A functor $T : C \to B$ is *faithful* (or an embedding) when to every pair c, c' of objects of C and to every pair $f_1, f_2 : c \to c'$ of parallel arrows of C the equality $Tf_1 = Tf_2 : Tc \to Tc'$ implies $f_1 = f_2$. Again, composites of faithful functors are faithful. For example, the forgetful functor **Grp**→**Set** is faithful but not full and not a bijection on objects.

These two properties may be visualized in terms of hom-sets (see (2.5)). Given a pair of objects $c, c' \in C$, the arrow function of $T : C \to B$ assigns to each $f : c \to c'$ an arrow $Tf : Tc \to Tc'$ and so defines a function

$$T_{c,c'} : \hom(c, c') \to \hom(Tc, Tc'), \qquad f \mapsto Tf.$$

Then T is full when every such function is surjective, and faithful when every such function is injective. For a functor which is both full and faithful (i.e., "fully faithful"), every such function is a bijection, but this need not mean that the functor itself is an isomorphism of categories, for there may be objects of B not in the image of T.

A *subcategory* S of a category C is a collection of some of the objects and some of the arrows of C, which includes with each arrow f both the object dom f and the object cod f, with each object s its identity arrow 1_s and with each pair of composable arrows $s \to s' \to s''$ their composite. These conditions ensure that these collections of objects and arrows themselves constitute a category S. Moreover, the injection (inclusion) map $S \to C$ which sends each object and each arrow of S to itself (in C) is a functor, the *inclusion functor*. This inclusion functor is automatically faithful. We say that S is a *full subcategory* of C when the inclusion functor $S \to C$ is full. A full subcategory, given C, is thus determined by giving just the set of its objects, since the arrows between any two of these objects s, s' are all morphisms $s \to s'$ in C. For example, the category $\mathbf{Set}_f$ of all finite sets is a full subcategory of the category **Set**.

Exercises

1. Show how each of the following constructions can be regarded as a functor: The field of quotients of an integral domain; the Lie algebra of a Lie group.
2. Show that functors $\mathbf{1} \to C$, $\mathbf{2} \to C$, and $\mathbf{3} \to C$ correspond respectively to objects, arrows, and composable pairs of arrows in C.
3. Interpret "functor" in the following special types of categories: (a) A functor between two preorders is a function T which is monotonic (i.e., $p \leqq p'$ implies $Tp \leqq Tp'$). (b) A functor between two groups (one-object categories) is a morphism of groups. (c) If G is a group, a functor $G \to \mathbf{Set}$ is a permutation representation of G, while $G \to \mathbf{Matr}_K$ is a matrix representation of G.
4. Prove that there is no functor **Grp**→**Ab** sending each group G to its center (consider $S_2 \to S_3 \to S_2$, the symmetric groups).
5. Find two different functors T: **Grp**→**Grp** with object function $T(G) = G$ the identity for every group G.

sets (in some universe U). Every ordinal $n = \{0, 1, \ldots, n-1\}$ is a finite set, so the inclusion S is a functor $S : \mathbf{Finord} \to \mathbf{Set}_f$. On the other hand, each finite set X determines an ordinal number $n = \# X$, the number of elements in X; we may choose for each X a bijection $\theta_X : X \to \# X$. For any function $f : X \to Y$ between finite sets we may then define a corresponding function $\# f : \# X \to \# Y$ between ordinals by $\# f = \theta_Y f \theta_X^{-1}$; this ensures that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\theta_X} & \# X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \# f} \\ Y & \xrightarrow{\theta_Y} & \# Y \end{array}$$

will commute, and makes $\#$ a functor $\# : \mathbf{Set}_f \to \mathbf{Finord}$. If X is itself an ordinal number, we may take θ_X to be the identity. This ensures that the composite functor $\# \circ S$ is the identity functor I' of **Finord**. On the other hand, the composite $S \circ \#$ is not the identity functor $I : \mathbf{Set}_f \to \mathbf{Set}_f$, because it sends each finite set X to a special finite set – the ordinal number n with the same number of elements as X. However, the square diagram above does show that $\theta : I \dot{\to} S\,\#$ is a natural isomorphism. All told we have $I \cong S \circ \#$, $I' = \# \circ S$.

More generally, an *equivalence* between categories C and D is defined to be a pair of functors $S : C \to D$, $T : D \to C$ together with natural isomorphisms $I_C \cong T \circ S$, $I_D \cong S \circ T$. This example shows that this notion (to be examined in § IV.4) allows us to compare categories which are "alike" but of very different "sizes".

We shall use many other examples of naturality. As Eilenberg-Mac Lane first observed, "category" has been defined in order to be able to define "functor" and "functor" has been defined in order to be able to define "natural transformation".

