Petri Nets and Higher-Dimensional Automata

Amazigh Amrane[1] Hugo Bazille[1] Uli Fahrenberg[12]
Loïc Hélouët[2] Philipp Schlehuber-Caissier[3]

EPITA Research Lab (LRE), Rennes/Paris, France
IRISA & Inria Rennes, France
SAMOVAR, Télécom SudParis, Institut Polytechnique de Paris, France

Petri Nets 2025

Concurrent Semantics of Petri Nets via ~~and~~ Higher-Dimensional Automata

Amazigh Amrane[1] Hugo Bazille[1] Uli Fahrenberg[12]
Loïc Hélouët[2] Philipp Schlehuber-Caissier[3]

EPITA Research Lab (LRE), Rennes/Paris, France
IRISA & Inria Rennes, France
SAMOVAR, Télécom SudParis, Institut Polytechnique de Paris, France

Petri Nets 2025

Higher-Dimensional Automata?

PN'23 UF, K. Ziemiański: *A Myhill-Nerode theorem for higher-dimensional automata*

PN'24 A. Amrane, H. Bazille, E. Clement, UF: *Languages of higher-dimensional timed automata*

PN'25 A. Amrane, H. Bazille, UF, L. Hélouët, P. Schlehuber-Caissier: *Petri Nets and higher-dimensional automata*

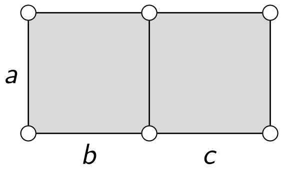
Higher-Dimensional Automata?

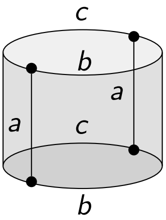
PN'23 UF, K. Ziemiański: *A Myhill-Nerode theorem for higher-dimensional automata*

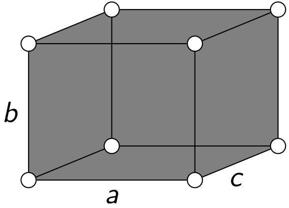
PN'24 A. Amrane, H. Bazille, E. Clement, UF: *Languages of higher-dimensional timed automata*

PN'25 A. Amrane, H. Bazille, UF, L. Hélouët, P. Schlehuber-Caissier: *Petri Nets and higher-dimensional automata*

- Rob van Glabbeek 2006: *On the expressiveness of higher-dimensional automata*

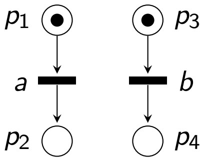






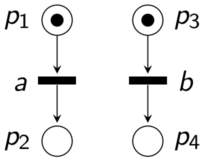
Semantics of Petri Nets

Petri net (S, T, F): places S; transitions T; weighted flows $F : S \times T \cup T \times S \to \mathbb{N}$



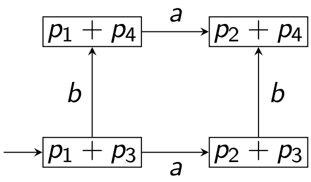
Semantics of Petri Nets

Petri net (S, T, F): places S; transitions T; weighted flows $F : S \times T \cup T \times S \to \mathbb{N}$

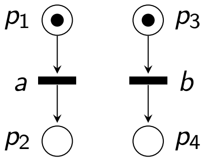


Interleaved semantics (reachability graph) (V, E):

- $V = \mathbb{N}^S$: all markings
- $E \subseteq V \times T \times V$: one transition at a time
- $E = \{(m, t, m') \mid {}^\bullet t \le m, m' = m - {}^\bullet t + t^\bullet\}$
- initial marking $\implies$ initial state; take reachable part



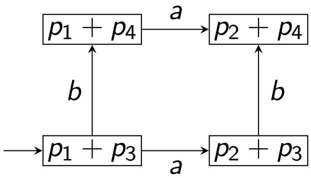
Semantics of Petri Nets



Petri net (S, T, F): places S; transitions T; weighted flows $F : S \times T \cup T \times S \to \mathbb{N}$

