# Discrete and Continuous Models for Concurrent Systems

## 1. The Geometry of Concurrency

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POPL 2026 Tutorial, Rennes, France

# Introduction

¿ Discrete and Continuous Models for Concurrent Systems ?

# Introduction

Discrete and Continuous Models for Concurrent Systems

# Introduction

Discrete and Continuous Models for Concurrent Systems

- plane ticket reservation system
- fleet of robots
- your computer!

# Introduction

Discrete and Continuous Models for Concurrent Systems

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Discrete and Continuous Models for Concurrent Systems

- networks of automata
- Petri nets
- event structures
- higher-dimensional automata

# Introduction

Discrete and Continuous Models for Concurrent Systems

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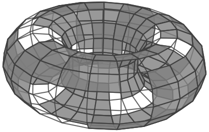
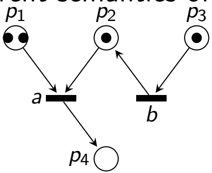
Discrete and Continuous Models for Concurrent Systems

- Variants of directed topological spaces

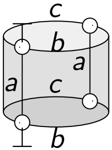
1. The geometry of concurrency



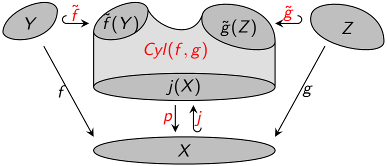
2. Concurrent semantics of Petri nets




3. Languages of higher-dimensional aut.




4. Advanced topics



# Viinistu

Based on lectures given at the 2025 Estonian Winter School in Computer Science

1. Introduction
2. Geometric Semantics
3. Directed Algebraic Topology

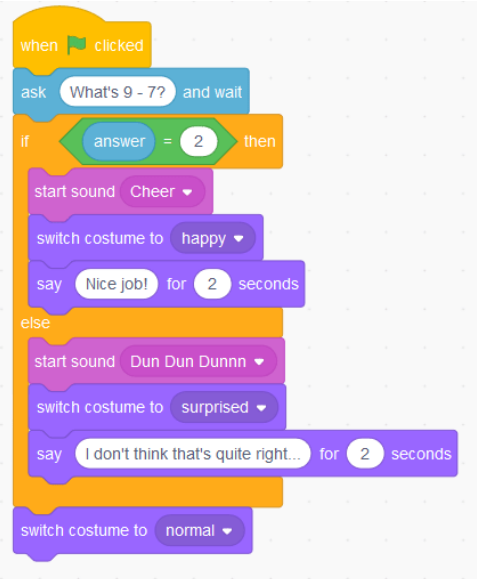
# Algebraic View

A program is

- a sequence of instructions
- plus branches
- and loops

Kleene algebra: set $S$ with

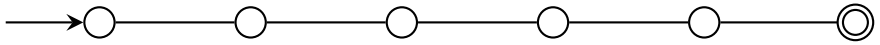
- concatenation $\otimes$
- choice $\oplus$
- repetition $^*$
- idempotent semiring with unary $^*$ which computes fixed points



# Geometric View

A program is a sequence of instructions

- ignoring branches and loops for now



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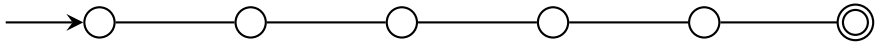

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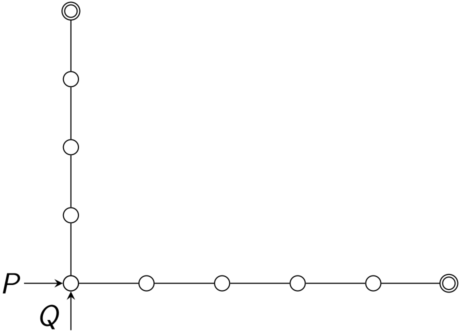

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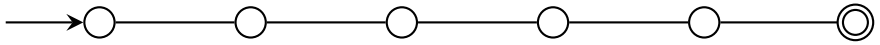

