Star-Continuous Ésik Algebras

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Motivation

What is the minimum amount of battery required for the satellite to always be able to send and receive messages?

- The theory of weighted automata is very powerful
- Here: an application to energy problems

Semirings

A semiring is a structure $(S, \oplus, \otimes, 0, 1)$ such that

- $(S, \oplus, 0)$ is a commutative monoid,
  - $x \oplus y = y \oplus x$, $x \oplus (y \oplus z) = (x \oplus y) \oplus z$, $x \oplus 0 = x$
- $(S, \otimes, 1)$ is a monoid,
  - $x \otimes (y \otimes z) = (x \otimes y) \otimes z$, $x \otimes 1 = 1 \otimes x = x$
- and which satisfies distributive and annihilation laws:
  - $x(y \oplus z) = xy \oplus xz$, $(x \oplus y)z = xz \oplus yz$
  - $x \otimes 0 = 0 \otimes x = 0$

Examples:

- natural numbers: $(\mathbb{N}, +, \cdot, 0, 1)$
- the boolean semiring: $(\{\mathbf{ff}, \mathbf{tt}\}, \vee, \wedge, \mathbf{ff}, \mathbf{tt})$
- max-plus algebra: $(\mathbb{R} \cup \{-\infty\}, \max, +, -\infty, 0)$
- min-plus algebra: $(\mathbb{R} \cup \{\infty\}, \min, +, \infty, 0)$
- languages over some alphabet Σ: $(2^{\Sigma^*}, \cup, \cdot, \emptyset, \{\epsilon\})$
- etc.

Weighted Automata

A weighted automaton (over a semiring S) is a structure (Q, I, K, T):

- Q: finite set of states, $I, K \subseteq Q$ initial / accepting states
- $T \subseteq Q \times S \times Q$

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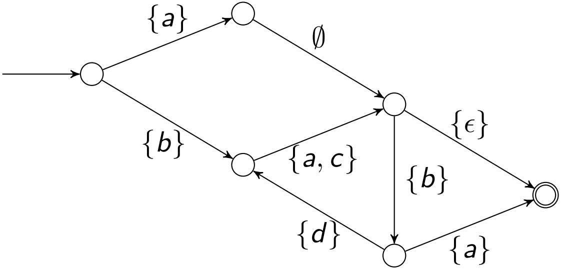
Examples:



$(2^{\Sigma^*}, \cup, \cdot, \emptyset, \{\epsilon\})$

- along paths: $\cdot$
- choice between paths: $\cup$
- usual automata

$|A| = \{b\}\{a,c\} \cup \{b\}\{a,c\}\{b\}\{a\} \cup \cdots$

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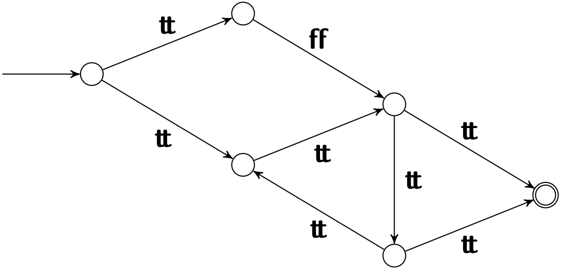

$(\{\mathbf{ff}, \mathbf{tt}\}, \vee, \wedge, \mathbf{ff}, \mathbf{tt})$

- along paths: $\wedge$
- choice between paths: $\vee$
- digraphs

$|A| = (\circledcirc \text{ is reachable})$

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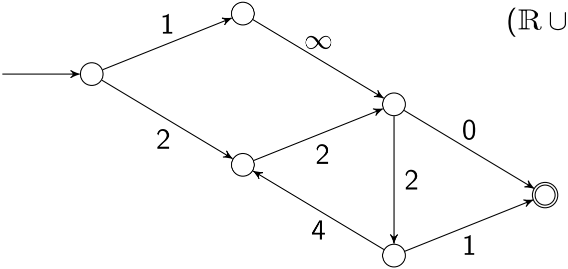

