Higher-Dimensional Automata Theory

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Language theory of higher-dimensional automata

- Languages of Higher-Dimensional Automata [MSCS 2021]
- Posets with Interfaces as a Model for Concurrency [I&C 2022]
- Kleene Theorem for Higher-Dimensional Automata [LMCS 2024]
- Myhill-Nerode Theorem for Higher-Dimensional Automata [FI 2024]

- Decision and Closure Properties for Higher-Dimensional Automata [ICTAC 2023]
- Logic and Languages of Higher-Dimensional Automata [DLT 2024]
- Presenting Interval Pomsets with Interfaces [RAMiCS 2024]
- Bisimulations and Logics for Higher-Dimensional Automata. [ICTAC 2024]

Today:

1. What are HDAs?
2. How do they relate to Petri nets?
3. What are languages of HDAs?
4. What can I do with languages of HDAs?

Nice people

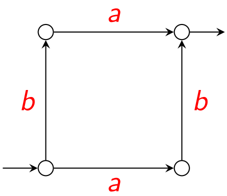
- Eric Goubault, Paris
- Christian Johansen, Gjøvik
- Georg Struth, Sheffield
- Krzysztof Ziemiański, Warsaw

- Amazigh Amrane (Paris), Hugo Bazille (Rennes), Emily Clement (Paris), Jérémy Dubut (Paris), Marie Fortin (Paris), Loïc Hélouët (Rennes), Jérémy Ledent (Paris), Safa Zouari (Gjøvik) ...

- See also `https://ulifahrenberg.github.io/pomsetproject/`

Higher-dimensional automata

semantics of "*a* parallel *b*":



$a.b + b.a$

Higher-dimensional automata

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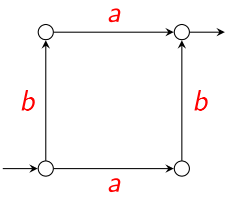
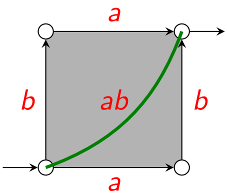


$a.b + b.a$



a and *b* are independent

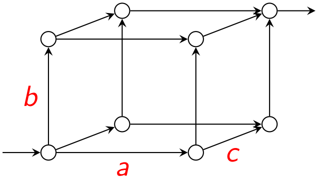
Higher-dimensional automata & concurrency

HDAs as a model for concurrency:

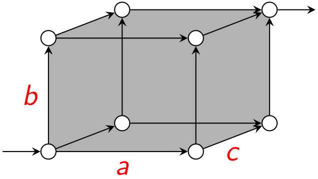
- points: states
- edges: transitions
- squares, cubes etc.: independency relations / concurrently executing events
- two-dimensional automata $\cong$ asynchronous transition systems [Bednarczyk]

[van Glabbeek 2006, TCS]: Up to history-preserving bisimilarity, HDAs "generalize the main models of concurrency proposed in the literature" (notably, event structures and Petri nets)

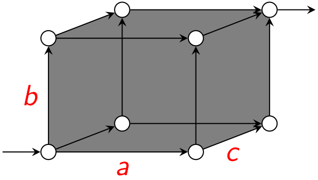
Examples



no concurrency



two out of three



full concurrency

Higher-dimensional automata

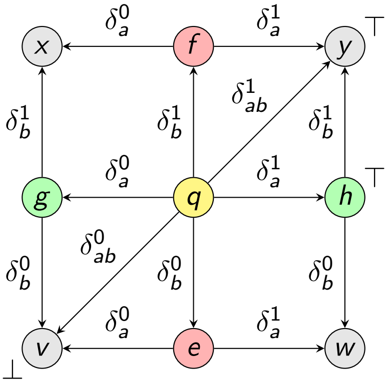
A conclist is a finite, ordered and Σ-labelled set. (a list of events)

A precubical set X consists of:

- A set of cells X (cubes)
- Every cell $x \in X$ has a conclist $\mathrm{ev}(x)$ (list of events active in x)
- We write $X[U] = \{x \in X \mid \mathrm{ev}(x) = U\}$ for a conclist U (cells of type U)
- For every conclist U and $A \subseteq U$ there are:
  upper face map $\delta_A^1 : X[U] \to X[U \setminus A]$ (terminating events A)
  lower face map $\delta_A^0 : X[U] \to X[U \setminus A]$ ("unstarting" events A)
- Precube identities: $\delta_A^\mu \delta_B^\nu = \delta_B^\nu \delta_A^\mu$ for $A \cap B = \emptyset$ and $\mu, \nu \in \{0, 1\}$

