \bullet b \bullet \\ \bullet a \bullet \end{bmatrix}$

Lemma (Janicki-Koutny 93; reformulated)

An ipomset is interval iff it has a decomposition into discrete ipomsets.

Decompositions

Lemma (Janicki-Koutny 93)

A poset $(P, <)$ is an interval order iff the order defined on its maximal antichains defined by $A \preceq B \iff \forall a \in A, b \in B : b \not< a$ is total.

Corollary

An ipomset is interval iff it has a decomposition into discrete ipomsets.

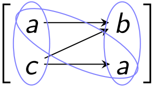
Lemma

Any discrete ipomset is a gluing of a starter and a terminator.

$$\begin{bmatrix} \bullet a \\ \bullet b \bullet \\ a \bullet \end{bmatrix} = \begin{bmatrix} \bullet a \bullet \\ \bullet b \bullet \\ a \bullet \end{bmatrix} * \begin{bmatrix} \bullet a \\ \bullet b \bullet \\ \bullet a \bullet \end{bmatrix}$$

Corollary

Any interval ipomset has a decomposition as a sequence of starters and terminators.



$$= \begin{bmatrix} a \bullet \\ c \end{bmatrix} * \begin{bmatrix} \bullet a \\ a \bullet \end{bmatrix} * \begin{bmatrix} b \\ \bullet a \end{bmatrix} = \begin{bmatrix} a \bullet \\ c \bullet \end{bmatrix} * \begin{bmatrix} \bullet a \bullet \\ \bullet c \end{bmatrix} * \begin{bmatrix} \bullet a \bullet \\ a \bullet \end{bmatrix} * \begin{bmatrix} \bullet a \\ \bullet a \bullet \end{bmatrix} * \begin{bmatrix} b \bullet \\ \bullet a \bullet \end{bmatrix} * \begin{bmatrix} \bullet b \\ \bullet a \end{bmatrix}$$

Unique decompositions

Notation: St: set of starters ${}_S U_U$
Te: set of terminators ${}_U U_T$
$\text{Id} = \text{St} \cap \text{Te}$: set of identities ${}_U U_U$
$\Omega = \text{St} \cup \text{Te}$

Definition

A word $w = (S_1, U_1, T_1) \dots (S_n, U_n, T_n) \in \Omega^+$ is coherent if $T_i = S_{i+1}$ for all i.

Definition

A coherent word is sparse if proper starters and proper terminators are alternating.

- additionally, all $w \in \text{Id} \subseteq \Omega^+$ are sparse
- so that's $\text{Id} \cup (\text{St} \setminus \text{Id})\big((\text{Te} \setminus \text{Id})(\text{St} \setminus \text{Id})\big)^* \cup (\text{Te} \setminus \text{Id})\big((\text{St} \setminus \text{Id})(\text{Te} \setminus \text{Id})\big)^*$

Lemma

Any interval ipomset P has a unique decomposition $P = P_1 * \cdots * P_n$ such that $P_1 \dots P_n \in \Omega^+$ is sparse.

Step sequences

Let $\sim$ be the congruence on Ω^+ generated by the relation

$$ {}_S U_U \cdot {}_U T_T \sim {}_S T_T \qquad {}_S S_U \cdot {}_U U_T \sim {}_S S_T $$

- (compose subsequent starters and subsequent terminators)

Definition

A step sequence is a $\sim$-equivalence class of coherent words in Ω^+.

Lemma

Any step sequence has a unique sparse representant.

Theorem

The category of interval ipomsets is isomorphic to the category of step sequences.

Categories?

Definition (Category iiPoms)

objects: conclists U (discrete ipomsets without interfaces)

morphisms in $\mathsf{iiPoms}(U, V)$: interval ipomsets P with sources U and targets V

composition: gluing

identities $\mathrm{id}_U = {}_U U_U$

Definition (Category Coh)

objects: conclists U (discrete ipomsets without interfaces)

morphisms in $\mathsf{Coh}(U, V)$: step sequences $\big[(S_1, U_1, T_1) \dots (S_n, U_n, T_n)\big]_\sim$ with $S_1 = U$ and $T_n = V$

composition: concatenation

identities $\mathrm{id}_U = {}_U U_U$

- Coh is category generated from (directed multi)graph Ω under relations $\sim$
- isomorphisms $\Phi : \mathsf{iiPoms} \leftrightarrow \mathsf{Coh} : \Psi$ provided by
  - $\Phi(P) = [w]_\sim$, where w is any step decomposition of P;
  - $\Psi([P_1 \dots P_n]_\sim) = P_1 * \dots * P_n$ (needs lemma)

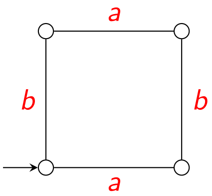
Algebra

This is not cancellative:

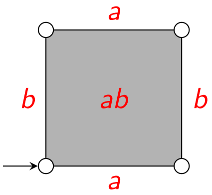
$$a\bullet \begin{bmatrix} \bullet a\bullet \\ a\bullet \end{bmatrix} = a\bullet \begin{bmatrix} a\bullet \\ \bullet a\bullet \end{bmatrix} = \begin{bmatrix} a\bullet \\ a\bullet \end{bmatrix}$$