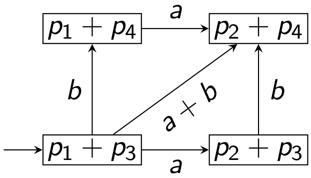
Interleaved semantics (reachability graph) (V, E):

- $V = \mathbb{N}^S$: all markings
- $E \subseteq V \times T \times V$: one transition at a time
- $E = \{(m, t, m') \mid {}^\bullet t \leq m, m' = m - {}^\bullet t + t^\bullet\}$
- initial marking $\implies$ initial state; take reachable part

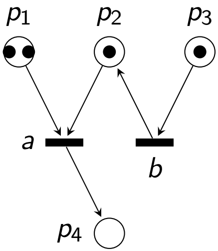


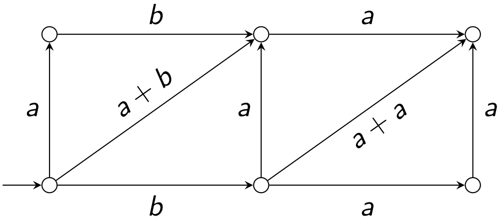
Concurrent step reachability graph (V, E'):

- $V = \mathbb{N}^S$
- $E' \subseteq V \times \mathbb{N}^T \times V$: multisets of transitions
- $E' = \{(m, U, m') \mid {}^\bullet U \leq m, m' = m - {}^\bullet U + U^\bullet\}$



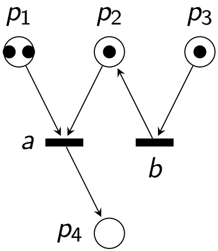
Another Example

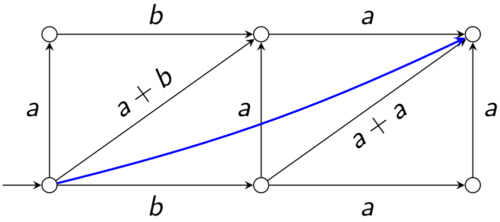




- after firing b, a is auto-concurrent

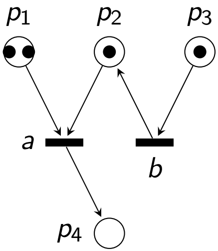
Another Example

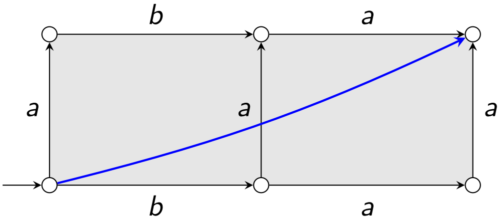




- after firing b, a is auto-concurrent
- semantics misses some behavoir?
  - start a – start b – finish b – start another a – etc.

Another Example





- after firing b, a is auto-concurrent
- semantics misses some behavoir?
  - start a – start b – finish b – start another a – etc.
- enter higher-dimensional automata
  - replace multi-transitions by squares

Higher-Dimensional Automata

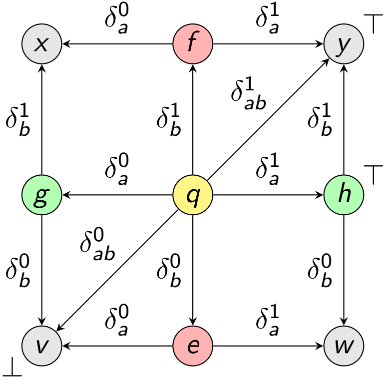
A conclist is a finite, totally ordered, Σ-labeled set. (a list of labeled events)

A precubical set X consists of:

- A set of cells X (cubes)
- Every cell $x \in X$ has a conclist $\mathrm{ev}(x)$ (list of events active in x)
- We write $X[U] = \{x \in X \mid \mathrm{ev}(x) = U\}$ for a conclist U (cells of type U)
- For every conclist U and $A \subseteq U$ there are:
  upper face map $\delta_A^1 : X[U] \to X[U \setminus A]$ (terminating events A)
  lower face map $\delta_A^0 : X[U] \to X[U \setminus A]$ (“unstarting” events A)
- Precube identities: $\delta_A^\mu \delta_B^\nu = \delta_B^\nu \delta_A^\mu$ for $A \cap B = \emptyset$ and $\mu, \nu \in \{0, 1\}$