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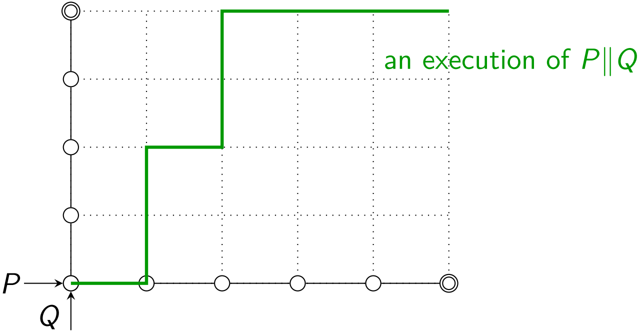

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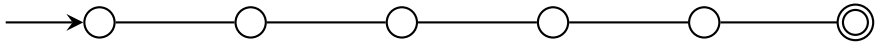

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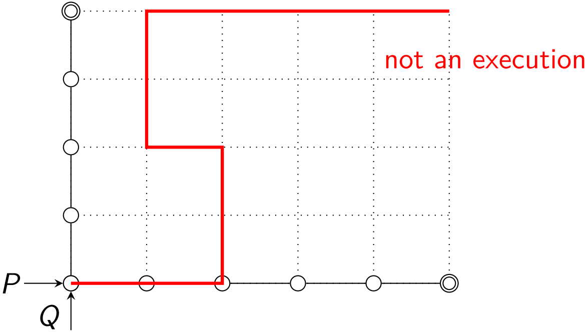

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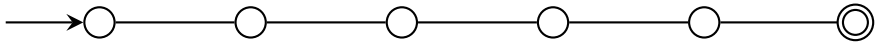

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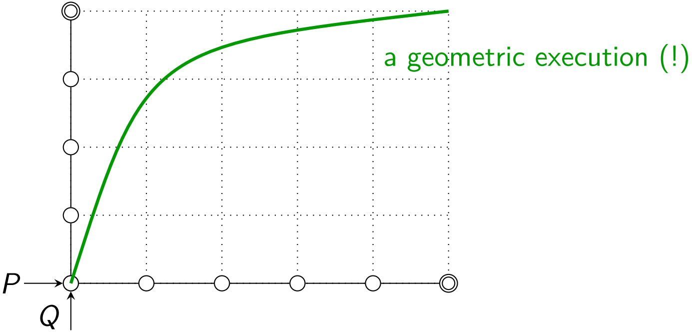

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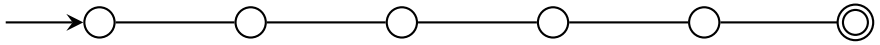

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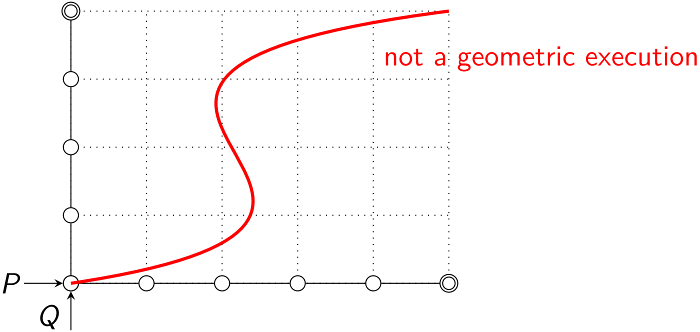
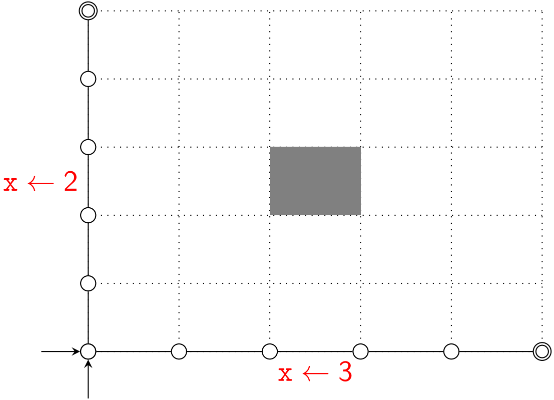


# Holes

Adding mutual exclusion:



