Discrete and Continuous Models for Concurrent Systems

4. Advanced Topics

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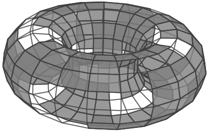
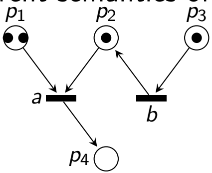
Laboratoire Méthodes Formelles

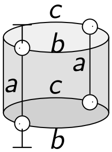
1. The geometry of concurrency



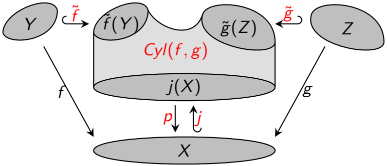
2. Concurrent semantics of Petri nets



3. Languages of higher-dimensional aut.



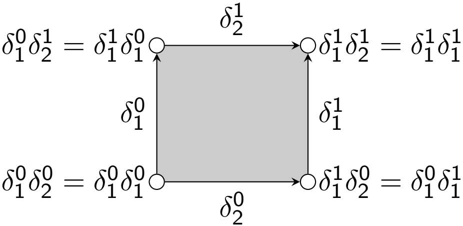
4. Advanced topics



Geometric Realisation

Definition

A precubical set is a graded set $X = \{X_n\}_{n\geq 0}$ together with face maps $\delta_i^0, \delta_i^1 : X_n \to X_{n-1}$, for $i = 1, \dots, n$, satisfying $\delta_i^\nu \delta_j^\mu = \delta_{j-1}^\mu \delta_i^\nu$ for $i < j$.



Definition

The geometric realisation of a precubical set X is the d-space $|X| = \bigsqcup_{n\geq 0} X_n \times \vec{I}^n /_\sim$, where $\sim$ is the equivalence generated by

$(\delta_i^\nu x, (t_1, \dots, t_{n-1})) \sim (x, (t_1, \dots, t_{i-1}, \nu, t_{i+1}, \dots, t_{n-1}))$.

Geometric Realisation

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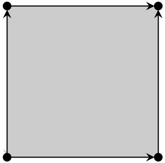
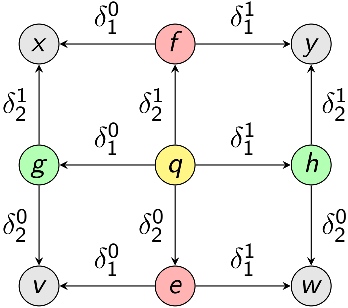
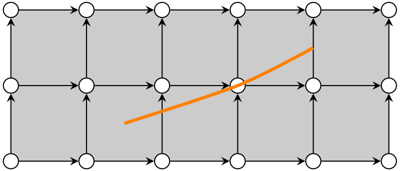
- usual coend definition; left adjoint to singular precubical set functor
- actually, $|X|$ is an lpo-space





Dipaths in Geometric Realisations

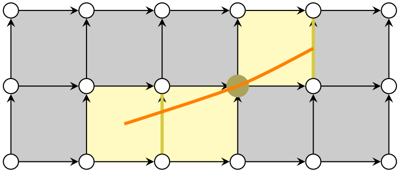


Let $p : \vec{I} \to |X|$ be a dipath in the geometric realisation of precubical set X.

- let $C_p = \{x \in X \mid \mathrm{im}(p) \cap |x| \neq \emptyset\}$ – all cells touched by p
- organize C_p into a sequence $c_p = (x_1, \dots, x_m)$ s.t. $\forall i$:

$$x_i = \delta^0_+ x_{i+1} \qquad \text{or} \qquad x_{i+1} = \delta^1_+ x_i \quad \text{(iterated face maps)}$$

Dipaths in Geometric Realisations



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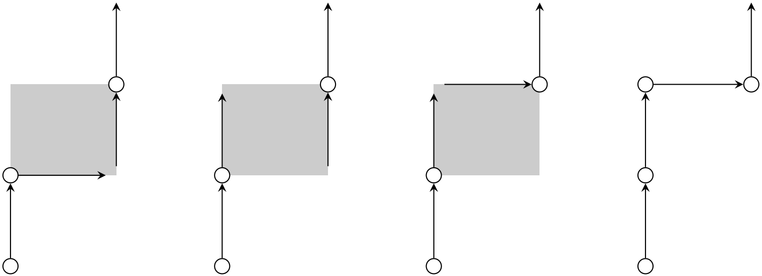
$\Longrightarrow$ the carrier sequence of p: a combinatorial path

