ω-Algebras for Real-Time Energy Problems

Uli Fahrenberg

Formal Methods Laboratory & Paris-Saclay University

March 26, 2026

What is the minimum amount of battery required for the satellite to always be able to send and receive messages?

Personal story, spanning 20 years

Featuring (in rough chronological order)

- Kim G. Larsen,
- Patricia Bouyer, Nicolas Markey, Jiří Srba,
- Claus Thrane,
- Zoltan Ésik, Axel Legay, Karin Quaas,
- David Cachera,
- Sven Dziadek, Philipp Schlehuber-Caissier

2007

Infinite Runs in Weighted Timed Automata with Energy Constraints

Patricia Bouyer, Uli Fahrenberg, Kim G. Larsen,
Nicolas Markey, Jiří Srba

Dept. of Computer Science, Aalborg University, Denmark
Lab. Spécification et Vérification, Ecole Normale Supérieure Cachan, France

MT-LAB Meeting, September 2009

(With slides by Nicolas Markey)

One-slide summary

Goal:

Find infinite schedules in priced timed automata which satisfy constraints on total cost

- When should I plan to re-charge my laptop battery if I want to be sure to be able to watch YouTube videos during all my travel?
- How should I re-fill my oil tank so that it never runs out of oil and never runs over?

Results: mixed...
For some problems schedules computable in P, for some uncomputable.

Slogan:

Hybridization of timed automata

Energy Constraints

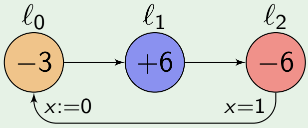
Energy is not only consumed, but can be regained.

$\rightsquigarrow$ "prices" can be negative;

$\rightsquigarrow$ the aim is to continuously satisfy cost constraints

$\rightsquigarrow$ in this paper, we focus on infinite runs.

Example



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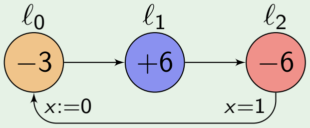

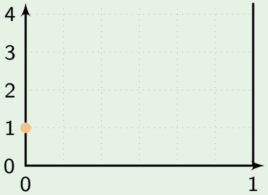

lower-bound problem

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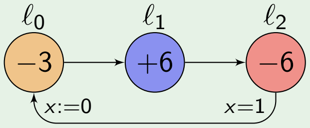

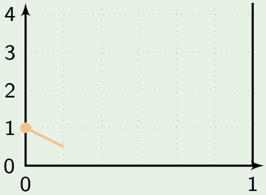

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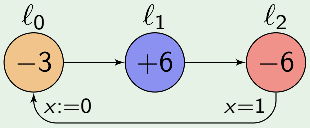

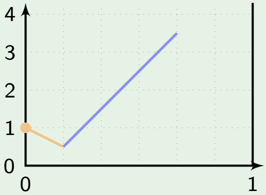

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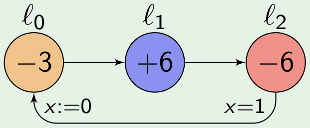

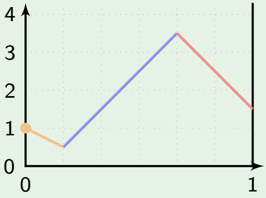

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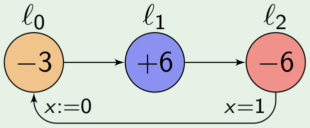

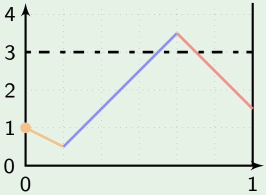

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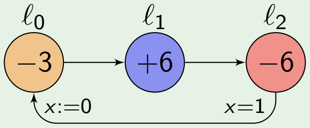

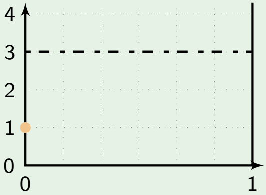

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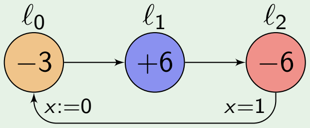

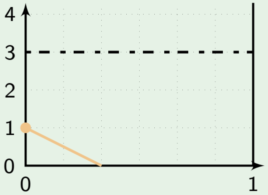

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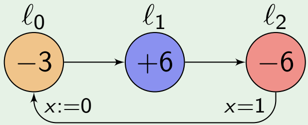

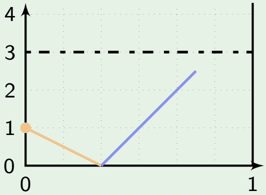

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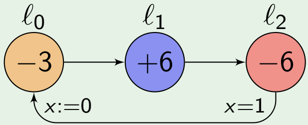

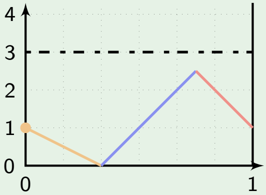

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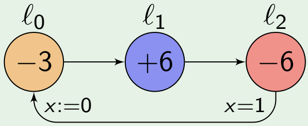

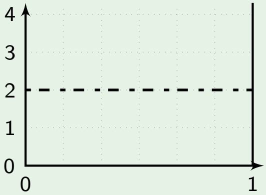

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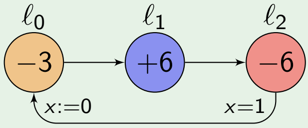