Exercises

1. Let S be a fixed set, and X^S the set of all functions $h : S \to X$. Show that $X \mapsto X^S$ is the object function of a functor $\mathbf{Set} \to \mathbf{Set}$, and that evaluation $e_X : X^S \times S \dot{\to} X$, defined by $e(h, s) = h(s)$, the value of the function h at $s \in S$, is a natural transformation.
2. If H is a fixed group, show that $G \mapsto H \times G$ defines a functor $H \times - : \mathbf{Grp} \to \mathbf{Grp}$, and that each morphism $f : H \to K$ of groups defines a natural transformation $H \times - \dot{\to} K \times -$.
3. If B and C are groups (regarded as categories with one object each) and $S, T : B \to C$ are functors (homomorphisms of groups), show that there is a natural transformation $S \dot{\to} T$ if and only if S and T are conjugate; i.e., if and only if there is an element $h \in C$ with $Tg = h(Sg)h^{-1}$ for all $g \in B$.

Exercises

1. Find a category with an arrow which is both epi and monic, but not invertible (e.g., dense subset of a topological space).
2. Prove that the composite of monics is monic, and likewise for epis.
3. If a composite $g\ f$ is monic, so is f. Is this true of g?
4. Show that the inclusion $\mathbf{Z} \to \mathbf{Q}$ is epi in the category **Rng**.
5. In **Grp** prove that every epi is surjective (Hint. If $\varphi: G \to H$ has image M not H, use the factor group H/M if M has index 2. Otherwise, let Perm H be the group of all permutations of the set H, choose three different cosets M, Mu and Mv of M, define $\sigma \in \operatorname{Perm} H$ by $\sigma(xu) = xv$, $\sigma(xv) = xu$ for $x \in M$, and σ otherwise the identity. Let $\psi: H \to \operatorname{Perm} H$ send each h to left multiplication ψ_h by h, while $\psi'_h = \sigma^{-1}\psi_h\sigma$. Then $\psi\varphi = \psi'\varphi$, but $\psi \neq \psi'$).
6. In **Set**, show that all idempotents split.
7. An arrow $f: a \to b$ in a category C is *regular* when there exists an arrow $g: b \to a$ such that $fgf = f$. Show that f is regular if it has either a left or a right inverse, and prove that every arrow in **Set** with $a \neq \varnothing$ is regular.
8. Consider the category with objects $\langle X, e, t\rangle$, where X is a set, $e \in X$, and $t: X \to X$, and with arrows $f: \langle X, e, t\rangle \to \langle X', e', t'\rangle$ the functions f on X to X' with $fe = e'$ and $ft = t'f$. Prove that this category has an initial object in which X is the set of natural numbers, $e = 0$, and t is the successor function.
9. If the functor $T: C \to B$ is faithful and Tf is monic, prove f monic.

6. Foundations

One of the main objectives of category theory is to discuss properties of totalities of Mathematical objects such as the "set" of all groups or the "set" of all homomorphisms between any two groups. Now it is the custom to regard a group as a set with certain added structure, so we are here proposing to consider a set of *all* sets with some given structure. This amounts to applying a comprehension principle: Given a property $\varphi(x)$ of sets x, form the set $\{x \mid \varphi(x)\}$ of *all* sets x with this property. However such a principle cannot be adopted in this generality, since it would lead to some of the famous paradoxical sets, such as the set of all sets not members of themselves.

For this reason, the standard practice in naive set theory, with the usual membership relation $\in$, is to restrict the application of the comprehension principle. One allows the formation from given sets u, v of the set $\{u, v\}$ (the set with exactly u and v as elements), of the ordered pair $\langle u, v\rangle$, of an infinite set (the set $\omega = \{0, 1, 2, \ldots\}$ of all finite ordinals), and of

The Cartesian Product	$u \times v = \{\langle x, y\rangle \mid x \in u \text{ and } y \in v\}$,
The Power Set	$\mathscr{P}u = \{v \mid v \subset u\}$,
The Union (of a set x of sets)	$\cup x = \{y \mid y \in z \text{ for some } z \in x\}$.