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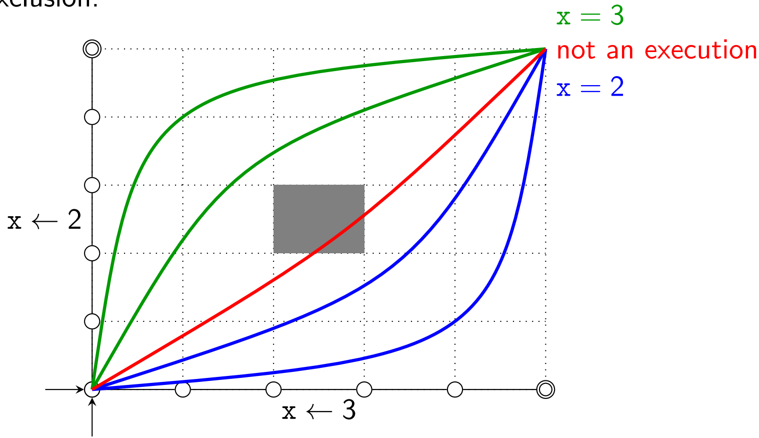

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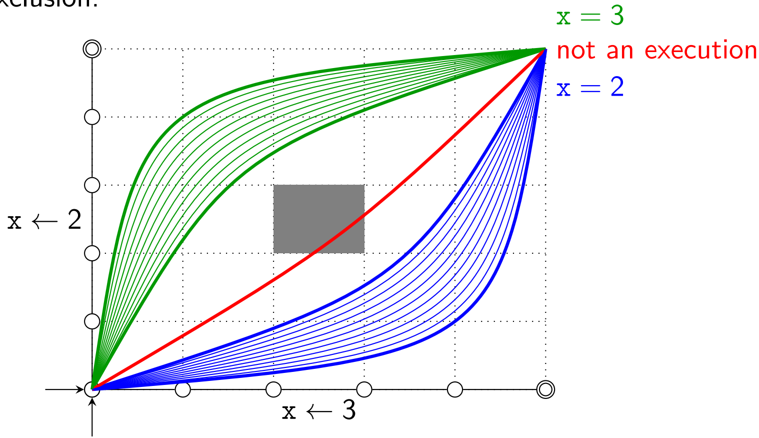
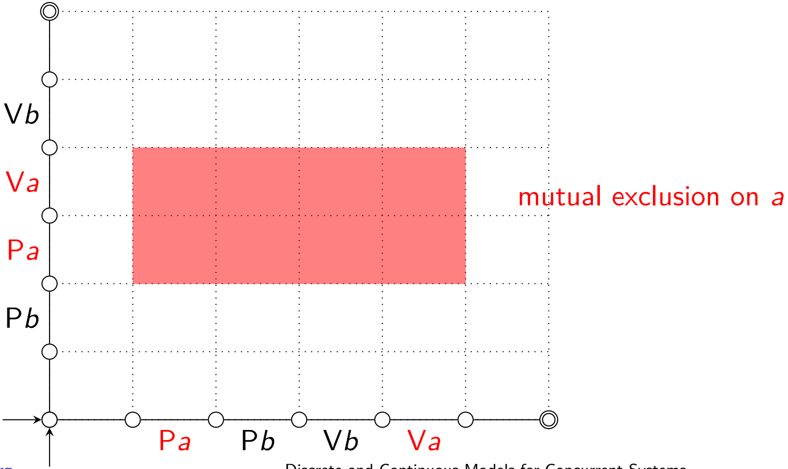



- homotopic paths ≙ equivalent executions

# More Holes

Semaphores à la Dijkstra ($\mathsf{P} \triangleq$ acquire; $\mathsf{V} \triangleq$ release):



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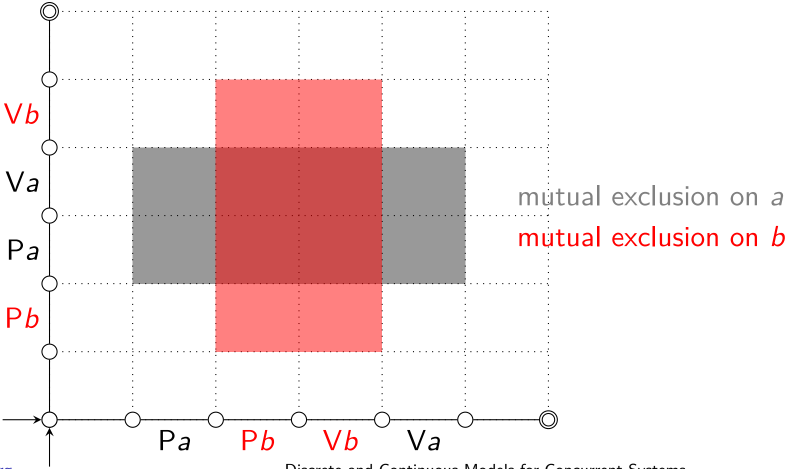