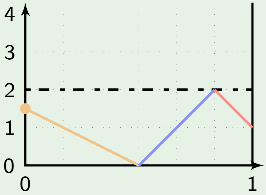

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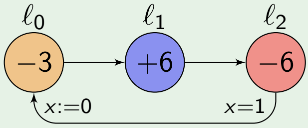

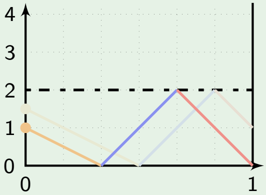

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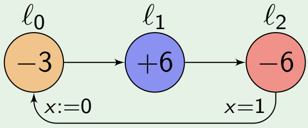

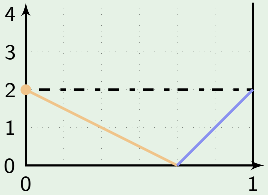

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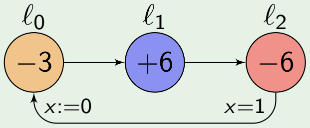

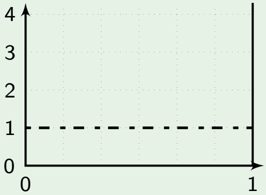

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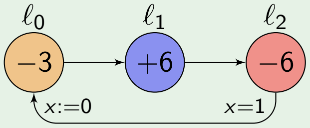

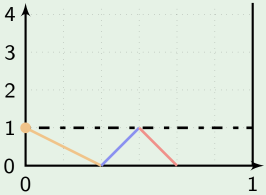

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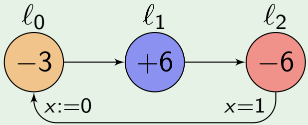
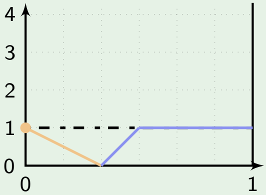


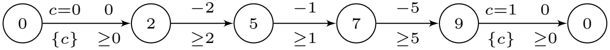


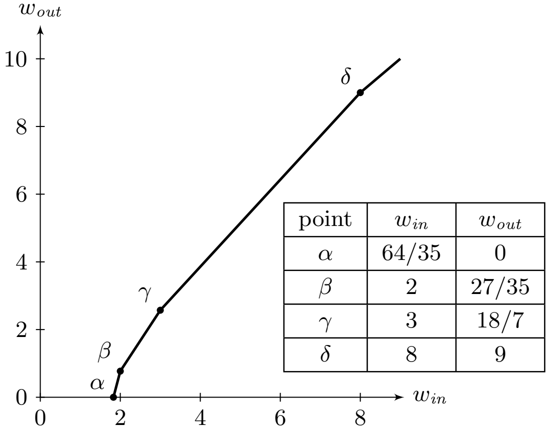
lower-weak-upper-bound problem

2009

after (w_n, ω_n).

Example 1. *We consider the following example, which is already in normal form. The corresponding function f_π then looks as depicted on Figure 5:*





point	w_{in}	w_{out}
α	64/35	0
β	2	27/35
γ	3	18/7
δ	8	9

Figure 5: Function f_π for example with linear observer

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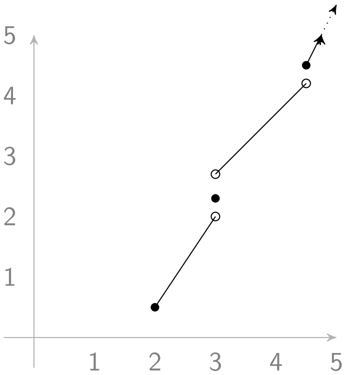
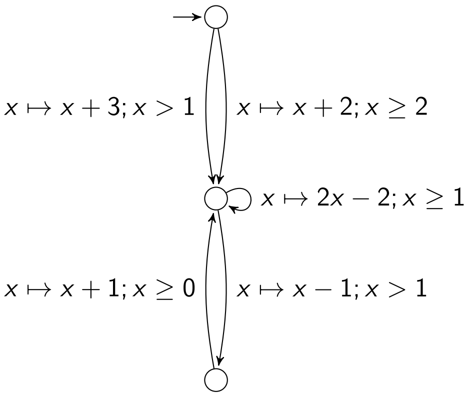
2011

Energy Automata, Examples



a simple energy function



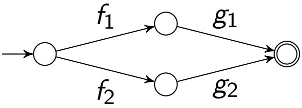
a simple energy automaton

Energy Function Semiring

Interest: reachability and Büchi acceptance

- Given a set F of accept states and $x_0 \in \mathbb{R}_{\geq 0}$: does there exist a run with initial energy x_0 which reaches F? does there exist one which visits F infinitely often?

Operations on energy functions: max and $\circ$



becomes $\max(g_1 \circ f_1, g_2 \circ f_2)$

The set $\mathcal{E}$ of energy functions with operations max and $\circ$ is a semiring, with $\mathbf{0} = \lambda x.\bot$, $\mathbf{1} = \lambda x.x$

- without "$f' \geq 1$" condition, only "near-semiring"
- idempotent, positively ordered, complete

Loops for Reachability

Star: $f^* = \sup_{n\geq 0} f^n$

- for loops which can be taken an arbitrary number of times
- $f^*(x) = \begin{cases} x & \text{if } f(x) \leq x \\ \infty & \text{if } f(x) > x \end{cases}$

Theorem: Always, $gf^*h = \sup_{n\geq 0} gf^nh$

- i.e. $\mathcal{E}$ is a star-continuous Kleene algebra

Corollary: Let M be the (transposed) transition matrix of an energy automaton

- i.e. M_{ji} is the transition label from state s_i to state s_j.