A higher dimensional automaton (HDA) is a precubical set X with start cells $\bot \subseteq X$ and accept cells $\top \subseteq X$ (not necessarily vertices)

Example



$X[\emptyset] = \{v, w, x, y\}$

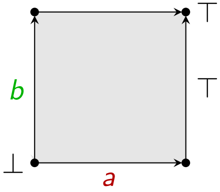
$X[a] = \{e, f\}$

$X[b] = \{g, h\}$

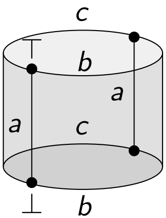
$X[ab] = \{q\}$

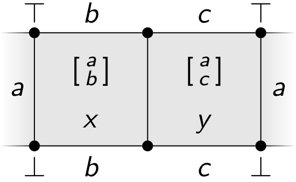
$\bot_X = \{v\}$

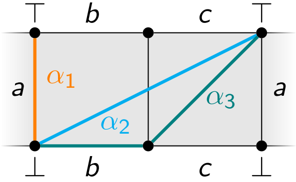
$\top_X = \{h, y\}$



Another one







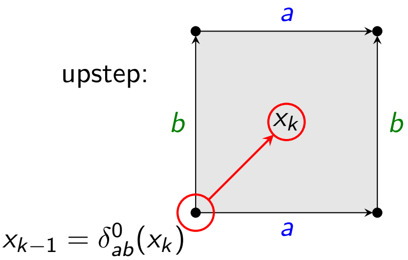
$$a \parallel (bc)^*$$

Computations of HDAs

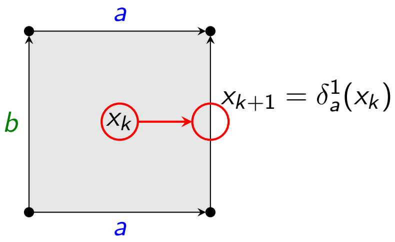
A path on an HDA X is a sequence $(x_0, \varphi_1, x_1, \dots, x_{n-1}, \varphi_n, x_n)$ such that for every k, $(x_{k-1}, \varphi_k, x_k)$ is either

- $(\delta^0_A(x_k), \nearrow^A, x_k)$ for $A \subseteq \text{ev}(x_k)$ or (upstep: start A)
- $(x_{k-1}, \searrow_B, \delta^1_B(x_{k-1}))$ for $B \subseteq \text{ev}(x_{k-1})$ (downstep: terminate B)

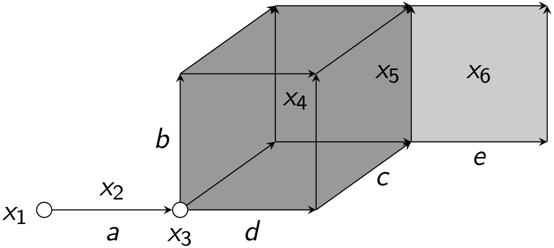
upstep:



downstep:

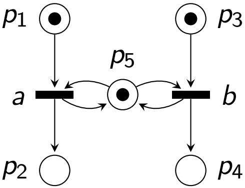


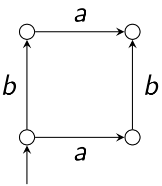
Example

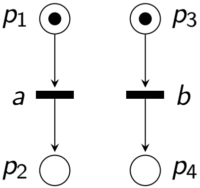


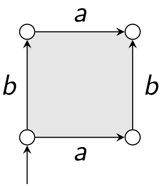
$$(x_1 \nearrow^{a} x_2 \searrow_{a} x_3 \nearrow^{\{b,c,d\}} x_4 \searrow_{\{c,d\}} x_5 \nearrow^{e} x_6)$$

Concurrency

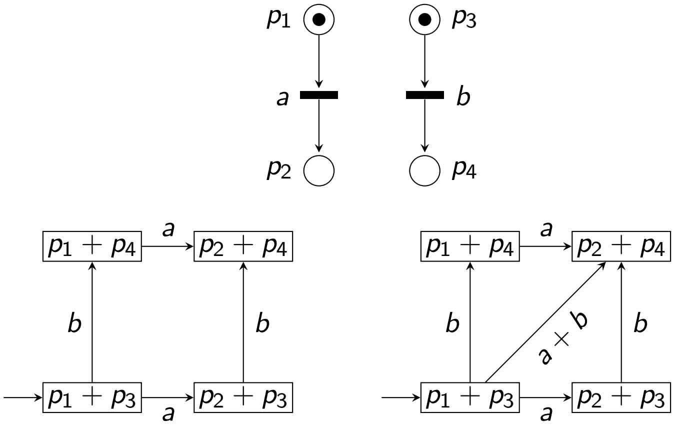








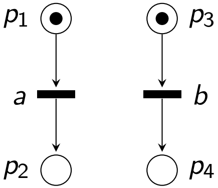
Reachability and concurrent step reachability