A higher dimensional automaton (HDA) is a precubical set X with initial cells $\bot \subseteq X$ and accepting cells $\top \subseteq X$ (not necessarily vertices)

Example



$$X[\emptyset] = \{v, w, x, y\}$$

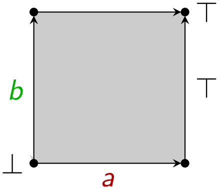
$$X[a] = \{e, f\}$$

$$X[b] = \{g, h\}$$

$$X[[\,{}^{a}_{b}\,]] = \{q\}$$

$$\bot_X = \{v\}$$

$$\top_X = \{h, y\}$$



Higher-Dimensional Automata & Concurrency Theory

HDAs as a model for concurrency:

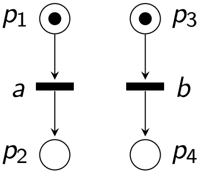
- points: states
- edges: transitions
- squares, cubes etc.: independency relations / concurrently executing events
- two-dimensional automata ≅ asynchronous transition systems [Shields'85] [Bednarczyk'88]

- Introduced in [van Glabbeek'89]
- Generalize all main models of concurrency proposed in the literature
- (event structures; Petri nets; communicating automata; etc.)
- [van Glabbeek'06]: translations from Petri nets
  - individual vs. collective tokens; autoconcurrency or not

Our Contributions

- Update van Glabbeek's translation to our new event-based HDA formalism
- Implement translation in new tool `pn2HDA`
- Extend to inhibitor arcs
- Extend to generalized self-modifying nets

Concurrent Semantics of Petri Nets



Petri net (S, T, F): places S; transitions T;
weighted flows $F : S \times T \cup T \times S \to \mathbb{N}$

- presets & postsets: ${}^\bullet t(s) = F(s, t)$, $t^\bullet(s) = F(t, s)$

Interleaved semantics (V, E): $V = \mathbb{N}^S$; $E \subseteq V \times T \times V$

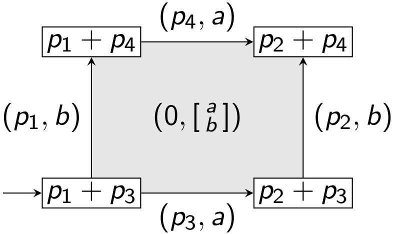
- $E = \{(m, t, m') \mid {}^\bullet t \le m, m' = m - {}^\bullet t + t^\bullet\}$

Concurrent semantics as HDA:
$\square = \square(T)$, $X = \mathbb{N}^S \times \square$, $\text{ev}(m, \tau) = \tau$

- for $x = (m, \tau) \in X[\tau]$ with $\tau = (t_1, \dots, t_n)$:

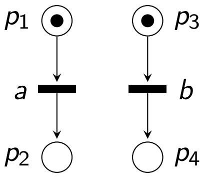
$$\delta^0_{t_i}(x) = (m + {}^\bullet t_i, (t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n))$$
$$\delta^1_{t_i}(x) = (m + t_i^\bullet, (t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n))$$

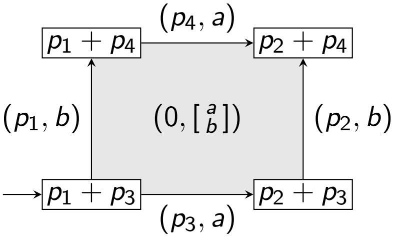
- initial marking $\Longrightarrow$ initial cell; take reachable part
- (no accepting cells)