Compute $M^* = \sup_{n\geq 0} M^n$
Then s_j is reachable from s_i with initial energy x_0 iff $M^*_{ji}(x_0) \neq \bot$.

Loops for Infinite Runs

Omega: " $f^\omega = \lim_{n\to\infty} f^n$ "

- for loops which are taken infinitely often
- $f^\omega(x) = \begin{cases} \bot & \text{if } f(x) < x \quad \text{or } x = \bot \\ \top & \text{if } f(x) \ge x \quad \text{and } x \ne \bot \end{cases}$
- important: two-valued; $\mathcal{V}$: energy functions into $\{\bot, \top\}$

Theorem: $(\mathcal{E}, \mathcal{V})$ is a Conway semiring-semimodule pair

Corollary: Let M be the (transposed) transition matrix of an energy automaton

- i.e. M_{ji} is the transition label from state s_i to state s_j.

Compute " $M^\omega = \lim_{n\to\infty} M^n$ "
Then there is an infinite run from s_i with initial energy x_0 iff $M_i^\omega(x_0) \ne \bot$.

Some Technical Details for Reachability

(Applying work by S. Bloom, Z. Ésik, W. Kuich and others)

For a matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, with $a \in \mathcal{E}^{k\times k}$ and $d \in \mathcal{E}^{m\times m}$ (and $k + m = n$), let

$$M^* = \begin{bmatrix} (a \vee bd^* c)^* & (a \vee bd^* c)^* bd^* \\ (d \vee ca^* b)^* ca^* & (d \vee ca^* b)^* \end{bmatrix} \in \mathcal{E}^{n\times n}$$

Lemma: M^* does not depend on k and m, and

always $NM^* P = \sup_n NMP$.

- can also use (generalized) Floyd-Warshall algorithm to compute M^*; generally faster

Theorem: For any $\mathcal{E}$-automaton (S, M) with $S = \{1, \dots, n\}$, $F = \{1, \dots, k\}$, $k \le n$, $s_0 \le n$, and $x_0 \in \mathbb{R}_{\ge 0}$,
$\text{Reach}(s_0, x_0, F) = \mathbf{tt}$ iff ${}_t F^{\le k} M^* I^{s_0}(x_0) \neq \bot$.

Some Technical Details for Büchi Acceptance

(Extending work by S. Bloom, Z. Ésik, W. Kuich and others)

For a matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, with $a \in \mathcal{E}^{k \times k}$ and $d \in \mathcal{E}^{m \times m}$ (and $k + m = n$), let

$$M^\omega = {}_{\mathrm{t}}\begin{bmatrix} (a \vee bd^* c)^\omega \vee d^\omega c(a \vee bd^* c)^* \\ (d \vee ca^* b)^\omega \vee a^\omega b(d \vee ca^* b)^* \end{bmatrix} \in \mathcal{E}^{1 \times n}$$

$$M^{\omega_k} = {}_{\mathrm{t}}\begin{bmatrix} (a \vee bd^* c)^\omega \\ (a \vee bd^* c)^\omega bd^* \end{bmatrix} \in \mathcal{E}^{1 \times n}$$

Theorem: For any $\mathcal{E}$-automaton (S, M) with $S = \{1, \dots, n\}$, $F = \{1, \dots, k\}$, $k \le n$, $s_0 \le n$, and $x_0 \in \mathbb{R}_{\ge 0}$, $\text{Büchi}(s_0, x_0, F) = \mathbf{tt}$ iff $M^{\omega_k} I^{s_0}(x_0) \neq \bot$.

Conclusion, 2011

- Energy problems can be solved using the theory of semiring-weighted automata and semiring-semimodule pairs
  - for reachability, use star; for Büchi, use omega
- Extensions to multi-dimension or games: semiring techniques do not seem to apply
  - but techniques from well-structured transition systems do
  - for multi-dimensional games, undecidability is quickly reached
- Extension to energy automata with discrete inputs?
  - modeling discrete control problems

2014

Semirings

A semiring is a structure $(S, \oplus, \otimes, 0, 1)$ such that

- $(S, \oplus, 0)$ is a commutative monoid,
  - $x \oplus y = y \oplus x$, $x \oplus (y \oplus z) = (x \oplus y) \oplus z$, $x \oplus 0 = x$
- $(S, \otimes, 1)$ is a monoid,
  - $x \otimes (y \otimes z) = (x \otimes y) \otimes z$, $x \otimes 1 = 1 \otimes x = x$
- and which satisfies distributive and annihilation laws:
  - $x(y \oplus z) = xy \oplus xz$, $(x \oplus y)z = xz \oplus yz$
  - $x \otimes 0 = 0 \otimes x = 0$

Examples:

- natural numbers: $(\mathbb{N}, +, \cdot, 0, 1)$
- the boolean semiring: $(\{\mathbf{ff}, \mathbf{tt}\}, \vee, \wedge, \mathbf{ff}, \mathbf{tt})$
- max-plus algebra: $(\mathbb{R} \cup \{-\infty\}, \max, +, -\infty, 0)$
- min-plus algebra: $(\mathbb{R} \cup \{\infty\}, \min, +, \infty, 0)$
- languages over some alphabet Σ: $(2^{\Sigma^*}, \cup, \cdot, \emptyset, \{\epsilon\})$
- etc.