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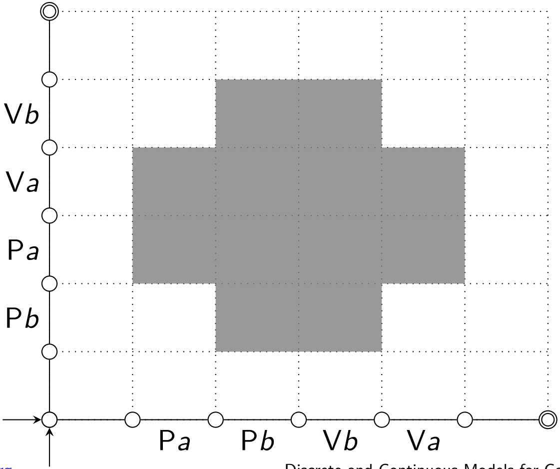

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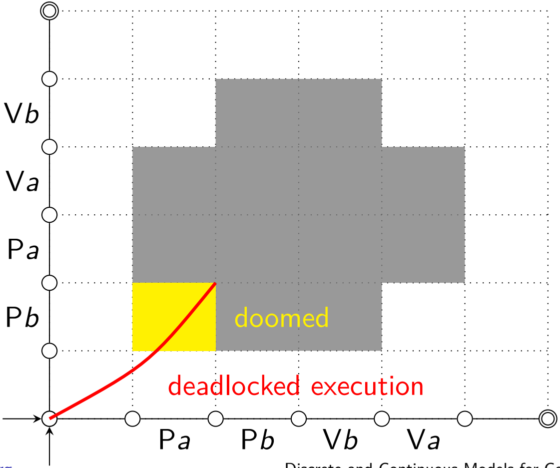

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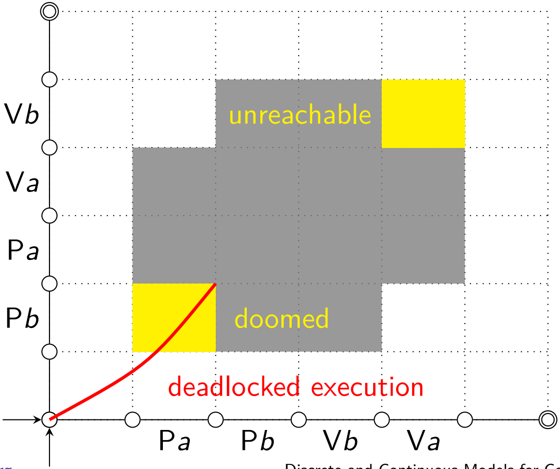
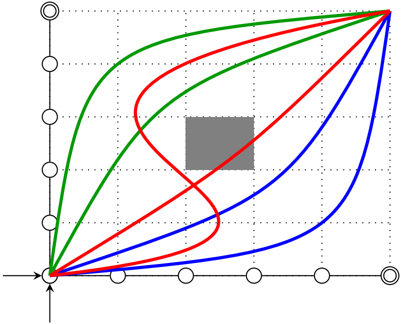
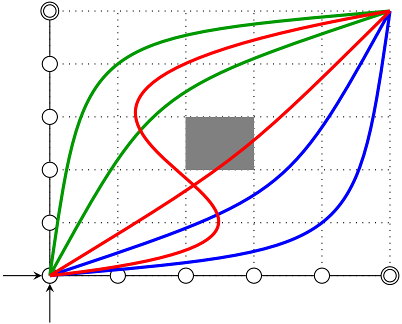


# Summing Up



- A program is a topological space
- An execution is a path through said space
- Two executions are equivalent iff their paths are homotopic
- Deadlocks and unreachable states are concave corners

# Summing Up



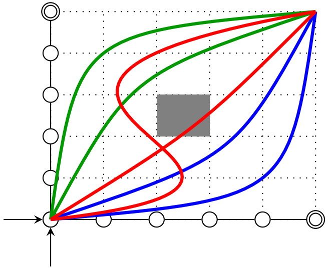
- A program is a directed topological space
- An execution is a directed path through said space
- Two executions are equivalent iff their dipaths are dihomotopic
- Deadlocks and unreachable states are concave corners

# Directed Spaces

## Definition (po-space)

A partially ordered space is a topological space $X$ together with a partial order $\leq$ on $X$ such that $\leq \subseteq X \times X$ is *closed* in the product topology.
A morphism of po-spaces is a $\leq$-preserving continuous function.