Lemma

- *any combinatorial path c gives rise to dipath p_c (non-unique) with $c_{p_c} = c$*
- *if $c_p = c_q$, then p and q are dihomotopic*

Combinatorial Homotopy

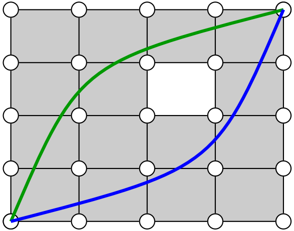


- equivalence relation on combinatorial paths generated by local replacements

Lemma

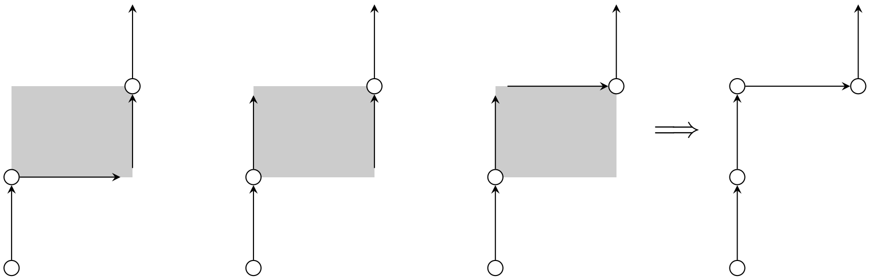
- *dipaths p, q are dihomotopic iff c_p and c_q are homotopic*
- *combinatorial paths c, d are homotopic iff p_c and p_d are dihomotopic*

Summing Up



- precubical sets: combinatorial models of directed spaces
- linked to directed spaces via geometric realisation
- dipaths $\hat{=}$ combinatorial paths $\hat{=}$ executions
- dihomotopy $\hat{=}$ combinatorial homotopy $\hat{=}$ equivalence of executions

Combinatorial Homotopy vs Subsumption



- subsumption $\sqsubseteq$: preorder generated by local replacements

$\Longrightarrow$ combinatorial homotopy $\simeq$ is the equivalence relation generated by subsumption

Lemma

- $\alpha \sqsubseteq \beta \implies \mathsf{ev}(\alpha) \sqsubseteq \mathsf{ev}(\beta)$

??? $\alpha \simeq \beta \implies \mathsf{ev}(\alpha)$??? $\mathsf{ev}(\beta)$

Precubical Sets?

Definition (5 mins ago)

A precubical set is a graded set $X = \{X_n\}_{n\geq 0}$ together with face maps $\delta_i^0, \delta_i^1 : X_n \to X_{n-1}$, for $i = 1, \dots, n$, satisfying $\delta_i^\nu \delta_j^\mu = \delta_{j-1}^\mu \delta_i^\nu$ for $i < j$.

Definition (30 mins ago)

A precubical set is a set X together with a mapping $\mathrm{ev} : X \to \text{(conclists)}$ and with face maps $\delta_A^0, \delta_A^1 : X[U] \to X[U \setminus A]$ satisfying $\delta_A^\mu \delta_B^\nu = \delta_B^\nu \delta_A^\mu$ for $A \cap B = \emptyset$.

Presheaves Presheaves Presheaves

(augmented) precube category $\square$	large (augmented) precube category $\boxdot$
objects $\{0,1\}^n$ for $n \geq 0$	objects totally ordered finite sets
morphisms injections of 0s and 1s	morphisms A, B-injections
skeletal	isos are unique

Lemma

The inclusion $\square \hookrightarrow \boxdot$ is an equivalence of categories with a unique left inverse.

Corollary

The presheaf categories $\mathsf{Set}^{\square^{\text{op}}}$ and $\mathsf{Set}^{\boxdot^{\text{op}}}$ are uniquely naturally isomorphic.

- precubical sets: $\mathsf{Set}^{\square^{\text{op}}}$ or $\mathsf{Set}^{\boxdot^{\text{op}}}$; makes no difference

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- (but, no labels in $\boxdot$; fundamental theorem takes care of this: For $\mathcal{C}$ a presheaf cat and $\Sigma \in \mathcal{C}$, also the slice $\mathcal{C}/\Sigma$ is a presheaf cat.)