Weighted Automata

A weighted automaton (over a semiring S) is a structure (Q, I, K, T):

- Q: finite set of states, $I, K \subseteq Q$ initial / accepting states
- $T \subseteq Q \times S \times Q$

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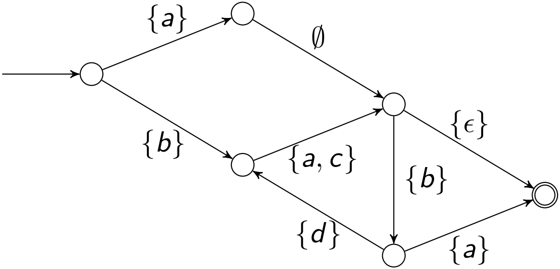

$(2^{\Sigma^*}, \cup, \cdot, \emptyset, \{\epsilon\})$

- along paths: $\cdot$
- choice between paths: $\cup$
- usual automata

$|A| = \{b\}\{a, c\} \cup \{b\}\{a, c\}\{b\}\{a\} \cup \cdots$

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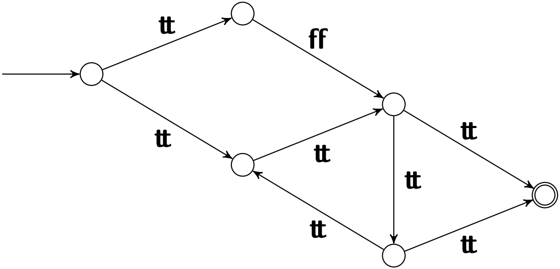
- $T \subseteq Q \times S \times Q$

Examples:



$(\{\mathbf{ff}, \mathbf{tt}\}, \vee, \wedge, \mathbf{ff}, \mathbf{tt})$

- along paths: $\wedge$
- choice between paths: $\vee$
- digraphs

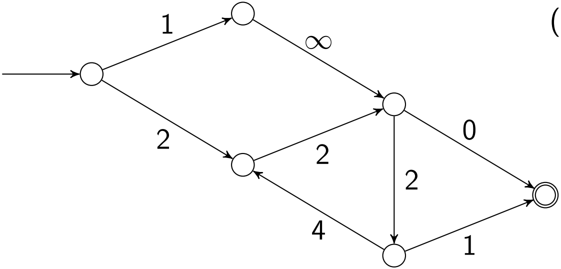
$|A| = (\circledcirc \text{ is reachable})$

Weighted Automata

A weighted automaton (over a semiring S) is a structure (Q, I, K, T):

- Q: finite set of states, $I, K \subseteq Q$ initial / accepting states
- $T \subseteq Q \times S \times Q$

Examples:



$(\mathbb{R} \cup \{\infty\}, \min, +, \infty, 0)$

- along paths: $+$
- choice between paths: min
- shortest path

$|A| = 4$

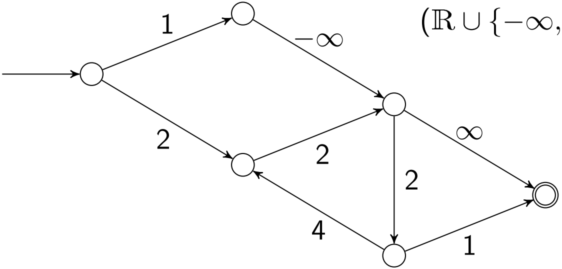
Weighted Automata

A weighted automaton (over a semiring S) is a structure (Q, I, K, T):

- Q: finite set of states, $I, K \subseteq Q$ initial / accepting states
- $T \subseteq Q \times S \times Q$

Examples:

$(\mathbb{R} \cup \{-\infty, \infty\}, \max, \min, -\infty, \infty)$



- along paths: min
- choice between paths: max
- maximum flow

$|A| = 2$

Reachability in Weighted Automata

Let $S = (S, \oplus, \otimes, 0, 1)$ be a semiring and $A = (Q, I, K, T)$ a weighted automaton over S.

- a path in A: $\pi = q_1 \xrightarrow{x_1} q_2 \xrightarrow{x_2} \cdots \xrightarrow{x_{n-1}} q_n$ with all $(q_i, x_i, q_{i+1}) \in T$
- the value of π: $|\pi| = x_1 \otimes x_2 \otimes \cdots \otimes x_{n-1}$
- π accepting if $q_1 \in I$ and $q_n \in K$

Definition

The reachability value of A is

$$|A| = \bigoplus \big\{ |\pi| \bigm| \pi \text{ accepting path in } A \big\}$$

- $\otimes$ along paths; $\oplus$ between paths
- needs some provision for infinite sums!

Complete Semirings

Definition (repeat)

The reachability value of A is

$$|A| = \bigoplus \big\{ |\pi| \;\big|\; \pi \text{ accepting path in } A \big\}$$

- needs some provision for infinite sums!