- a dipath in a po-space is a continuous & monotone path
  - a morphism $\vec{I} \to X$
- directed interval $\vec{I} = [0, 1]$ with usual order
- directed square $\vec{I} \times \vec{I}$, cube, etc.
- concatenation $\otimes$, branching $\oplus$
- no loops



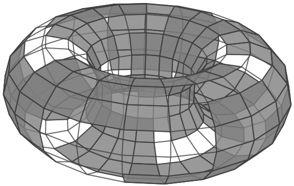
# Directed Spaces

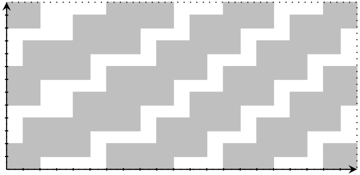
## Definition (lpo-space)

A locally partially ordered space is a *Hausdorff* topological space $X$ together with a relation $\leq$ on $X$ in which any $x \in X$ has an open neighborhood $U \ni x$ such that the restriction of $\leq$ to $U$ is a closed partial order.

A morphism of po-spaces is a continuous function which is *locally* $\leq$-preserving.

- $\leq$ may be taken globally reflexive and transitive





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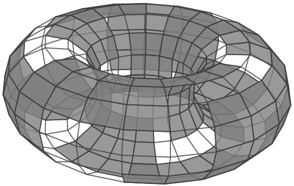
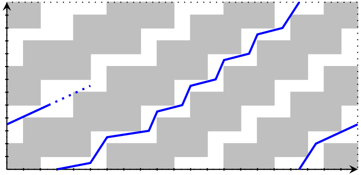




# Directed Spaces

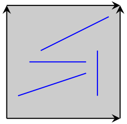
## Definition (d-space)

A directed space is a topological space $X$ together with a set $\vec{P}X$ of paths $I \to X$, called directed paths, such that

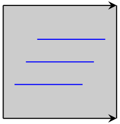
- all constant paths are directed,
- concatenations of directed paths are directed, and
- reparametrizations and restrictions of directed paths are directed.

A morphism of d-spaces is a continuous function which preserves directed paths.

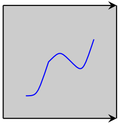
- a dipath is a morphism $p : \vec{I} \to X$, equivalently $p \in \vec{P}X$



$\vec{I} \times \vec{I}$



$\vec{I} \times I$



$\vec{I} \times \tilde{I}$

# Directed Spaces

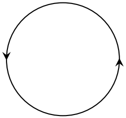
## Definition (d-space; unpacked)

A directed space is a topological space $X$ together with a set $\vec{P}X \subseteq \mathrm{Top}(I, X)$ such that

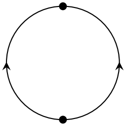
- $\forall p \in X : \lambda x.p \in \vec{P}X$ [const]
- $\forall \alpha, \beta \in \vec{P}X : \alpha(1) = \beta(0) \implies \alpha * \beta \in \vec{P}X$ [conc]
- $\forall \alpha \in \vec{P}X, \rho : \vec{I} \to \vec{I} : \alpha \circ \rho \in \vec{P}X$ [rep-rest]

A morphism of d-spaces $X$, $Y$ is $f \in \mathrm{Top}(X, Y)$ such that $\forall \alpha \in \vec{P}X : f \circ \alpha \in \vec{P}Y$.

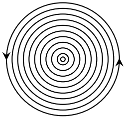
Resulting category dTop is complete and cocomplete (and dLCHaus is cartesian closed).