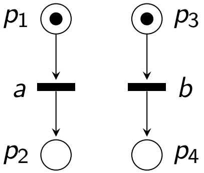
Event Order

- trouble with symmetry:
  have a cell $(0, [\begin{smallmatrix} a \\ b \end{smallmatrix}])$, but also $(0, [\begin{smallmatrix} b \\ a \end{smallmatrix}])$ (not shown)
- solution: fix an arbitrary order $\preccurlyeq$ on T
- and use $\square = \left\{ \begin{bmatrix} t_1 \\ \vdots \\ t_n \end{bmatrix} \,\middle|\, \forall i = 1, \dots, n-1 : t_i \preccurlyeq t_{i+1} \right\}$
  instead of $\square(T)$
- order $\preccurlyeq$ may be chosen (and re-chosen) at will
- here: lexicographic $a \prec b \prec \dots$

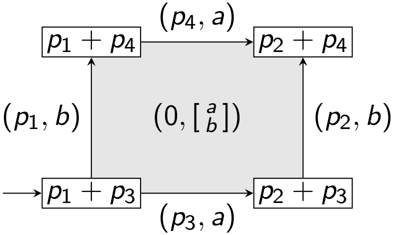


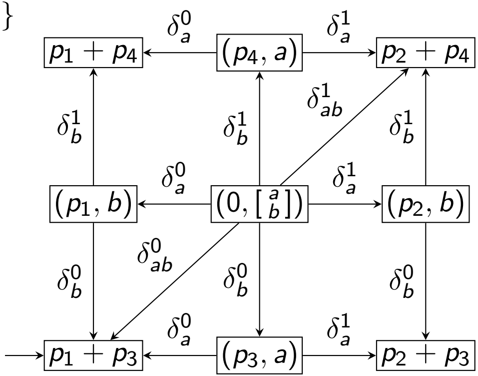


Example, Complete

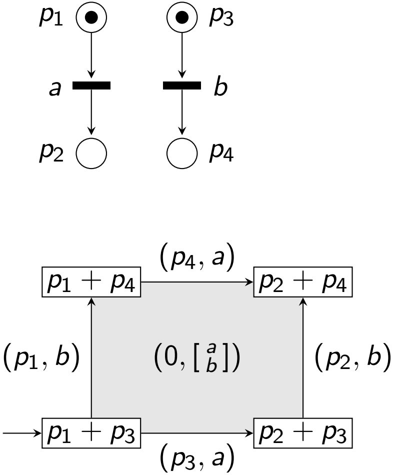


$$X[\emptyset] = \{p_1 + p_3, p_2 + p_3, p_1 + p_4, p_2 + p_4\}$$
$$X[a] = \{(p_3, a), (p_4, a)\}$$
$$X[b] = \{(p_1, b), (p_2, b)\}$$
$$X[[\begin{smallmatrix} a \\ b \end{smallmatrix}]] = \{(0, [\begin{smallmatrix} a \\ b \end{smallmatrix}])\}$$

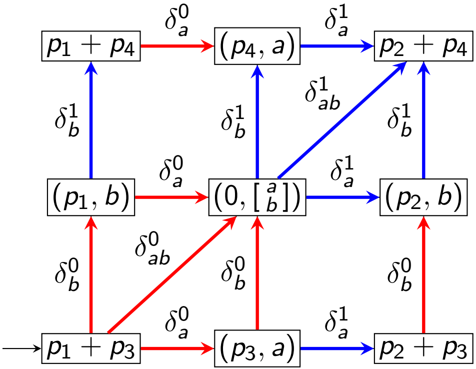




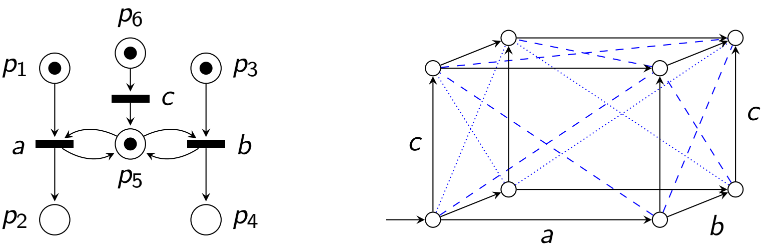
Example, Complete



– for computations, *invert* lower face maps



Another Example



- initially, p_5 is a mutex place: it disables concurrency of a and b
- after c fires, p_5 holds two tokens, so a and b become concurrent