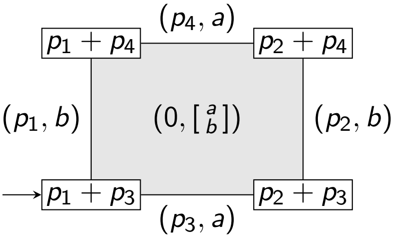


Definitions

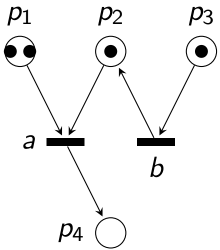
- Petri net: places S, transitions T, $S \cap T = \emptyset$,
  weighted flow relation $F : S \times T \cup T \times S \to \mathbb{N}$
- marking: multiset $m : S \to \mathbb{N}$
- for $t \in T$: preset ${}^\bullet t(s) = F(s, t)$, postset $t^\bullet(s) = F(t, s)$
- interleaved semantics: $[\![N]\!]_1 = (V, E)$ with $V = \mathbb{N}^S$ and $E = \{(m, t, m') \in V \times T \times V \mid {}^\bullet t \leq m, m' = m - {}^\bullet t + t^\bullet\}$
- HDA semantics: $\square = \square(T)$, $X = \mathbb{N}^S \times \square$, $\text{ev}(m, \tau) = \tau$
  - for $x = (m, \tau) \in X[\tau]$ with $\tau = (t_1, \dots, t_n)$:

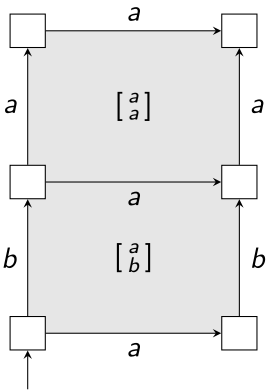
$$\delta^0_{t_i}(x) = (m + {}^\bullet t_i, (t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n)),$$
$$\delta^1_{t_i}(x) = (m + t_i^\bullet, (t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n)).$$

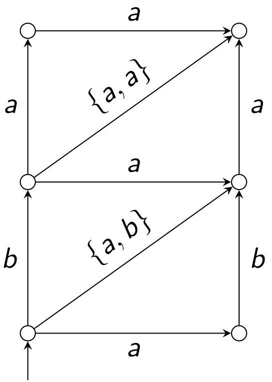




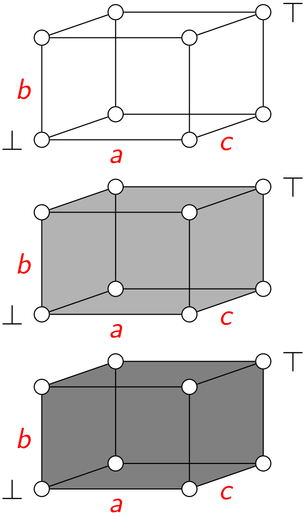
Another example







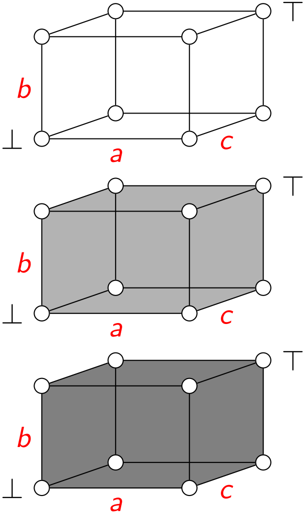
Languages of HDAs: Examples



$$L_1 = \{abc, acb, bac, bca, cab, cba\}$$

$$L_3 = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \dots \right\}$$

Languages of HDAs: Examples



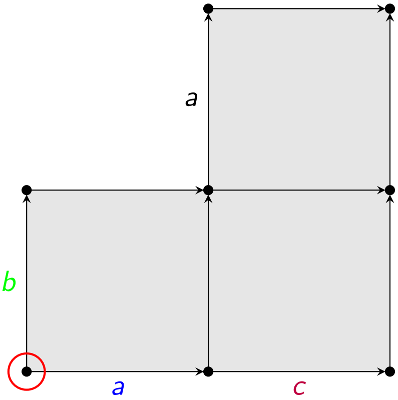
$$L_1 = \{abc, acb, bac, bca, cab, cba\}$$

$$L_2 = \left\{ \begin{bmatrix} a \\ b \to c \end{bmatrix}, \begin{bmatrix} a \\ c \to b \end{bmatrix}, \begin{bmatrix} b \\ a \to c \end{bmatrix}, \begin{bmatrix} b \\ c \to a \end{bmatrix}, \begin{bmatrix} c \\ a \to b \end{bmatrix}, \begin{bmatrix} c \\ b \to a \end{bmatrix} \right\} \cup L_1 \cup \dots$$