5. If a monoid M is regarded as a discrete category, with objects the elements $x \in M$, then the multiplication of M is a bifunctor $\mu : M \times M \to M$. If M is a group, show that the group inverse provides right adjoints for the functors $\mu(x, -)$ and $\mu(-, y) : M \to M$. Conversely, does the presence of such adjoints make a monoid into a group?
6. Describe units and counits for pushout and pullback.
7. If the category J is a disjoint union (coproduct) $\amalg J_k$ of categories J_k, for index k in some set K, with $I_k : J_k \to J$ the injections of the coproduct, then each functor $F : J \to C$ determines functors $F_k = F I_k : J_k \to C$.
   (a) Prove that $\operatorname{Lim} F \cong \Pi_k \operatorname{Lim} F_k$, if the limits on the right exist.
   (b) Show that every category J is a disjoint union of connected categories (called the *connected components* of J).
   (c) Conclude that all limits can be obtained from products and limits over connected categories.
8. (a) If the category J is connected, prove for any $c \in C$ that $\operatorname{Lim} \Delta c = c$ and $\operatorname{Colim} \Delta c = c$, where $\Delta c : J \to C$ is the constant functor.
   (b) Describe the unit for the right adjoint to $\Delta : C \to C^J$.
9. (Smythe.) Show that the functor $O : \mathbf{Cat} \to \mathbf{Set}$ assigning to each category C the set of its objects has a left adjoint D which assigns to each set X the discrete category on X, and that D in turn has a left adjoint assigning to each category the set of its connected components. Also show that O has a right adjoint which assigns to each set X a category with objects X and exactly one arrow in every hom-set.
10. If a category C has both cokernel pairs and equalizers, show that the functor $K : C^2 \to C^{\downarrow\downarrow}$ which assigns to each arrow of C its cokernel pair has as right adjoint the functor which assigns to each parallel pair of arrows its equalizing arrow.
11. If C has finite coproducts and $a \in C$, prove that the projection $Q : (a \downarrow C) \to C$ of the comma category ($Q(a \to c) = c$) has a left adjoint, with $c \mapsto (a \to a \amalg c)$.
12. If X is a set and C a category with powers and copowers, prove that the copower $c \mapsto X \cdot c$ is left adjoint to the power $c \mapsto c^X$.

3. Reflective Subcategories

For many of the forgetful functors $U : A \to X$ listed in § 2, the counit $\varepsilon : FU \dot{\to} I_A$ of the adjunction assigns to each $a \in A$ the epimorphism $\varepsilon_a : F(Ua) \to a$ which gives the standard representation of a as a quotient of a free object. This is a general fact: Whenever a right adjoint G is faithful, every counit ε_a of the adjunction is epi.

Theorem 1. *For an adjunction* $\langle F, G, \eta, \varepsilon \rangle : X \rightharpoonup A$: (i) G *is faithful if and only if every component* ε_a *of the counit* ε *is epi,* (ii) G *is full if and only if every* ε_a *is a split monic. Hence* G *is full and faithful if and only if each* ε_a *is an isomorphism* $FGa \cong a$.

The proof depends on a lemma.

Duality theorems in functional analysis are often instances of equivalences. For example, let **CAb** be the category of compact topological abelian groups, and let P assign to each such group G its character group PG, consisting of all continuous homomorphisms $G \to \mathbf{R}/\mathbf{Z}$. The Pontrjagin duality theorem asserts that $P: \mathbf{CAb} \to \mathbf{Ab}^{\text{op}}$ is an equivalence of categories. Similarly, the Gelfand-Naimark theorem states that the functor C which assigns to each compact Hausdorff space X its abelian C^*-algebra of continuous complex-valued functions is an equivalence of categories (see Negrepontis [1971]).

Exercises

1. Prove: (a) Any two skeletons of a category C are isomorphic.
   (b) If A_0 is a skeleton of A and C_0 a skeleton of C, then A and C are equivalent if and only if A_0 and C_0 are isomorphic.
2. (a) Prove: the composite of two equivalences $D \to C$, $C \to A$ is an equivalence.
   (b) State and prove the corresponding fact for adjoint equivalences.
3. If $S: A \to C$ is full, faithful, and surjective on objects (each $c \in C$ is $c = Sa$ for some $a \in A$), prove that there is an adjoint equivalence $\langle T, S; 1, \varepsilon \rangle : C \to A$ with unit the identity (and thence that T is a left-adjoint-right-inverse of S).
4. Given a functor $G: A \to X$, prove the three following conditions logically equivalent:
   (a) G has a left-adjoint-left-inverse.
   (b) G has a left adjoint, and is full, faithful, and injective on objects.
   (c) There is a full reflective subcategory Y of X and an isomorphism $H: A \cong Y$ such that $G = KH$, where $K: Y \to X$ is the insertion.
5. If J is a connected category and $\Delta: C \to C^J$ has a left adjoint (colimit), show that this left adjoint can be chosen to be a left-adjoint-left-inverse.

5. Adjoints for Preorders

Recall that a preorder P is a set $P = \{p, p', \ldots\}$ equipped with a reflexive and transitive binary relation $p \leqq p'$, and that preorders may be regarded as categories so that order-preserving functions become functors. An order-reversing function $\bar{L}$ on P to Q is then a functor $L: P \to Q^{\text{op}}$.

Theorem 1 *(Galois connections are adjoint pairs). Let P, Q be two preorders and $L: P \to Q^{\text{op}}$, $R: Q^{\text{op}} \to P$ two order-preserving functions. Then L (regarded as a functor) is a left adjoint to R if and only if, for all $p \in P$ and $q \in Q$,*

$$Lp \geqq q \text{ in } Q \text{ if and only if } p \leqq Rq \text{ in } P. \tag{1}$$

When this is the case, there is exactly one adjunction φ making L the left adjoint of R. For all p and q, $p \leqq RLp$ and $LRq \geqq q$; hence also

$$Lp \geqq LRLp \geqq Lp, \quad Rq \leqq RLRq \leqq Rq. \tag{2}$$