- semantically, a hollow cube without bottom face
- the five faces: front: $(p_3, [\,{}^{a}_{c}\,])$, back: $(p_4, [\,{}^{a}_{c}\,])$
  left: $(p_1, [\,{}^{b}_{c}\,])$, right: $(p_2, [\,{}^{b}_{c}\,])$
  top: $(0, [\,{}^{a}_{b}\,])$

Petri Nets With Inhibitor Arcs

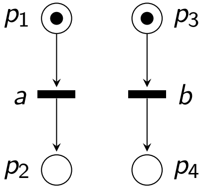
PNI (S, T, F, I): places S; transitions T; weighted flows F; inhibitor arcs $I \subseteq S \times T$

- inhibitor presets: ${}^\circ t(s) = \{s \in S \mid (s, t) \in I\}$

Transition t can fire if its inhibitor places are empty: $\forall s \in {}^\circ t : m(s) = 0$.

Example:

- without inhibitors: $a.b$, $b.a$, $a + b$



Petri Nets With Inhibitor Arcs

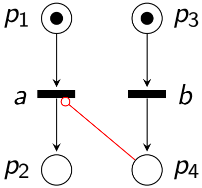
PNI (S, T, F, I): places S; transitions T; weighted flows F; inhibitor arcs $I \subseteq S \times T$

- inhibitor presets: $^\circ t(s) = \{s \in S \mid (s, t) \in I\}$

Transition t can fire if its inhibitor places are empty: $\forall s \in {}^\circ t : m(s) = 0$.

Example:

- without inhibitors: $a.b$, $b.a$, $a + b$
- now, $b.a$ is forbidden

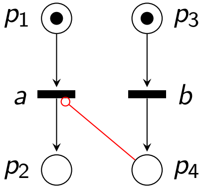


Petri Nets With Inhibitor Arcs

PNI (S, T, F, I): places S; transitions T; weighted flows F; inhibitor arcs $I \subseteq S \times T$

- inhibitor presets: ${}^{\circ}t(s) = \{s \in S \mid (s, t) \in I\}$

Transition t can fire if its inhibitor places are empty: $\forall s \in {}^{\circ}t : m(s) = 0$.



Example:

- without inhibitors: $a.b$, $b.a$, $a + b$
- now, $b.a$ is forbidden
- but what about $a + b$?

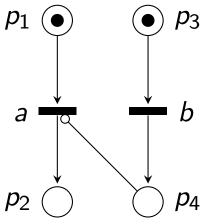
$\Longrightarrow$ This gives rise to two different semantics!

Petri Nets With Inhibitor Arcs: *a-posteriori* semantics

Conservative: *a-posteriori*, forbidding $a + b$.

It forbids all concurrent steps in which the post-places and inhibitor places intersect:

1. ${}^\bullet U \le m,\ m' = m - {}^\bullet U + U^\bullet$
2. $\forall t \in U : \forall s \in {}^\circ t : m(s) = 0$
3. $\forall t_1, t_2 \in U : t_1^\bullet \cap {}^\circ t_2 = \emptyset$ (if $t_1 \neq t_2$ or $U(t_1) \ge 2$)



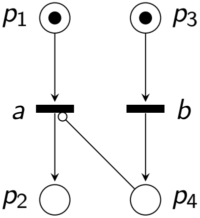
Petri Nets With Inhibitor Arcs: *a-posteriori* semantics

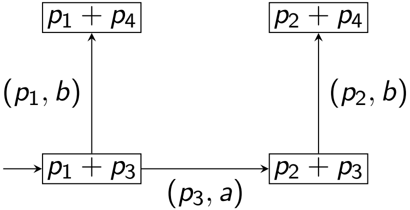
Conservative: *a-posteriori*, forbidding $a + b$.