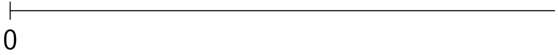
sets of pomsets

$$L_3 = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right\} \cup L_2$$

Event ipomset of a path



Lifetimes of events



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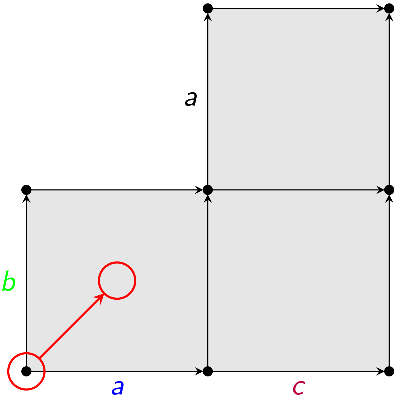

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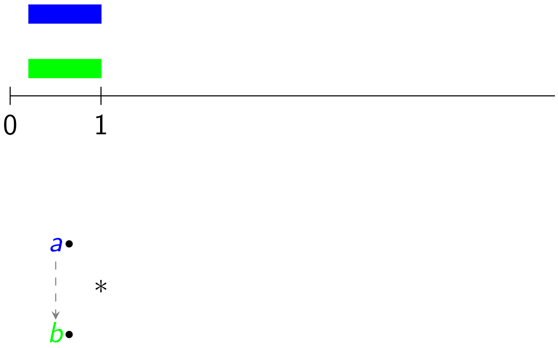
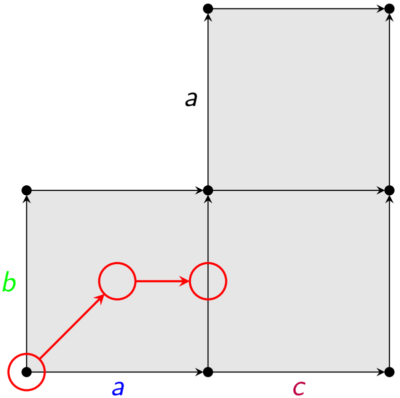
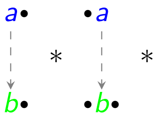