Context

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augmented presimplex category Δ	large augmented presimplex category $\dot{\Delta}$
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category of ordinals	category of combinatorial species $\mathbb{B}$
objects $\{1, \dots, n\}$ for $n \geq 0$	objects finite sets
morphisms permutations	morphisms bijections
skeletal	isos are not unique

The Zoo of Cubes (It's HoTT!)

precubical set: graded set $X = \{X_n\}_{n \geq 0}$ plus face maps $\delta_i^0, \delta_i^1 : X_n \to X_{n-1}$

- plus degeneracies $\epsilon_i : X_n \to X_{n+1}$: cubical set
- with connections $\gamma_i^0, \gamma_i^1 : X_n \to X_{n+1}$
- with transpositions $\sigma_i : X_n \to X_n$
- with diagonals $\Delta : X_n \to X_{n-1}$

- many subsets of these are in use
- all are presheaves
- diagonals are important in cubical homotopy type theory

- cubical ω-categories with connections are equivalent to globular ω-categories

Monoids, Semirings, Kleene Algebra

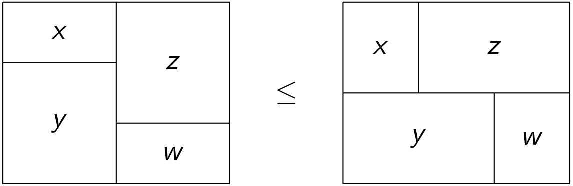
- a monoid: set S, operation $\cdot$ on S, associative with unit $1 \in S$
  - free monoid on Σ: Σ^*; $\cdot$ is concatenation
- a semiring: set S, operations $+$ (associative, commutative, unit 0) and $\cdot$ (associative, unit 1), plus distributivity $(x+y)z = xz + yz$ and $z(x+y) = zx + zy$
  - idempotent if $x + x = x$
  - free idempotent semiring on Σ: finite subsets of Σ^*
  - (powerset lifting; general principle of adding idempotent $+$ to algebraic structure)
- a Kleene algebra: idempotent semiring plus (unary) * operation
  - * "computes loops" / "computes least fixed points"
  - (different axiomatisations possible)

Concurrent Kleene Algebra

Definition

A concurrent monoid $(S, \cdot, \|, 1, \leq)$:

- S set, $\cdot$ and $\|$ associative operations with shared unit 1
  - $\cdot$ concatenation, $\|$ parallel composition
- $\leq$ partial order on S such that $\cdot$ and $\|$ are monotone
  - $x \leq y \implies x \cdot z \leq y \cdot z$ and $x \| z \leq y \| z$ etc.
- and such that $(x \| y) \cdot (z \| w) \leq (x \cdot z) \| (y \cdot w)$
  - lax interchange:



Concurrent Kleene Algebra?

- concurrent monoid: $(S, \cdot, \|, 1, \leq)$
- free concurrent monoid on Σ: series-parallel pomsets
  - pomsets obtained from $a \in \Sigma$ by series and parallel composition
- Theorem (Valdes-Tarjan-Lawler '82): pomset P is series-parallel iff P contains no induced N

Concurrent Kleene Algebra?

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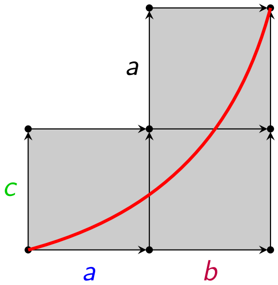
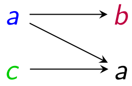
- Theorem (Valdes-Tarjan-Lawler '82): pomset P is series-parallel iff P contains no induced N
- but look we have N's:





$\Longrightarrow$ standard CKA does not seem suited for HDA languages!

Special ipomsets

Definition

An ipomset $(P, <, \dashrightarrow, S, T, \lambda)$ is

- discrete if $<$ is empty (hence $\dashrightarrow$ is total) $\left[\begin{smallmatrix} \bullet a \\ \bullet b \bullet \\ a \bullet \end{smallmatrix}\right]$
  - also written ${}_S P_T$
- a conclist ("concurrency list") if it is discrete and $S = T = \emptyset$ $\left[\begin{smallmatrix} a \\ b \\ a \end{smallmatrix}\right]$
- a starter if it is discrete and $T = P$ $\left[\begin{smallmatrix} \bullet a \bullet \\ \bullet b \bullet \\ a \bullet \end{smallmatrix}\right]$
- a terminator if it is discrete and $S = P$ $\left[\begin{smallmatrix} \bullet a \\ \bullet b \bullet \\ \bullet a \bullet \end{smallmatrix}\right]$
- an identity if it is both a starter and a terminator $\left[\begin{smallmatrix} \bullet a \bullet \\ \bullet b \bullet \\ \bullet a \bullet \end{smallmatrix}\right]$

Lemma (Janicki-Koutny 93; reformulated)

An ipomset is interval iff it has a decomposition into discrete ipomsets.