It forbids all concurrent steps in which the post-places and inhibitor places intersect:

1. ${}^\bullet U \leq m, m' = m - {}^\bullet U + U^\bullet$
2. $\forall t \in U : \forall s \in {}^\circ t : m(s) = 0$
3. $\forall t_1, t_2 \in U : t_1^\bullet \cap {}^\circ t_2 = \emptyset$ (if $t_1 \neq t_2$ or $U(t_1) \geq 2$)

Transitions of concurrent step can be executed in any order, they are closed under substeps:
$(m, U, m'') \implies (m, U_1, m')$ and (m', U_2, m'')
for $U = U_1 \uplus U_2$





Petri Nets With Inhibitor Arcs: *a-priori* semantics

Less conservative: *a-priori*, allowing $a + b$.

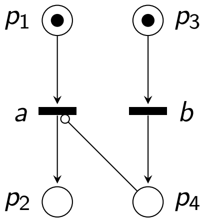
Omits the third rule:

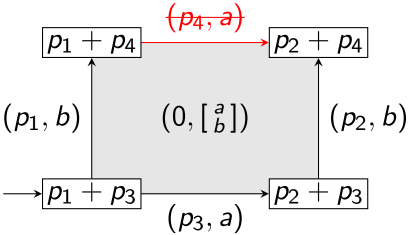
3. ~~$\forall t_1, t_2 \in U : t_1^\bullet \cap {}^\circ t_2 = \emptyset$ (if $t_1 \neq t_2$ or $U(t_1) \geq 2$)~~

This allows to start a while b is active.

However we need to forbid the top face, as otherwise $b.a$ would be allowed.

We obtain a partial HDA.





Partial Higher-Dimensional Automata

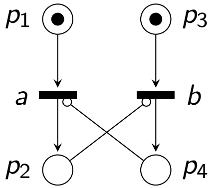
A partial precubical set X consists of:

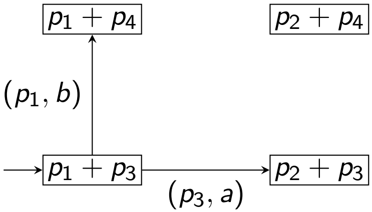
- A set of cells X (cubes)
- Every cell $x \in X$ has a conclist $\mathrm{ev}(x)$ (list of events active in x)
- We write $X[U] = \{x \in X \mid \mathrm{ev}(x) = U\}$ for a conclist U (cells of type U)
- For every conclist U and $A \subseteq U$ there are:
  partial upper face map $\delta_A^1 : X[U] \rightharpoonup X[U \setminus A]$ (terminating events A)
  partial lower face map $\delta_A^0 : X[U] \rightharpoonup X[U \setminus A]$ (unstarting events A)
- $\delta_A^\mu \delta_B^\nu = \delta_B^\nu \delta_A^\mu$ for $A \cap B = \emptyset$ and $\mu, \nu \in \{0, 1\}$ if these are defined

A partial higher dimensional automaton (pHDA) is a partial precubical set X with initial cells $\bot \subseteq X$ and accepting cells $\top \subseteq X$

Mutual Inhibition

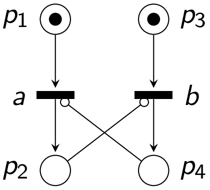
a-posteriori semantics

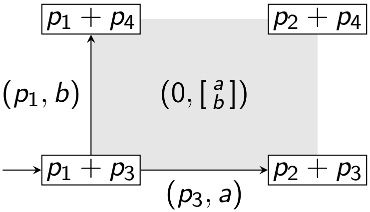




Mutual Inhibition

a-priori semantics



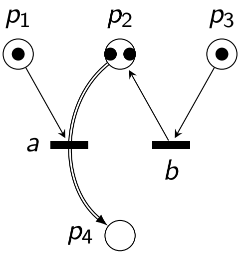


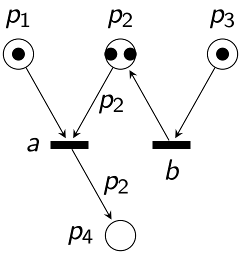
There is a (concurrent) path to the final marking!