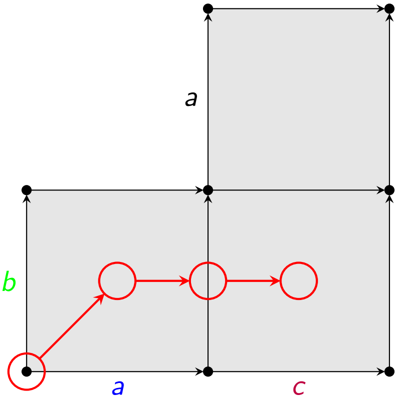
Event ipomset of a path



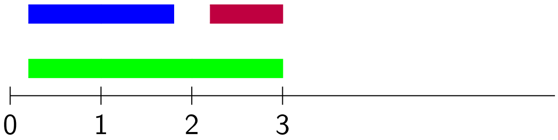
Lifetimes of events

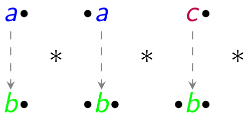


Event ipomset of a path

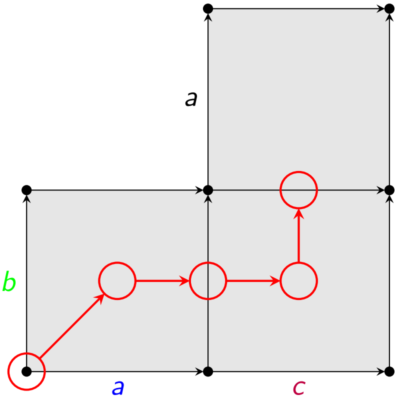


Lifetimes of events

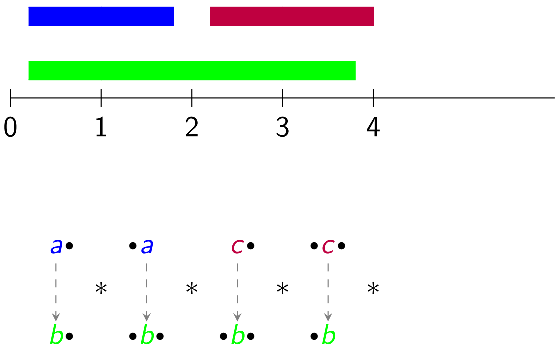




Event ipomset of a path



Lifetimes of events



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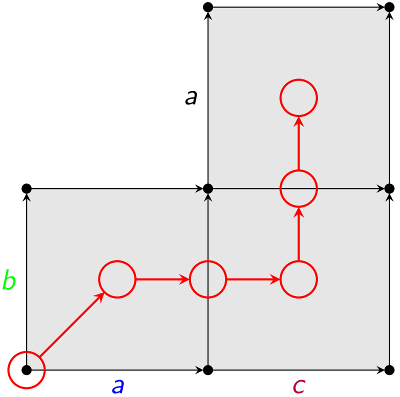

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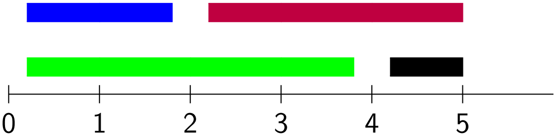

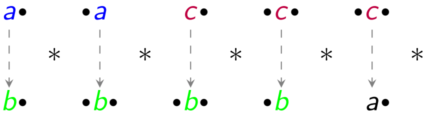

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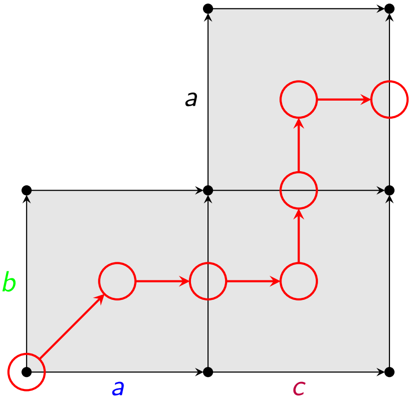
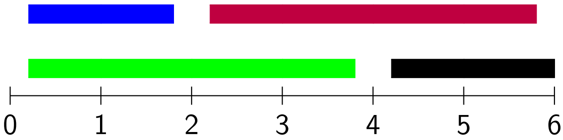
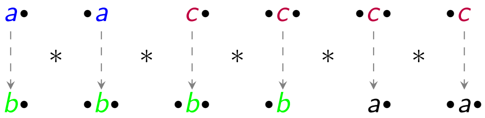


Lifetimes of events





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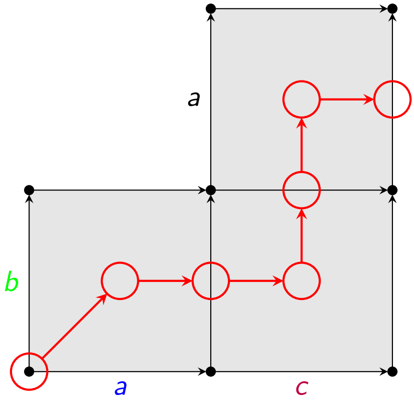
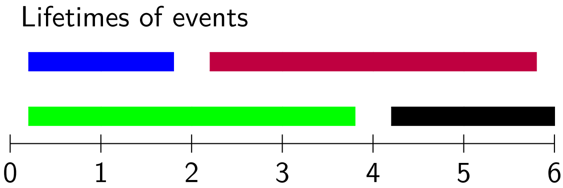
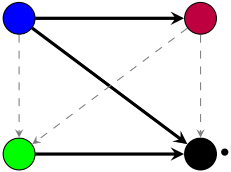






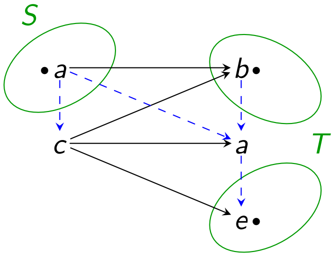
(not series-parallel!)

Pomsets with interfaces

Definition

A pomset with interfaces (ipomset): $(P, <, \dashrightarrow, S, T, \lambda)$:

- finite set P;
- two partial orders $<$ (precedence order), $\dashrightarrow$ (event order)
  - s.t. $< \cup \dashrightarrow$ is a *total relation*;
- $S, T \subseteq P$ source and target interfaces
  - s.t. S is $<$-minimal and T is $<$-maximal.