Decompositions

Lemma (Janicki-Koutny 93)

A poset $(P, <)$ is an interval order iff the order defined on its maximal antichains defined by $A \preceq B \iff \forall a \in A, b \in B : b \not< a$ is total.

Corollary

An ipomset is interval iff it has a decomposition into discrete ipomsets.

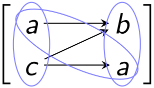
Lemma

Any discrete ipomset is a gluing of a starter and a terminator.

$$\begin{bmatrix} \bullet a \\ \bullet b \bullet \\ a \bullet \end{bmatrix} = \begin{bmatrix} \bullet a \bullet \\ \bullet b \bullet \\ a \bullet \end{bmatrix} * \begin{bmatrix} \bullet a \\ \bullet b \bullet \\ \bullet a \bullet \end{bmatrix}$$

Corollary

Any interval ipomset has a decomposition as a sequence of starters and terminators.



$$= \begin{bmatrix} a \bullet \\ c \end{bmatrix} * \begin{bmatrix} \bullet a \\ a \bullet \end{bmatrix} * \begin{bmatrix} b \\ \bullet a \end{bmatrix} = \begin{bmatrix} a \bullet \\ c \bullet \end{bmatrix} * \begin{bmatrix} \bullet a \bullet \\ \bullet c \end{bmatrix} * \begin{bmatrix} \bullet a \bullet \\ a \bullet \end{bmatrix} * \begin{bmatrix} \bullet a \\ \bullet a \bullet \end{bmatrix} * \begin{bmatrix} b \bullet \\ \bullet a \bullet \end{bmatrix} * \begin{bmatrix} \bullet b \\ \bullet a \end{bmatrix}$$

Unique decompositions

Notation: St: set of starters ${}_S U_U$
Te: set of terminators ${}_U U_T$
$\mathsf{Id} = \mathsf{St} \cap \mathsf{Te}$: set of identities ${}_U U_U$
$\Omega = \mathsf{St} \cup \mathsf{Te}$

Definition

A word $w = (S_1, U_1, T_1) \dots (S_n, U_n, T_n) \in \Omega^+$ is coherent if $T_i = S_{i+1}$ for all i.

Definition

A coherent word is sparse if proper starters and proper terminators are alternating.

- additionally, all $w \in \mathsf{Id} \subseteq \Omega^+$ are sparse
- so that's $\mathsf{Id} \cup (\mathsf{St} \setminus \mathsf{Id})((\mathsf{Te} \setminus \mathsf{Id})(\mathsf{St} \setminus \mathsf{Id}))^* \cup (\mathsf{Te} \setminus \mathsf{Id})((\mathsf{St} \setminus \mathsf{Id})(\mathsf{Te} \setminus \mathsf{Id}))^*$

Lemma

Any interval ipomset P has a unique decomposition $P = P_1 * \cdots * P_n$ such that $P_1 \dots P_n \in \Omega^+$ is sparse.

Step sequences

Let $\sim$ be the congruence on Ω^+ generated by the relation

$$_S U_U \cdot {}_U T_T \sim {}_S T_T \qquad {}_S S_U \cdot {}_U U_T \sim {}_S S_T$$

- (compose subsequent starters and subsequent terminators)

Definition

A step sequence is a $\sim$-equivalence class of coherent words in Ω^+.

Lemma

Any step sequence has a unique sparse representant.

Theorem

The category of interval ipomsets is isomorphic to the category of step sequences.

Categories?

Definition (Category iiPoms)

- **objects:** conclists U (discrete ipomsets without interfaces)
- **morphisms** in $\mathsf{iiPoms}(U, V)$: interval ipomsets P with sources U and targets V
- **composition:** gluing
- **identities** $\mathrm{id}_U = {}_U U_U$

Definition (Category Coh)

- **objects:** conclists U (discrete ipomsets without interfaces)
- **morphisms** in $\mathsf{Coh}(U, V)$: step sequences $[(S_1, U_1, T_1) \dots (S_n, U_n, T_n)]_\sim$ with $S_1 = U$ and $T_n = V$
- **composition:** concatenation
- **identities** $\mathrm{id}_U = {}_U U_U$

- Coh is category generated from (directed multi)graph Ω under relations $\sim$
- isomorphisms $\Phi : \mathsf{iiPoms} \leftrightarrow \mathsf{Coh} : \Psi$ provided by
  - $\Phi(P) = [w]_\sim$, where w is any step decomposition of P;
  - $\Psi([P_1 \dots P_n]_\sim) = P_1 * \dots * P_n$ (needs lemma)

Algebra?