$(\mathbb{R} \cup \{\infty\}, \min, +, \infty, 0)$

- along paths: $+$
- choice between paths: min
- shortest path

$|A| = 4$

Weighted Automata

A weighted automaton (over a semiring S) is a structure (Q, I, K, T):

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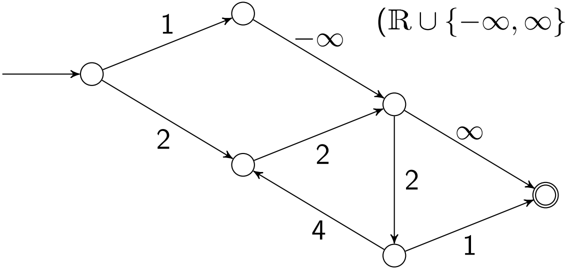


$(\mathbb{R} \cup \{-\infty, \infty\}, \max, \min, -\infty, \infty)$

- along paths: min
- choice between paths: max
- maximum flow

$|A| = 2$

Reachability in Weighted Automata

Let $S = (S, \oplus, \otimes, 0, 1)$ be a semiring and $A = (Q, I, K, T)$ a weighted automaton over S.

- a path in A: $\pi = q_1 \xrightarrow{x_1} q_2 \xrightarrow{x_2} \cdots \xrightarrow{x_{n-1}} q_n$ with all $(q_i, x_i, q_{i+1}) \in T$
- the value of π: $|\pi| = x_1 \otimes x_2 \otimes \cdots \otimes x_{n-1}$
- π accepting if $q_1 \in I$ and $q_n \in K$

Definition

The reachability value of A is

$$|A| = \bigoplus \{ |\pi| \mid \pi \text{ accepting path in } A \}$$

- $\otimes$ along paths; $\oplus$ between paths
- needs some provision for infinite sums!

Complete Semirings

Definition (repeat)

The reachability value of A is

$$|A| = \bigoplus \{|\pi| \mid \pi \text{ accepting path in } A\}$$

- needs some provision for infinite sums!

Definition

A semiring $(S, \oplus, \otimes, 0, 1)$ is complete if all infinite sums $\bigoplus X$ for $X \subseteq S$ exist.

- now the definition of $|A|$ makes sense
- but completeness is a rather restrictive condition
- we'll do something different

Continuous Kleene Algebras

From now on, restrict to idempotent semirings $(S, \oplus, \otimes, 0, 1)$.

- that is, $x \oplus x = x$ for all $x \in S$
- $\mathbb{N}$ is not idempotent, but $\mathbb{B}$, max-plus, min-plus, max-min, 2^{Σ^*} are, as are most other important examples
- write $\vee = \oplus$ and $\bot = 0$ for emphasis

Definition

A continuous Kleene algebra is an idempotent semiring $(S, \vee, \otimes, \bot, 1)$ in which $\bigvee X$ exists for all $X \subseteq S$, and such that for all $Y \subseteq S$, $x, z \in S$, $x(\bigvee Y)z = \bigvee xYz$.

- a complete idempotent semiring in which multiplication distributes over infinite suprema
- again, too restrictive

Star-Continuous Kleene Algebras

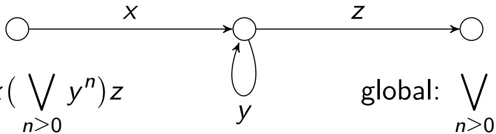
Definition (repeat)

A continuous Kleene algebra is an idempotent semiring $(S, \vee, \otimes, \bot, 1)$ in which $\bigvee X$ exists for all $X \subseteq S$, and such that for all $Y \subseteq S$, $x, z \in S$, $x(\bigvee Y)z = \bigvee xYz$.

Definition

A star-continuous Kleene algebra is an idempotent semiring $(S, \vee, \otimes, \bot, 1)$ in which $\bigvee\{x^n \mid n \geq 0\}$ exists for all $x \in S$, and such that for all $x, y, z \in S$, $x(\bigvee\{y^n \mid n \geq 0\})z = \bigvee x\{y^n \mid n \geq 0\}z$.