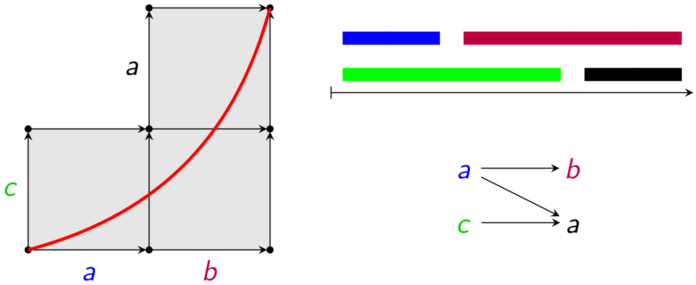


Interval orders

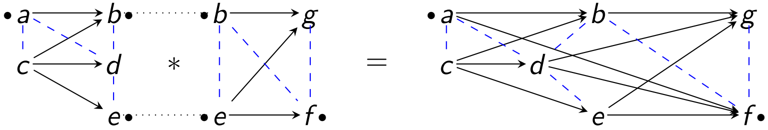
Definition

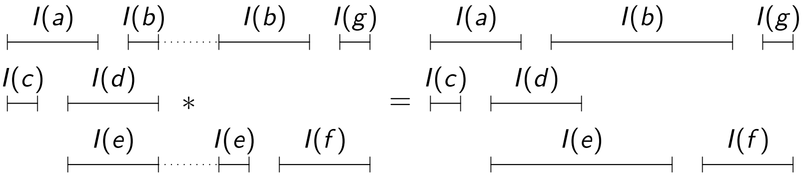
An ipomset $(P, <_P, \dashrightarrow, S, T, \lambda)$ is interval if $(P, <_P)$ has an interval representation: functions $b, e : P \to \mathbb{R}$ s.t.

- $\forall x \in P : b(x) \leq_\mathbb{R} e(x)$;
- $\forall x, y \in P : e(x) <_\mathbb{R} b(y) \iff x <_P y$



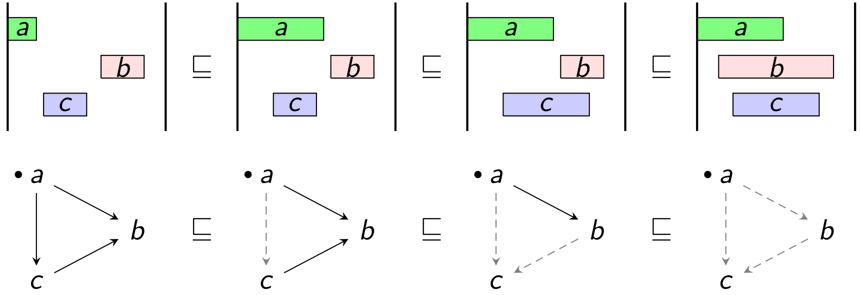
Gluing composition





- Gluing $P * Q$: P before Q, except for interfaces (which are identified)
- (also have parallel composition $P \parallel Q$: disjoint union)

Subsumption



P refines Q / Q subsumes P / $P \sqsubseteq Q$ iff

- P and Q have same interfaces
- P has more $<$ than Q
- Q has more $\dashrightarrow$ than P

Languages of HDAs

Definition

The language of an HDA X is the set of event ipomsets of all accepting paths:

$$L(X) = \{\text{ev}(\pi) \mid \pi \in \text{Paths}(X), \text{src}(\pi) \in \bot_X, \text{tgt}(\pi) \in \top_X\}$$

- $L(X)$ contains only interval ipomsets,
- is closed under subsumption,
- and has finite width

Definition

A language $L \subseteq \text{iiPoms}$ is regular if there is an HDA X with $L = L(X)$.

Theorems

Definition (Rational Languages over Σ)

- Generated by $\emptyset$, $\{\epsilon\}$, and all $\{[a]\}$, $\{[\bullet\, a]\}$, $\{[a\, \bullet]\}$, $\{[\bullet\, a\, \bullet]\}$ for $a \in \Sigma$
- under operations $\cup$, $*$, $\|$ and (Kleene plus) $^+$
- (these need to take subsumption closure into account)

Definition (Monadic Second-Order Logics over Ipomsets)

$$\psi ::= a(x) \mid \mathsf{s}(x) \mid \mathsf{t}(x) \mid x < y \mid x \dashrightarrow y \mid x \in X \mid$$
$$\exists x.\, \psi \mid \forall x.\, \psi \mid \exists X.\, \psi \mid \forall X.\, \psi \mid \psi_1 \wedge \psi_2 \mid \psi_1 \vee \psi_2 \mid \neg \psi$$

Theorem (à la Kleene [LMCS 2024])

A language is rational iff it is regular.