Generalized Self-Modifying Nets

"Can you do read arcs/transfer arcs/reset arcs/etc.?"

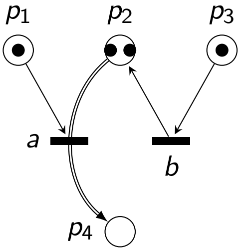
G-net (S, T, F): places S; transitions T; weighted flows $F : S \times T \cup T \times S \to \mathbb{N}[S]$

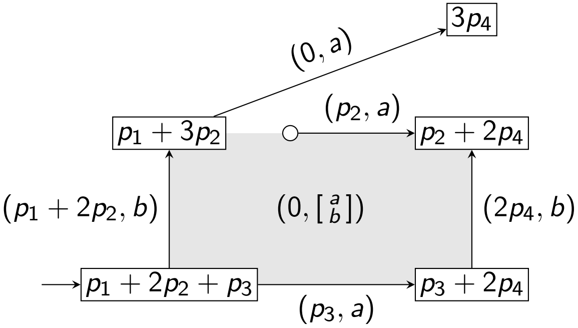




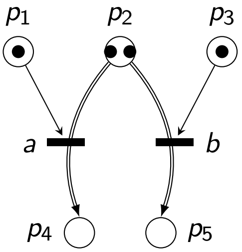
- partial-HDA semantics needs a notion of memory:
  remember state of the net before transitions fire

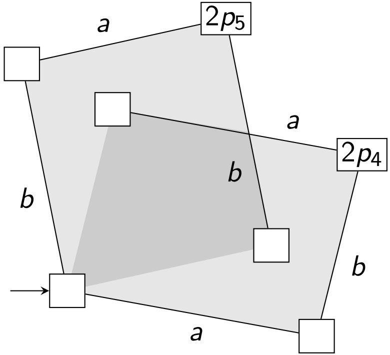
Transfer Arcs Are Funny





Transfer Arcs Are Funny



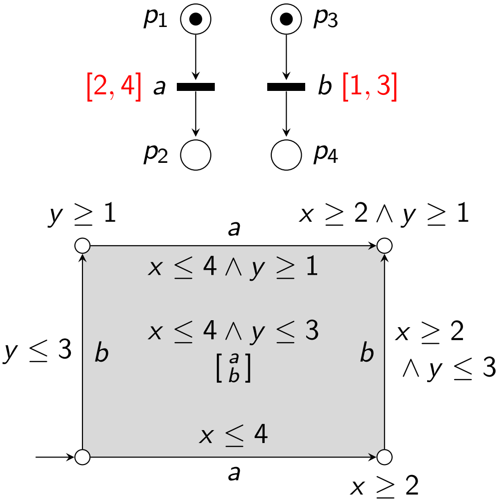


Conclusion

- higher-dimensional automata (HDAs) are useful to express concurrent semantics of Petri nets
- new tool `pn2HDA`
- partial HDAs for extensions
  - ¿ are pHDAs sufficient for all G-nets ?
- useful for reasoning about transformations
  - Example (Busi): Primitive systems (with inhibitors) may be simulated by Petri nets without inhibitors, but not if you want to respect concurrency.
- next step: time!

Conclusion

- higher-dimensional automata (HDAs) are useful to express concurrent semantics of Petri nets
- new tool `pn2HDA`
- partial HDAs for extensions
    - ¿ are pHDAs sufficient for all G-nets ?
- useful for reasoning about transformations
    - Example (Busi): Primitive systems (with inhibitors) may be simulated by Petri nets without inhibitors, but not if you want to respect concurrency.
- next step: time!



Conclusion

- higher-dimensional automata (HDAs) are useful to express concurrent semantics of Petri nets
- new tool pn2HDA
- partial HDAs for extensions
  - ¿ are pHDAs sufficient for all G-nets ?
- useful for reasoning about transformations
  - Example (Busi): Primitive systems (with inhibitors) may be simulated by Petri nets without inhibitors, but not if you want to respect concurrency.
- next step: time!