- loop abstraction:



local: $x(\bigvee_{n\geq 0} y^n)z$ global: $\bigvee_{n\geq 0}(xy^nz)$

Star-Continuous Kleene Algebras

For $x \in S$ in a star-continuous Kleene algebra S, define

$$x^* = \bigvee_{n \geq 0} x^n$$

- for languages, that's the Kleene star
- poor man's inverse: the equation

$$x^* = 1 \oplus x \oplus x^2 \oplus \cdots = \frac{1}{1 - x}$$

does make surprisingly much sense!

Matrix Semirings

Let S be a semiring and $n \geq 1$.

- $S^{n \times n}$: semiring of $n \times n$ matrices over S
- (with matrix addition and multiplication)
- If S is a star-continuous Kleene algebra, then so is $S^{n \times n}$
- with $M^*_{i,j} = \bigvee_{m \geq 0} \bigvee_{1 \leq k_1, \dots, k_m \leq n} M_{i,k_1} M_{k_1,k_2} \cdots M_{k_m,j}$
- and for $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ any partition,

  $$M^* = \begin{bmatrix} (a \vee bd^*c)^* & (a \vee bd^*c)^* bd^* \\ (d \vee ca^*b)^* ca^* & (d \vee ca^*b)^* \end{bmatrix}$$

  (recursively)
- "generalized Floyd-Warshall"

Reachability in Weighted Automata, II

Let $S = (S, \vee, \otimes, \perp, 1)$ be a star-continuous Kleene algebra and $A = (Q, I, K, T)$ a weighted automaton over S.

- transform A to matrix form:
  - recall $T : Q \times Q \to S$
  - write $Q = \{1, \dots, n\}$
  - then $I, K \subseteq Q$ become $\iota, \kappa \in \{\perp, 1\}^n$
  - and $T \in S^{n \times n}$: the transition matrix
- recall $|A| = \bigoplus \{ |\pi| \mid \pi \text{ accepting path in } A \}$

Theorem

$|A| = \iota T^* \kappa$

Motivation: Büchi Conditions in Weighted Automata

Let $A = (Q, I, K, T)$ be a weighted automaton over a semiring S

- an infinite path in A: $\pi = q_1 \xrightarrow{x_1} q_2 \xrightarrow{x_2} \cdots$ with all $(q_i, x_i, q_{i+1}) \in T$
- π Büchi if $q_1 \in I$ and

$$\{q \in Q \mid \forall n \geq 0 : \exists i \geq n : q_i = q\} \cap K \neq \emptyset$$

Goal: make sense of the "definition"

$$\|A\| = \bigoplus \{\|\pi\| \mid \pi \text{ Büchi path in } A\}$$

- but what is the value $\|\pi\|$ of an infinite path? an infinite product?
- and, how to compute the sum $\bigoplus$?

Semiring-Semimodule Pairs

- semiring $S = (S, \oplus, \otimes, 0, 1)$
- plus commutative monoid $V = (V, \oplus, 0)$
- left S-action $S \times V \to V$, $(s, v) \mapsto sv$
- such that for all $s, s' \in S$, $v \in V$:

$$
\begin{aligned}
(s \oplus s')v &= sv \oplus s'v & \qquad s(v \oplus v') &= sv \oplus sv' \\
(ss')v &= s(s'v) & 0s &= 0 \\
s0 &= 0 & 1v &= v
\end{aligned}
$$

- (think of vector spaces over fields, or modules over rings)

Ésik Algebras

Definition

An Ésik algebra is an idempotent semiring-semimodule pair (S, V) together with an infinite product $\prod : S^\omega \to V$, such that

- for all $x_0, x_1, \ldots \in S$, $\prod x_n = x_0 \prod x_{n+1}$;
- for any sequence $x_0, x_1, \ldots \in S$ and any sequence $0 = n_0 \le n_1 \le \cdots$ which increases without a bound, let $y_k = x_{n_k} \cdots x_{n_{k+1}-1}$ for all $k \ge 0$; then $\prod x_n = \prod y_k$.