Theorem (à la Büchi-Elgot-Trakhtenbrot [DLT 2024])

A language is rational iff it is MSO-definable, of finite width, and subsumption-closed.

More theorems

Theorem (à la Myhill-Nerode [FI 2024])

A language is rational iff it has finite prefix quotient.

Proof: $\Longrightarrow$ as usual

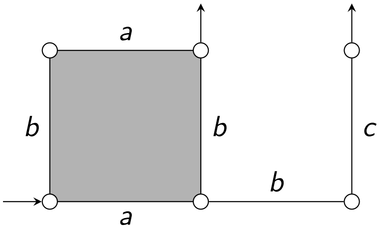
$\Longleftarrow$: usual MN congruence, but taking terminating interfaces into account

Theorem (Closure properties [ICTAC 2023])

Rational languages are closed under intersection
– but not under complement

- because $L = L{\downarrow}$ implies $\overline{L} = \overline{L}{\uparrow}$; also $\overline{L}$ has infinite width
- bounded-width pseudo-complement $\overline{L}^k = \{P \in \overline{L} \mid \text{wid}(P) \leq k\}$ is useful

Determinizability & ambiguity



Proposition

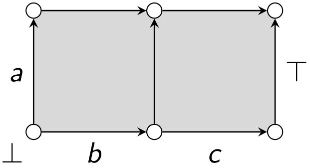
Not all HDAs are determinizable.

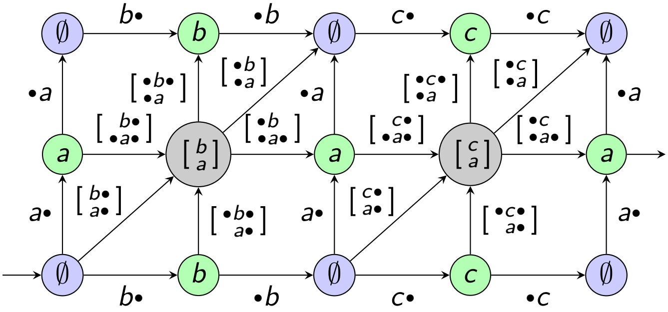
Proposition

There is a rational language which is inherently infinitely ambiguous.

$$\left(\left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right] cd + ab \left[\begin{smallmatrix} c \\ d \end{smallmatrix} \right] \right)^{+}$$

Important tool: ST-automata & step decompositions





$$\begin{bmatrix} b \to c \\ a\bullet \end{bmatrix}$$

$$\left[\begin{smallmatrix} b\bullet \\ a\bullet \end{smallmatrix}\right]\left[\begin{smallmatrix} \bullet b \\ \bullet a\bullet \end{smallmatrix}\right]\left[\begin{smallmatrix} c\bullet \\ \bullet a\bullet \end{smallmatrix}\right]\left[\begin{smallmatrix} \bullet c \\ \bullet a\bullet \end{smallmatrix}\right]$$

Special ipomsets

Definition

An ipomset $(P, <, \dashrightarrow, S, T, \lambda)$ is

- discrete if $<$ is empty (hence $\dashrightarrow$ is total) $\begin{bmatrix} \bullet a \\ \bullet b \bullet \\ a \bullet \end{bmatrix}$
  - also written ${}_S P_T$
- a starter if it is discrete and $T = P$ $\begin{bmatrix} \bullet a \bullet \\ \bullet b \bullet \\ a \bullet \end{bmatrix}$
- a terminator if it is discrete and $S = P$ $\begin{bmatrix} \bullet a \\ \bullet b \bullet \\ \bullet a \bullet \end{bmatrix}$
- an identity if it is both a starter and a terminator $\begin{bmatrix} \bullet a \bullet \\ \bullet b \bullet \\ \bullet a \bullet \end{bmatrix}$

Lemma (Janicki-Koutny 93; reformulated)

An ipomset is interval iff it has a decomposition into discrete ipomsets.

Decompositions

Lemma (Janicki-Koutny 93)

A poset $(P, <)$ is an interval order iff the order defined on its maximal antichains defined by $A \preceq B \iff \forall a \in A, b \in B : b \not< a$ is total.

Corollary

An ipomset is interval iff it has a decomposition into discrete ipomsets.