- this is not cancellative:

$$a\bullet \begin{bmatrix} \bullet a\bullet \\ a\bullet \end{bmatrix} = a\bullet \begin{bmatrix} a\bullet \\ \bullet a\bullet \end{bmatrix} = \begin{bmatrix} a\bullet \\ a\bullet \end{bmatrix}$$

- "categorical concurrent Kleene algebra"?
- (what to do about subsumptions? 2-categories?)

Path Objects

Theorem

For every ipomset P there exists an HDA $\square^P := X$ such that $L(X) = \{P\}\downarrow$.

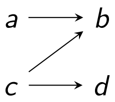
- the path object of P
- ad-hoc construction

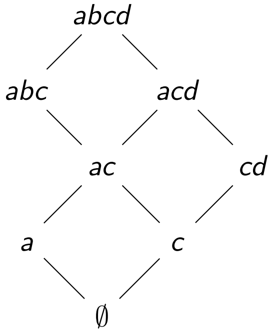
Lemma (very useful!)

For any ipomset P and HDA X, $P \in L(X) \iff \exists f : \square^P \to X$.

- express languages using morphisms

Now Look At Those Down-Closed Subsets!





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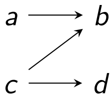

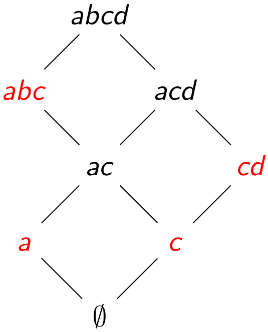

- join-irreducibles $\hat{=}$ principal downsets

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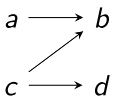

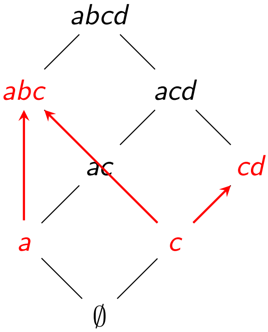

- join-irreducibles $\hat{=}$ principal downsets
- induced order isomorphic to P

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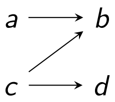

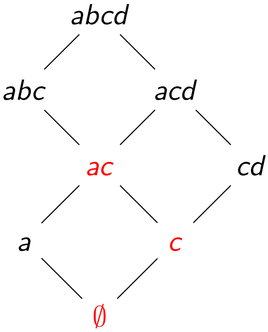

- join-irreducibles $\hat{=}$ principal downsets
- induced order isomorphic to P
- principal strict downsets totally ordered $\iff$ P interval order

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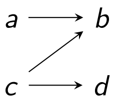

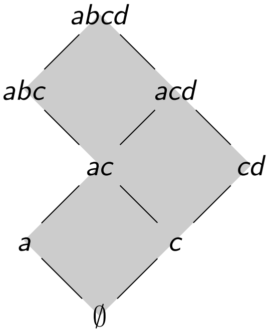

- join-irreducibles $\hat{=}$ principal downsets
- induced order isomorphic to P
- principal strict downsets totally ordered $\iff$ P interval order
- Conjecture: path object $\square^P$ $\hat{=}$ CAT0-closure of subset lattice

Conclusion

1. The Geometry of Concurrency
2. Concurrent Semantics of Petri Nets
3. Languages of Higher-Dimensional Automata
4. Advanced Topics

- Geometry and topology of concurrency provide intuition and methodology
- Petri nets are a nice and useful model for concurrent systems
- Higher-dimensional automata are a powerful model for concurrent systems with a nice language theory
- Non-interleaving concurrency is both nice and necessary
- Categories, functors, and presheavs are everywhere

Selected Bibliography

- L.Fajstrup. *Dicovering spaces.* Homology Homotopy Appl. 2003
- L.Fajstrup. *Dipaths and dihomotopies in a cubical complex.* Adv.Appl.Math. 2005
- F.A.Al-Agl, R.Brown, R.Steiner. *Multiple categories: the equivalence of a globular and a cubical approach.* Adv.Math. 2002
- S.Awodey. *Cartesian cubical model categories.* arxiv 2023
- T.Hoare, B.Möller, G.Struth, I.Wehrman. *Concurrent Kleene algebra and its foundations.* J.Log.Alg.Prog. 2011
- C.Calk, L.Santocanale. *Complete congruences of completely distributive lattices.* RAMiCS 2024