- that is, $\prod$ generalizes the finite product in S
- a new name for an old notion (Zoltán Ésik passed away in 2016)
- S for values of finite paths; V for values of infinite paths:

$$\| q_1 \xrightarrow{x_1} q_2 \xrightarrow{x_2} \cdots \| = \prod x_{n-1}$$

Continuous Ésik Algebras

Definition

An Ésik algebra $(S, V, \prod)$ is continuous if

- S is a continuous Kleene algebra and V is a complete lattice,
- the S-action on V preserves all suprema in either argument, and
- for all $X_0, X_1, \ldots \subseteq S$, $\prod(\bigvee X_n) = \bigvee \{ \prod x_n \mid x_n \in X_n, n \geq 0 \}$.

- Ésik-Kuich 2004
- "continuous" $\implies$ too restrictive

Star-Continuous Ésik Algebras

Definition

An Ésik algebra $(S, V, \prod)$ is star-continuous if

- S is a star-continuous Kleene algebra,
- for all $x, y \in S$, $v \in V$, $xy^* v = \bigvee_{n \geq 0} xy^n v$,
- for all $x_0, x_1, \dots, y, z \in S$, $\prod (x_n(y \vee z)) = \bigvee_{x'_0, x'_1, \dots \in \{y, z\}} \prod x_n x'_n$,
- for all $x, y_0, y_1, \dots \in S$, $\prod x^* y_n = \bigvee_{k_0, k_1, \dots \geq 0} \prod x^{k_n} y_n$.

- Ésik-Fahrenberg-Legay-Quaas 2015

Matrix Semiring-Semimodule Pairs

Let (S, V) be a semiring-semimodule pair and $n \geq 1$.

- $(S^{n\times n}, V^n)$ is again a semiring-semimodule pair
- (the action is matrix-vector product)
- if (S, V) is a star-continuous Ésik algebra, then there is an operation ${}^\omega : S^{n\times n} \to V^n$ given by
  $M_i^\omega = \bigvee_{1\leq k_1,k_2,\ldots\leq n} M_{i,k_1} M_{k_1,k_2} \cdots$
  - (not a general infinite product)
- and for $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ any partition,

$$M^\omega = \begin{bmatrix} (a \vee bd^*c)^\omega \vee (a \vee bd^*c)^* bd^\omega \\ (d \vee ca^*b)^\omega \vee (d \vee ca^*b)^* ca^\omega \end{bmatrix}$$

(recursively)

Büchi Conditions in Weighted Automata, II

Let (S, V) be a star-continuous Ésik algebra and $A = (n, \iota, \kappa, T)$ a weighted automaton over S.

- reorder $Q = \{1, \dots, n\}$ so that $\kappa = (1, \dots, 1, \bot, \dots, \bot)$
  - that is, now the first $k \le n$ states are accepting
- write $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, with $a \in S^{k \times k}$

Theorem

$$\|A\| = \iota \begin{bmatrix} (a + bd^* c)^\omega \\ d^* c(a + bd^* c)^\omega \end{bmatrix}$$

Energy Problems

What is the minimum amount of battery required for the satellite to always be able to send and receive messages?

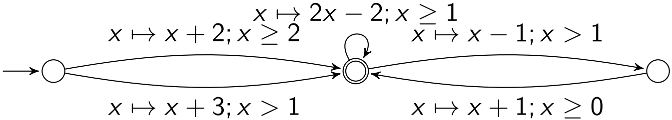
Energy Automata

Energy function:

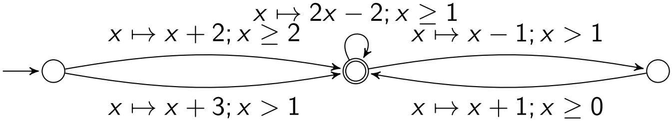
- partial function $f : \mathbb{R}_{\geq 0} \hookrightarrow \mathbb{R}_{\geq 0}$
- which is defined on some closed interval $[l_f, \infty[$ or on some open interval $]l_f, \infty[$,
- and such that for all $x \leq y$ for which f is defined,

$$f(y) - f(x) \geq y - x$$

Energy automaton: finite automaton labeled with energy functions



Energy Automata, Semantically



- start with initial energy x_0 and update at transitions according to label function
- if label function undefined on input, transition is disabled
- a discrete-time hybrid automaton (?)