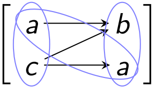
Lemma

Any discrete ipomset is a gluing of a starter and a terminator.

$$\begin{bmatrix} \bullet a \\ \bullet b \bullet \\ a \bullet \end{bmatrix} = \begin{bmatrix} \bullet a \bullet \\ \bullet b \bullet \\ a \bullet \end{bmatrix} * \begin{bmatrix} \bullet a \\ \bullet b \bullet \\ \bullet a \bullet \end{bmatrix}$$

Corollary

Any interval ipomset has a decomposition as a sequence of starters and terminators.



$$= \begin{bmatrix} a \bullet \\ c \end{bmatrix} * \begin{bmatrix} \bullet a \\ a \bullet \end{bmatrix} * \begin{bmatrix} b \\ \bullet a \end{bmatrix} = \begin{bmatrix} a \bullet \\ c \bullet \end{bmatrix} * \begin{bmatrix} \bullet a \bullet \\ \bullet c \end{bmatrix} * \begin{bmatrix} \bullet a \bullet \\ a \bullet \end{bmatrix} * \begin{bmatrix} \bullet a \\ \bullet a \bullet \end{bmatrix} * \begin{bmatrix} b \bullet \\ \bullet a \bullet \end{bmatrix} * \begin{bmatrix} \bullet b \\ \bullet a \end{bmatrix}$$

Unique decompositions

Notation: St: set of starters ${}_S U_U$
Te: set of terminators ${}_U U_T$
$\mathsf{Id} = \mathsf{St} \cap \mathsf{Te}$: set of identities ${}_U U_U$
$\Omega = \mathsf{St} \cup \mathsf{Te}$

Definition

A word $w = (S_1, U_1, T_1) \dots (S_n, U_n, T_n) \in \Omega^+$ is coherent if $T_i = S_{i+1}$ for all i.

Definition

A coherent word is sparse if proper starters and proper terminators are alternating.

- additionally, all $w \in \mathsf{Id} \subseteq \Omega^+$ are sparse

Lemma

Any interval ipomset P has a unique decomposition $P = P_1 * \cdots * P_n$ such that $P_1 \dots P_n \in \Omega^+$ is sparse.

Step sequences

Let $\sim$ be the congruence on Ω^+ generated by the relation

$$_SU_U \cdot {}_UT_T \sim {}_ST_T \qquad {}_SS_U \cdot {}_UU_T \sim {}_SS_T$$

- (compose subsequent starters and subsequent terminators)

Definition

A step sequence is a $\sim$-equivalence class of coherent words in Ω^+.

Lemma

Any step sequence has a unique sparse representant.

Theorem

The category of interval ipomsets is isomorphic to the category of step sequences.

One stone, several birds

The operational semantics of an HDA $(X, \bot, \top, \Sigma)$ is the ST-automaton with states X, (infinite) alphabet Ω, state labeling $\mathsf{ev} : X \to \square$, and transitions

$$E = \{\delta_A^0(\ell) \xrightarrow{A\uparrow \mathsf{ev}(\ell)} \ell \mid A \subseteq \mathsf{ev}(\ell)\} \cup \{\ell \xrightarrow{\mathsf{ev}(\ell)\downarrow A} \delta_A^1(\ell) \mid A \subseteq \mathsf{ev}(\ell)\}.$$

- (from ST-automata to HDAs: more complicated)

Lemma: regular $\implies$ rational

- HDA $\to$ ST-automaton $\to$ reg.ex. over Ω $\to$ reg.ex. over ipomsets

Lemma: regular $\implies$ MSO-definable

- HDA $\to$ ST-automaton $\to$ MSO over step sequences $\to$ MSO over ipomsets

Lemma: inclusion is decidable

- $L(X_1) \subseteq L(X_2) \iff L(\mathsf{ST}(X_1)) \subseteq L(\mathsf{ST}(X_2))$ and then width-bound to get finite automata

Conclusion

- Higher-Dimensional Automata: an automaton-like model for non-interleaving concurrency
- With a nice language theory!
- The trifecta Kleene–Myhill-Nerode–Büchi-Elgot-Trakhtenbrot is now complete for HDAs [LMCS 2024]–[FI 2024]–[DLT 2024]
- ST-automata provide a good combinatorial abstraction
- (and geometry & topology provide intuition)

Ongoing & Future work:

- Linear-time logic for HDAs — PhD Enzo Erlich
- Branching-time logics for HDAs — PhD Safa Zouari; [ICTAC 2024]
- Higher-dimensional timed automata — [Petri Nets 2024]
- HDAs over infinite ipomsets — M2 Luc Passemard
- from (extensions of) Petri nets to (partial) HDAs — [arxiv 2025]