Higher-dimensional automata

$a|b$



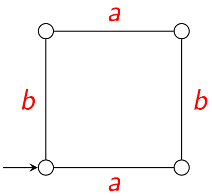
$a.b + b.a$



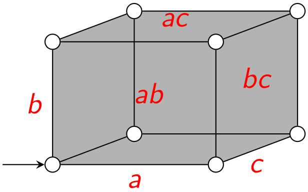
a and b are independent

Higher-dimensional automata

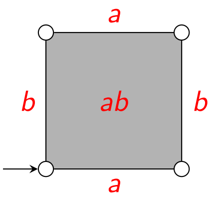
$a|b$

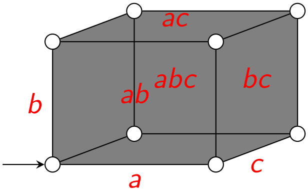


$a|b|c$



$a|b + a|c + b|c$





$\{a, b, c\}$ independent

Higher-dimensional automata

A conclist is a finite, ordered and Σ-labelled set. (a list of events)

A precubical set X consists of:

- A set of cells X (cubes)
- Every cell $x \in X$ has a conclist $\text{ev}(x)$ (list of events active in x)
- We write $X[U] = \{x \in X \mid \text{ev}(x) = U\}$ for a conclist U (cells of type U)
- For every conclist U and $A \subseteq U$ there are:
  upper face map $\delta_A^1 : X[U] \to X[U \setminus A]$ (terminating events A)
  lower face map $\delta_A^0 : X[U] \to X[U \setminus A]$ (unstarting events A)
- Precube identities: $\delta_A^\mu \delta_B^\nu = \delta_B^\nu \delta_A^\mu$ for $A \cap B = \emptyset$ and $\mu, \nu \in \{0, 1\}$

A higher dimensional automaton (HDA) is a precubical set X with start cells $\bot \subseteq X$ and accept cells $\top \subseteq X$ (not necessarily vertices)

Higher-dimensional automata

HDAs as a model for concurrency:

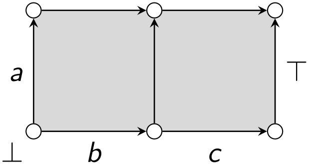
- vertices $x \in X[\emptyset]$: states
- edges $a \in X[\{a\}]$: labeled transitions
- n-squares $\alpha \in X[\{a_1, \dots, a_n\}]$ $(n \geq 2)$: independency relations / concurrently executing events

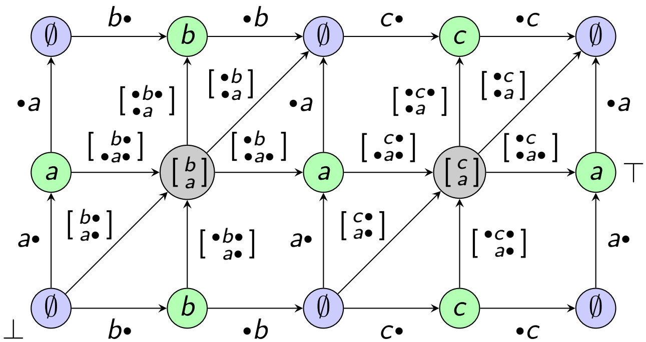
van Glabbeek (TCS 2006): Up to history-preserving bisimilarity, HDAs generalize "the main models of concurrency proposed in the literature"

Lots of recent activity on languages of HDAs:

- Kleene theorem
- Myhill-Nerode theorem
- Büchi-Elgot-Trakhtenbrot theorem
- Model checking
- ...

ST-automata





- The operational semantics of an HDA $(X, \bot, \top, \Sigma)$ is the "ST-automaton" with states X, alphabet Ω, state labeling $\mathsf{ev} : X \to \square$, and transitions

$$E = \{\delta_A^0(\ell) \xrightarrow{A\uparrow \mathsf{ev}(\ell)} \ell \mid A \subseteq \mathsf{ev}(\ell)\} \cup \{\ell \xrightarrow{\mathsf{ev}(\ell)\downarrow A} \delta_A^1(\ell) \mid A \subseteq \mathsf{ev}(\ell)\}.$$

- An automaton on the (infinite) graph alphabet Ω

Un peu de recul

What we have:

- alphabet Σ
- St: set of starters ${}_S U_U$ on Σ; Te: set of terminators ${}_U U_T$ on Σ; $\Omega = \mathrm{St} \cup \mathrm{Te}$
- (equivalently: St and Te are marked inclusions $\Sigma^* \hookrightarrow \Sigma^*$)
- ST-automaton: automaton on (infinite) graph alphabet Ω
- language of ST-automaton: subset of (morphisms of) Coh
- Coh: category generated from Ω under relation $\sim$ (not free)

What this looks like:

- Automata on graph alphabets $\triangleq$ languages of morphisms in categories
- Melliès-Zeilberger – König – Morvan – . . .
- Our cats are not freely generated, but almost:
  - The nerve functor N : Cat $\to$ SSet has a left adjoint "Ho"
  - $\Longrightarrow$ "category freely generated by a simplicial set"
  - Thank you, Vincent Moreau