Reachability: Given x_0, does there exist an accepting (finite) run with initial energy x_0?

Büchi: Given x_0, does there exist a Büchi (infinite) run with initial energy x_0?

Reachability in Energy Automata

- Let $L = [0, \infty]_\perp$: extended nonnegative real numbers plus $\perp$ (for "undefined")
  - (a complete lattice)
- Extended energy function: function $f : L \to L$
- with $f(\perp) = \perp$, and $f(\infty) = \infty$ unless $f(x) = \perp$ for all $x \in L$,
- and $f(y) - f(x) \geq y - x$ for all $x \leq y$.
- Set $\mathcal{E}$ of such functions is an idempotent semiring with operations $\vee$ (pointwise max) and $\circ$ (composition)
- in fact, a star-continuous Kleene algebra
  - $f^*(x) = x$ if $f(x) \leq x$; $f^*(x) = \infty$ if $f(x) > x$
  - not a continuous Kleene algebra

Theorem (Reachability)

There exists an accepting run from initial energy x_0 iff $|A|(x_0) \neq \perp$.

Büchi Runs in Energy Automata

- Let $\mathbf{2} = \{\mathbf{ff}, \mathbf{tt}\}$: the Boolean lattice
- Let $\mathcal{V}$ be the set of monotone and $\top$-continuous functions $L \to \mathbf{2}$
  - $f : L \to \mathbf{2}$ is called $\top$-continuous if $f(x) \equiv \mathbf{ff}$ or for all $X \subseteq L$ with $\bigvee X = \infty$, also $\bigvee f(X) = \mathbf{tt}$.
- $(\mathcal{E}, \mathcal{V})$ is an idempotent semiring-semimodule pair
- Define $\prod : \mathcal{E}^\omega \to \mathcal{V}$ by

$$(\prod f_n)(x) = \mathbf{tt} \text{ iff } \forall n \geq 0 : f_n(f_{n-1}(\cdots(x)\cdots)) \neq \bot$$

- Lemma: $\prod f_n$ is indeed $\top$-continuous for all $f_0, f_1, \ldots \in \mathcal{E}$
- Theorem: $(\mathcal{V}, \mathcal{E})$ is a star-continuous Ésik algebra
  - not a continuous Ésik algebra

Theorem (Büchi)

There exists a Büchi run from initial energy x_0 iff $\|A\|(x_0) \neq \mathbf{ff}$.

Computability

Let $\mathcal{E}' \subseteq \mathcal{E}$, $\mathcal{V}' \subseteq \mathcal{V}$ such that

- $\mathcal{E}'$ is closed under $\vee$, $\circ$, and *
- $\mathcal{V}'$ is closed under $\vee$ and contains all infinite products of elements of $\mathcal{E}'$
- all elements of $\mathcal{E}'$ and $\mathcal{V}'$ are finitely representable

Theorem

Reachability and Büchi acceptance are decidable for $\mathcal{E}'$-weighted energy automata.

The above holds for example for $\mathcal{E}'$ all piecewise linear energy functions.

Conclusion

- semirings and weighted automata: a very versatile framework
  - barely touched applications here
  - see Droste, Kuich, Vogler (eds.): Handbook of Weighted Automata, Springer 2009
- star-continuous Ésik algebras: a useful generalization of continuous Ésik algebras
  - (like star-continuous Kleene algebras are a useful generalization of continuous Kleene algebras)
- can be used to solve general energy problems

Ongoing work:

- real-time energy problems (FORMATS 2008; FM 2018; LMCS 2019; FAC 2025)
- hybrid systems?
- non-idempotent case?

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