$\vec{S}_1$



$\vec{O}_1$



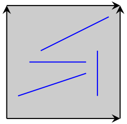
$C(\vec{S}_1)$

origin is a vortex

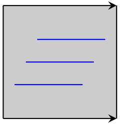
# Directed (?) Intervals

Three types of intervals:

- $\vec{I}$: $I = [0,1]$ with usual order
  - po-space; lpo-space
  - as d-space: $\vec{P}\vec{I}$: all monotone paths
- $I$: trivial order: $x \leq y$ iff $x = y$
  - po-space; lpo-space
  - as d-space: $\vec{P}I$: only constant paths
- $\tilde{I}$: chaotic d-space
  - d-structure: $\vec{P}\tilde{I}$: all paths
  - not an lpo-space (every point is a vortex!), neither a po-space



$\vec{I} \times \vec{I}$



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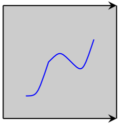
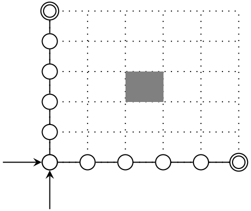


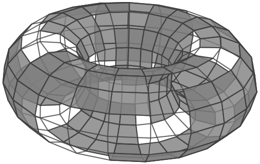
$\vec{I} \times \tilde{I}$

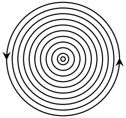
# Directed Spaces, Summary

- po-spaces: partially ordered topological spaces: no loops, nice but restrictive
- lpo-spaces: locally partially ordered top. spaces: loops OK, difficult to work with
- d-spaces: top. spaces with distinguished paths: nice category, but include vortices
- (other models exist)

- po-spaces $\hookrightarrow$ lpo-spaces $\hookrightarrow$ d-spaces (not full as categories)

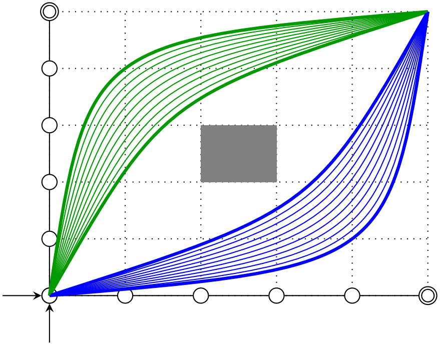






# Directed Paths and Homotopies

- the directed interval $\vec{I}$: $([0,1], \leq)$ (usual order): po-space
- dipaths in $X$: morphisms $\vec{I} \to X$
  - for d-space $(X, \vec{P}X)$: dipaths $\triangleq \vec{P}X$
- a dihomotopy $H : \tilde{I} \times \vec{I} \to X$:
  - all $H(s, \cdot)$ dipaths
  - $H : I \times I \to X$ continuous
  - $H(\cdot, 0)$ and $H(\cdot, 1)$ constant
  - (some variants exist)



# The Fundamental Category

- central object in algebraic topology: the fundamental group of a space $X$
- for $x \in X$, $\pi_1(X, x) = \{\alpha : I \to X \mid \alpha(0) = \alpha(1) = x\}$ modulo homotopy
- captures all information on homotopy of paths / 1-dimensional holes

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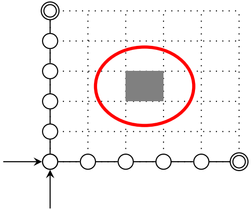
- captures all information on homotopy of paths / 1-dimensional holes
- in d-spaces, loops carry little information!



## Definition (fundamental category)

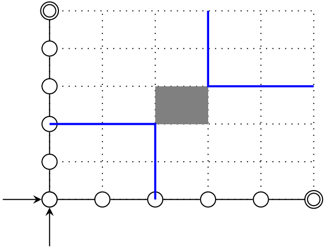
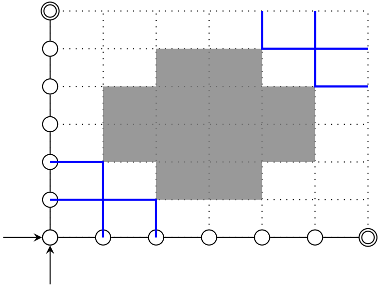
The fundamental category $\vec{\pi}_1(X)$ of a d-space $X$ has as

- objects points $x \in X$,
- morphisms $\vec{\pi}_1(X)(x, y) = \{\alpha : \vec{I} \to X \mid \alpha(0) = x, \alpha(1) = y\}$ modulo dihomotopy

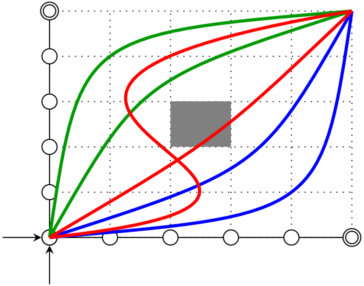
# The Fundamental Category

The fundamental cat $\vec{\pi}_1(X)$ of a d-space $X$ has as objects points $x \in X$ and as morphisms $\vec{\pi}_1(X)(x, y) = \{\alpha : \vec{I} \to X \mid \alpha(0) = x, \alpha(1) = y\}$ modulo dihomotopy.