Definition

A semiring $(S, \oplus, \otimes, 0, 1)$ is complete if all infinite sums $\bigoplus X$ for $X \subseteq S$ exist.

- now the definition of $|A|$ makes sense
- but completeness is a rather restrictive condition
- we'll do something different

Continuous Kleene Algebras

From now on, restrict to idempotent semirings $(S, \oplus, \otimes, 0, 1)$.

- that is, $x \oplus x = x$ for all $x \in S$
- $\mathbb{N}$ is not idempotent, but $\mathbb{B}$, max-plus, min-plus, max-min, 2^{Σ^*} are, as are most other important examples
- write $\vee = \oplus$ and $\bot = 0$ for emphasis

Definition

A continuous Kleene algebra is an idempotent semiring $(S, \vee, \otimes, \bot, 1)$ in which $\bigvee X$ exists for all $X \subseteq S$, and such that for all $Y \subseteq S$, $x, z \in S$, $x(\bigvee Y)z = \bigvee xYz$.

- a complete idempotent semiring in which multiplication distributes over infinite suprema
- again, too restrictive

Star-Continuous Kleene Algebras

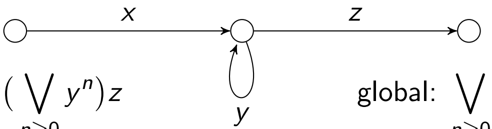
Definition (repeat)

A continuous Kleene algebra is an idempotent semiring $(S, \vee, \otimes, \bot, 1)$ in which $\bigvee X$ exists for all $X \subseteq S$, and such that for all $Y \subseteq S$, $x, z \in S$, $x(\bigvee Y)z = \bigvee xYz$.

Definition

A star-continuous Kleene algebra is an idempotent semiring $(S, \vee, \otimes, \bot, 1)$ in which $\bigvee\{x^n \mid n \geq 0\}$ exists for all $x \in S$, and such that for all $x, y, z \in S$, $x(\bigvee\{y^n \mid n \geq 0\})z = \bigvee x\{y^n \mid n \geq 0\}z$.

- loop abstraction:



local: $x(\bigvee_{n \geq 0} y^n)z$ global: $\bigvee_{n \geq 0} (xy^n z)$

Star-Continuous Kleene Algebras

For $x \in S$ in a star-continuous Kleene algebra S, define

$$x^* = \bigvee_{n \geq 0} x^n$$

- for languages, that's the Kleene star
- poor man's inverse: the equation

$$x^* = 1 \oplus x \oplus x^2 \oplus \cdots = \frac{1}{1 - x}$$

does make surprisingly much sense!

Matrix Semirings

Let S be a semiring and $n \geq 1$.

- $S^{n \times n}$: semiring of $n \times n$ matrices over S
- (with matrix addition and multiplication)
- If S is a star-continuous Kleene algebra, then so is $S^{n \times n}$
- with $M^*_{i,j} = \bigvee_{m \geq 0} \bigvee_{1 \leq k_1, \dots, k_m \leq n} M_{i,k_1} M_{k_1,k_2} \cdots M_{k_m,j}$
- and for $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ any partition,

  $$M^* = \begin{bmatrix} (a \vee bd^*c)^* & (a \vee bd^*c)^* bd^* \\ (d \vee ca^*b)^* ca^* & (d \vee ca^*b)^* \end{bmatrix}$$

  (recursively)
- "generalized Floyd-Warshall"

Reachability in Weighted Automata, II

Let $S = (S, \vee, \otimes, \bot, 1)$ be a star-continuous Kleene algebra and $A = (Q, I, K, T)$ a weighted automaton over S.

- transform A to matrix form:
  - recall $T : Q \times Q \to S$
  - write $Q = \{1, \dots, n\}$
  - then $I, K \subseteq Q$ become $\iota, \kappa \in \{\bot, 1\}^n$
  - and $T \in S^{n \times n}$: the transition matrix
- recall $|A| = \bigoplus \{ |\pi| \mid \pi \text{ accepting path in } A \}$

Theorem

$|A| = \iota T^* \kappa$

Motivation: Büchi Conditions in Weighted Automata

Let $A = (Q, I, K, T)$ be a weighted automaton over a semiring S

- an infinite path in A: $\pi = q_1 \xrightarrow{x_1} q_2 \xrightarrow{x_2} \cdots$ with all $(q_i, x_i, q_{i+1}) \in T$
- π Büchi if $q_1 \in I$ and

$$\{q \in Q \mid \forall n \geq 0 : \exists i \geq n : q_i = q\} \cap K \neq \emptyset$$

Goal: make sense of the "definition"

$$\|A\| = \bigoplus \Big\{ \|\pi\| \;\Big|\; \pi \text{ Büchi path in } A \Big\}$$

- but what is the value $\|\pi\|$ of an infinite path? an infinite product?
- and, how to compute the sum $\bigoplus$?

Semiring-Semimodule Pairs

- semiring $S = (S, \oplus, \otimes, 0, 1)$
- plus commutative monoid $V = (V, \oplus, 0)$
- left S-action $S \times V \to V$, $(s, v) \mapsto sv$
- such that for all $s, s' \in S$, $v \in V$:

$$(s \oplus s')v = sv \oplus s'v \qquad s(v \oplus v') = sv \oplus sv'$$
$$(ss')v = s(s'v) \qquad 0s = 0$$
$$s0 = 0 \qquad 1v = v$$

- (think of vector spaces over fields, or modules over rings)

Ésik Algebras

Definition

An Ésik algebra is an idempotent semiring-semimodule pair (S, V) together with an infinite product $\prod : S^\omega \to V$, such that

- for all $x_0, x_1, \ldots \in S$, $\prod x_n = x_0 \prod x_{n+1}$;
- for any sequence $x_0, x_1, \ldots \in S$ and any sequence $0 = n_0 \le n_1 \le \cdots$ which increases without a bound, let $y_k = x_{n_k} \cdots x_{n_{k+1}-1}$ for all $k \ge 0$; then $\prod x_n = \prod y_k$.

- that is, $\prod$ generalizes the finite product in S
- a new name for an old notion (Zoltán Ésik passed away in 2016)
- S for values of finite paths; V for values of infinite paths:

$$\left\| q_1 \xrightarrow{x_1} q_2 \xrightarrow{x_2} \cdots \right\| = \prod x_{n-1}$$

Continuous Ésik Algebras

Definition

An Ésik algebra $(S, V, \prod)$ is continuous if

- S is a continuous Kleene algebra and V is a complete lattice,
- the S-action on V preserves all suprema in either argument, and
- for all $X_0, X_1, \ldots \subseteq S$, $\prod(\bigvee X_n) = \bigvee \{ \prod x_n \mid x_n \in X_n, n \geq 0 \}$.

- Ésik-Kuich 2004
- "continuous" $\implies$ too restrictive

Star-Continuous Ésik Algebras

Definition

An Ésik algebra $(S, V, \prod)$ is star-continuous if

- S is a star-continuous Kleene algebra,
- for all $x, y \in S$, $v \in V$, $xy^*v = \bigvee_{n \geq 0} xy^n v$,
- for all $x_0, x_1, \dots, y, z \in S$, $\prod (x_n(y \vee z)) = \bigvee_{x_0', x_1', \dots \in \{y, z\}} \prod x_n x_n'$,
- for all $x, y_0, y_1, \dots \in S$, $\prod x^* y_n = \bigvee_{k_0, k_1, \dots \geq 0} \prod x^{k_n} y_n$.

- Ésik-Fahrenberg-Legay-Quaas 2015

Matrix Semiring-Semimodule Pairs

Let (S, V) be a semiring-semimodule pair and $n \geq 1$.

- $(S^{n \times n}, V^n)$ is again a semiring-semimodule pair
- (the action is matrix-vector product)
- if (S, V) is a star-continuous Ésik algebra, then there is an operation ${}^\omega : S^{n \times n} \to V^n$ given by $M^\omega_i = \bigvee_{1 \leq k_1, k_2, \ldots \leq n} M_{i,k_1} M_{k_1,k_2} \cdots$
  - (not a general infinite product)
- and for $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ any partition,

$$M^\omega = \begin{bmatrix} (a \vee bd^* c)^\omega \vee (a \vee bd^* c)^* bd^\omega \\ (d \vee ca^* b)^\omega \vee (d \vee ca^* b)^* ca^\omega \end{bmatrix}$$

(recursively)

Büchi Conditions in Weighted Automata, II

Let (S, V) be a star-continuous Ésik algebra and $A = (n, \iota, \kappa, T)$ a weighted automaton over S.

- reorder $Q = \{1, \dots, n\}$ so that $\kappa = (1, \dots, 1, \bot, \dots, \bot)$
  - that is, now the first $k \le n$ states are accepting
- write $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, with $a \in S^{k \times k}$

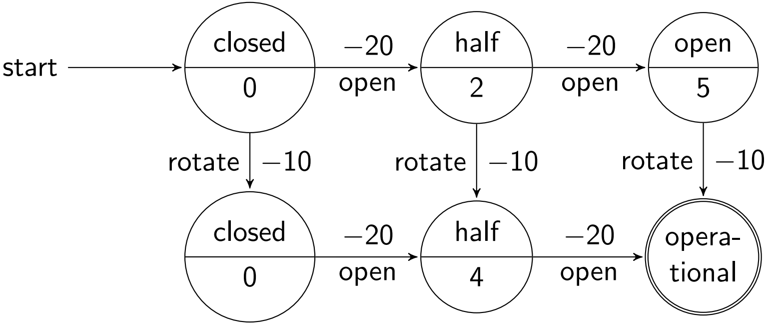
Theorem

$$\|A\| = \iota \begin{bmatrix} (a + bd^* c)^\omega \\ d^* c(a + bd^* c)^\omega \end{bmatrix}$$

2016

Example

Example



- given initial energy and time budget
- spend time in states to regain energy
- lose energy on transitions
- decide reachability / Büchi acceptance

Real-Time Energy Automata

A Real-Time Energy Automaton (S, s_0, F, T, r):

- finite set S of states,
- initial state $s_0 \in S$,
- accepting states $F \subseteq S$,
- transitions $T \subseteq S \times \mathbb{R}_{\le 0} \times \mathbb{R}_{\ge 0} \times S$
  - $s \xrightarrow[b]{p} s'$; p price, b bound

Semantics:

- configurations $(s, x, t) \in C = S \times \mathbb{R}_{\ge 0} \times \mathbb{R}_{\ge 0}$
  - x energy value; t available time
- $(s, x, t) \rightsquigarrow (s', x', t')$ iff $d = t - t' \ge 0$ and $\exists\, (s, p, b, s') \in T$ such that
  - $x + d\, r(s) \ge b$ and
  - $x' = x + d\, r(s) + p$

Problems

Let $A = (S, s_0, F, T, r)$ be a computable real-time energy automaton and $x_0, t, y \in [0, \infty]$ computable numbers.