- related: fundamental groupoid of topological spaces ("blow-up" of fundamental group)
- $\vec{\pi}_1(X)$ is huge $\implies$ identify components "where nothing happens"

# Summing Up



- A program is a directed topological space $X$
  - po-space, lpo-space, d-space, etc.
- An execution is a directed path $\vec{I} \to X$
- Two executions are equivalent iff they are related by a dihomotopy $\tilde{I} \times \vec{I} \to X$
- The fundamental category: useful invariant, but too big

- Directed homotopy equivalence; directed coverings; directed homology; directed topological complexity; etc.

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# Exercises!

**Exercise 1**: For each of the following objects: (1) make a drawing to understand it; (2) decide where it is situated in the hierarchy po-space $\hookrightarrow$ lpo-space $\hookrightarrow$ d-space; (3) draw a few dipaths.

1. The directed square $\vec{I} \times \vec{I}$
2. The "half-directed" square $\vec{I} \times I$, where $I$ is the po-space with order $x \leq y \iff x = y$
3. The "half-twiggly" square $\vec{I} \times \tilde{I}$, where $\tilde{I}$ is the d-space with $\vec{P}\tilde{I} = \text{Top}(I, I)$
4. $I \times \tilde{I}$
5. $\vec{S}_1$, where $S_1 = \{e^{it} \mid 0 \leq t \leq 2\pi\} \subseteq \mathbb{C}$ is the unit circle and $e^{it_1} \leq e^{it_2} \iff 0 \leq t_2 - t_1 \mod 2\pi < \pi$
6. $\vec{O}_1$, where $O_1 = S_1$ and $e^{it_1} \leq e^{it_2} \iff 0 \leq t_1 \leq t_2 \leq \pi$ or $\pi \leq t_2 \leq t_1 \leq 2\pi$
7. a finite automaton with language given by the regular expression $a(b + c)a$, seen as a "geometric graph" where the vertices are points and the edges, unit intervals
8. a finite automaton with language given by the regular expression $a(b + c)^*$

# Exercises!

**Exercise 2**: Show that for any d-space $X = (X, \vec{P}X)$, $\vec{P}X = \mathrm{dTop}(\vec{I}, X)$.

**Exercise 3**: Recall that in a po-space $(X, \leq)$, the relation $\leq$ is required to be closed in the product topology on $X \times X$. Show that any po-space is Hausdorff.

# Exercises!

**Exercise 4**: Let $X$ be the po-space-with-a-hole induced by the parallel composition of the PV programs P$a$ P$b$ V$b$ V$a$ and P$b$ P$a$ V$a$ V$b$. (We usually call it the Swiss Flag.)

1. How many dihomotopy classes of dipaths are there from initial to final state?
2. Say that a dipath $\alpha \in \vec{P}X$ is inessential if
   - for all $\beta \in \vec{P}X$ with $\beta(0) = \alpha(0)$ and $\alpha(1) \leq \beta(1)$, there is $\gamma \in \vec{P}X$ with $\gamma(0) = \alpha(1)$ and $\gamma(1) = \beta(1)$ such that $\beta$ and $\alpha * \gamma$ are dihomotopic;
   - for all $\beta \in \vec{P}X$ with $\beta(1) = \alpha(1)$ and $\beta(0) \leq \alpha(0)$, there is $\gamma \in \vec{P}X$ with $\gamma(1) = \alpha(0)$ and $\gamma(0) = \beta(0)$ such that $\beta$ and $\gamma * \alpha$ are dihomotopic.

The first condition says that if we can go from the beginning of $\alpha$ to some point $x = \beta(1)$, then we still can do so, and dihomotopically, once we have reached the end of $\alpha$: so taking $\alpha$ "does not make any choices". (The second condition says the same, but in reverse.)

Let $\cong$ be the equivalence relation induced on the points of $X$ by the existence of inessential dipaths. What does the partition of $X$ into $\cong$-equivalence classes look like?