State reachability: Does there exist a finite run $(s_0, x_0, t) \rightsquigarrow \cdots \rightsquigarrow (s, x, t')$ in A with $s \in F$?

Coverability: Does there exist a finite run $(s_0, x_0, t) \rightsquigarrow \cdots \rightsquigarrow (s, x, t')$ in A with $s \in F$ and $x \geq y$?

Büchi acceptance: Does there exist $s \in F$ and an infinite run $(s_0, x_0, t) \rightsquigarrow (s_1, x_1, t_1) \rightsquigarrow \cdots$ in A in which $s_n = s$ for infinitely many $n \geq 0$?

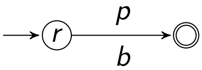
Real-Time Energy Functions

Idea: a real-time energy automaton computes a function

$$(x, t) \mapsto y$$

$$(\text{input energy}, \text{available time}) \mapsto \text{output energy}$$

Atomic functions:

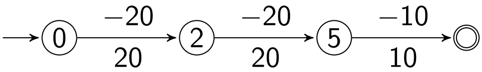


$$f(x,t) = \begin{cases} x + r\,t + p & \text{if } x + r\,t \geq b\,, \\ \bot & \text{otherwise} \end{cases}$$

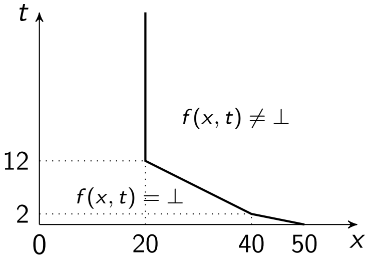
Composition:

$$f \triangleright g(x,t) = \bigvee_{t_1+t_2=t} g(f(x,t_1), t_2)$$

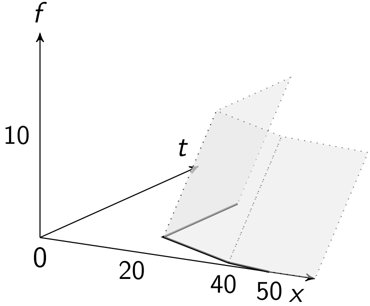
Example



−20
20
−20
20
−10
10
0
2
5



t
12
2
0
20
40
50
x
$f(x,t) \neq \bot$
$f(x,t) = \bot$



f
10
t
0
20
40
50
x

Operations on Real-Time Energy Functions

Composition: $f \rhd g(x,t) = \bigvee_{t_1+t_2=t} g(f(x,t_1),t_2)$

Maximum: $f \vee g(x,t) = \max(f(x,t), g(x,t))$

Star: $f^*(x,t) = \bigvee_{n\ge 0} f^n(x,t)$

Definition

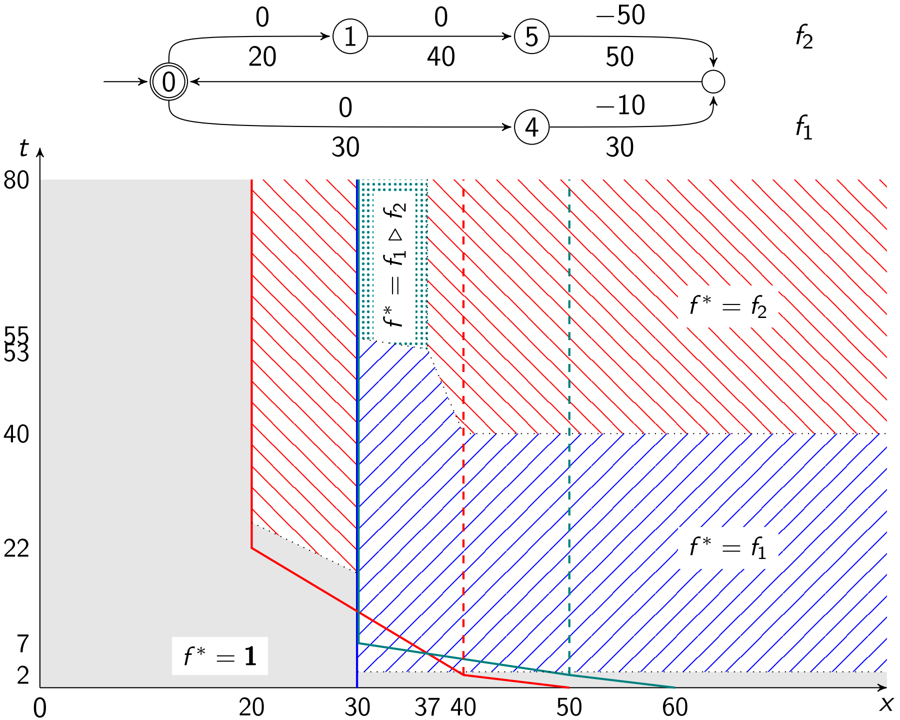
$\mathcal{E}$: set of functions generated by atomic functions under $\vee$ and $\rhd$.

Lemma

For every $f \in \mathcal{E}$ there exists $N \ge 0$ so that $f^* = \bigvee_{n=0}^{N} f^n$.

Corollary

$\mathcal{E}$ is locally closed, hence a *-continuous Kleene algebra.



0
1
5
4
20
40
50
−50
0
30
30
−10
f_2
f_1
t
80
55
53
40
22
7
2
0
20
30
37
40
50
60
x
$f^* = f_1 \rhd f_2$
$f^* = f_2$
$f^* = f_1$
$f^* = \mathbf{1}$

Reachability & Coverability

Let $A = (\alpha, M, \kappa)$ be a computable real-time energy automaton.

- (Recall: $\alpha \in \{\bot, 1\}^n$ initial vector, $\kappa \in \{\bot, 1\}^n$ accepting vector, $M \in S^{n \times n}$ transition matrix)

Compute $|A| = \alpha M^* \kappa$.

Let $x_0, t, y \in [0, \infty]$ computable numbers.

Theorem

There exists a finite run $(s_0, x_0, t) \leadsto \cdots \leadsto (s, x, t')$ in A with $s \in F$ iff $|A|(x_0, t) > \bot$.

Theorem

There exists a finite run $(s_0, x_0, t) \leadsto \cdots \leadsto (s, x, t')$ in A with $s \in F$ and $x \geq y$ iff $|A|(x_0, t) \geq y$.

Büchi Acceptance

- $\mathbb{B} = \{\mathbf{ff}, \mathbf{tt}\}$: the Boolean lattice, $\mathbf{ff} < \mathbf{tt}$
- $\mathcal{V}$: set of monotonic functions $[0, \infty] \times [0, \infty] \to \mathbb{B}$
- infinite product $\mathcal{E}^\omega \to \mathcal{V}$: $\prod_{n \ge 0} f_n(x, t) = \mathbf{tt}$ iff $\exists\, t_0, t_1, \ldots \in [0, \infty]$: $\sum_{n=0}^{\infty} t_n = t$ and $\forall\, n \ge 0,\ f_n(t_n) \circ \cdots \circ f_0(t_0)(x) \ne \bot$
- $\mathcal{U}$: subset of $\mathcal{V}$ generated by infinite products of $\mathcal{E}$-functions: a left $\mathcal{E}$-semimodule

Theorem

$(\mathcal{E}, \mathcal{U})$ forms a *-continuous Ésik algebra.

Büchi Acceptance

Let $A = (\alpha, M, \kappa)$ be a computable real-time energy automaton.

Write $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, with $a \in S^{k \times k}$, and compute

$$\|A\| = \alpha \begin{bmatrix} (a + bd^* c)^\omega \\ d^* c (a + bd^* c)^\omega \end{bmatrix}$$

Let $x_0, t, y \in [0, \infty]$ computable numbers.

Theorem

There exists $s \in F$ and an infinite run $(s_0, x_0, t) \leadsto (s_1, x_1, t_1) \leadsto \cdots$ in A in which $s_n = s$ for infinitely many $n \geq 0$ iff $\|A\|(x_0, t) = \mathbf{tt}$.

- Functions in $\mathcal{E}$ are computable piecewise linear, hence $|A|$ and $\|A\|$ are computable
- (probably in EXPTIME)

2018 –

Still going on

- UF, Larsen: *Energiautomater, energifunktioner og Kleene-algebra*. NIK 2018
- Bacci, Bouyer, UF, Larsen, Markey, Reynier: *Optimal and robust controller synthesis using energy timed automata with uncertainty*. FM 2018 & FAC 2021
- UF: *Une approche générique à la vérification quantitative*. Habilitation thesis, Paris-Saclay University
- Dziadek, UF, Schlehuber-Caissier: *Energy Büchi problems*. FM 2023
- Dziadek, UF, Schlehuber-Caissier: *ω-Regular energy problems*. FAC 2025

Conclusion?

State of the art, 2008:

6 Conclusion

A summary of the results proved in this paper is provided in the following table. The fields in gray remain open. Matching the complexity lower and upper bounds for some of the problems is left open: the lower-bound problems for finite games are strongly related to the well known open problem of complexity of mean-payoff games; closing the gap between NP-hardness and containment in PSPACE for the existential interval-bound problem seems intricate and it is a part of our future work.

	games		**existential problem**		**universal problem**	
	finite	**1-clock**	**finite**	**1-clock**	**finite**	**1-clock**
L	$\in$ UP $\cap$ coUP P-h (Th. 13)		$\in$ P (Th. 7)	$\in$ P (Th. 9)	$\in$ P (Th. 7)	$\in$ P (Th. 9)
L+W	$\in$ NP $\cap$ coNP P-h (Th. 13)		$\in$ P (Th. 7)	$\in$ P (Th. 9)	$\in$ P (Th. 7)	$\in$ P (Th. 9)
L+U	EXPTIME-c (Th. 14)	Undec. (Th. 15)	$\in$ PSPACE NP-h (Th. 14)		$\in$ P (Th. 14)	

Note that the results related to Theorem 9 hold for the initial credits arbitrarily close (for any given $\varepsilon > 0$) to the given ones.