# IV. Adjoints

## 1. Adjunctions

We now present a basic concept due to Kan, which provides a different formulation for the properties of free objects and other universal constructions. As motivation, we first reexamine the construction (§ III.1) of a vector space $V_X$ with basis $X$. For a fixed field $K$ consider the functors

$$\mathbf{Set} \underset{U}{\overset{V}{\rightleftarrows}} \mathbf{Vct}_K ,$$

where, for each vector space $W$, $U(W)$ is the set of all vectors in $W$, so that $U$ is the forgetful functor, while, for any set $X$, $V(X)$ is the vector space with basis $X$. The vectors of $V(X)$ are thus the formal finite linear combinations $\Sigma\, r_i x_i$ with scalar coefficients $r_i \in K$ and with each $x_i \in X$, with the evident vector operations. Each function $g: X \to U(W)$ extends to a unique linear transformation $f: V(X) \to W$, given explicitly by $f(\Sigma\, r_i x_i) = \Sigma\, r_i(g x_i)$ (i.e., formal linear combinations in $V(X)$ to actual linear combinations in $W$). This correspondence $\psi: g \mapsto f$ has an inverse $\varphi: f \mapsto f \mid X$, the restriction of $f$ to $X$, hence is a bijection

$$\varphi: \mathbf{Vct}_K(V(X), W) \cong \mathbf{Set}(X, U(W)) .$$

This bijection $\varphi = \varphi_{X,W}$ is defined "in the same way" for all sets $X$ and all vector spaces $W$. This means that the $\varphi_{X,W}$ are the components of a natural transformation $\varphi$ when both sides above are regarded as functors of $X$ and $W$. It suffices to verify naturality in $X$ and in $W$ separately. Naturality in $X$ means that for each arrow $h: X' \to X$ the diagram

$$\begin{array}{ccc} \mathbf{Vct}_K(V(X), W) & \xrightarrow{\varphi} & \mathbf{Set}(X, U(W)) \\ {\scriptstyle (Vh)^*}\downarrow & & \downarrow{\scriptstyle h^*} \\ \mathbf{Vct}_K(V(X'), W) & \xrightarrow{\varphi} & \mathbf{Set}(X', U(W)) , \end{array}$$

where $h^* g = g \circ h$, will commute. This commutativity follows from the definition of $\varphi$ by a routine calculation, as does also the naturality in $W$.

Note next several similar examples.

The free category $C = FG$ on a given (small) graph $G$ is a functor **Grph**$\to$**Cat**: it is related to the forgetful functor $U:\mathbf{Cat}\to\mathbf{Grph}$ by the fact (§ II.7) that each morphism $D: G\to UB$ of graphs extends to a unique map $D': FG\to B$ of categories; moreover, $D\mapsto D'$ is a natural isomorphism

$$\mathbf{Cat}(FG, B) \cong \mathbf{Grph}(G, UB).$$

In the category of small sets, each function $g: S\times T\to R$ of two variables can be treated as a function $\varphi g: S\to \hom(T,R)$ of one variable (in $S$) whose values are functions of a second variable (in $T$); explicitly, $[(\varphi g)s]t = g(s,t)$ for $s\in S$, $t\in T$. This describes $\varphi$ as a bijection

$$\varphi: \hom(S\times T, R) \cong \hom(S, \hom(T,R)).$$

It is natural in $S$, $T$, and $R$. If we hold the set $T$ fixed and define functors $F, G:\mathbf{Set}\to\mathbf{Set}$ by $F(S) = S\times T$, $G(R) = \hom(T, R)$, the bijection takes the form

$$\hom(F(S), R) \cong \hom(S, G(R))$$

natural in $S$ and $R$, and much like the previous examples.

For modules $A$, $B$, and $C$ over a commutative ring $K$ there is a similar isomorphism

$$\hom(A\otimes_K B, C) \cong \hom(A, \hom_K(B, C))$$

natural in all three arguments.

**Definition.** *Let $A$ and $X$ be categories. An adjunction from $X$ to $A$ is a triple $\langle F, G, \varphi\rangle: X\rightharpoonup A$, where $F$ and $G$ are functors*

$$X \underset{G}{\overset{F}{\rightleftarrows}} A,$$

*while $\varphi$ is a function which assigns to each pair of objects $x\in X$, $a\in A$ a bijection of sets*

$$\varphi = \varphi_{x,a}: A(Fx, a) \cong X(x, Ga) \qquad (1)$$

*which is natural in $x$ and $a$.*

Here the left hand side $A(Fx, a)$ is the bifunctor

$$X^{\mathrm{op}}\times A \xrightarrow{F^{\mathrm{op}}\times \mathrm{Id}} A^{\mathrm{op}}\times A \xrightarrow{\hom} \mathbf{Set}$$

which sends each pair of objects $\langle x, a\rangle$ to the hom-set $A(Fx, a)$, and the right hand side is a similar bifunctor $X^{\mathrm{op}}\times A\to\mathbf{Set}$. Therefore the naturality of the bijection $\varphi$ means that for all $k: a\to a'$ and $h: x'\to x$ both the diagrams:

$$\begin{array}{ccc} A(Fx,a) & \xrightarrow{\varphi} & X(x,Ga) \\ {\scriptstyle k_*}\downarrow & & \downarrow{\scriptstyle (Gk)_*} \\ A(Fx,a') & \xrightarrow{\varphi} & X(x,Ga') \end{array} \qquad \begin{array}{ccc} A(Fx,a) & \xrightarrow{\varphi} & X(x,Ga) \\ {\scriptstyle (Fh)^*}\downarrow & & \downarrow{\scriptstyle h^*} \\ A(Fx',a) & \xrightarrow{\varphi} & X(x',Ga) \end{array} \qquad (2)$$

will commute. Here $k_*$ is short for $A(Fx, k)$, the operation of composition with $k$, and $h^* = X(h, Ga)$.

This discussion assumes that all the hom-sets of $X$ and $A$ are small. If not, we just replace **Set** above by a suitable larger category **Ens** of sets.

An adjunction may also be described without hom-sets directly in terms of arrows. It is a bijection which assigns to each arrow $f: Fx \to a$ an arrow $\varphi f = \operatorname{rad} f: x \to Ga$, the *right adjunct of* $f$, in such a way that the naturality conditions of (2),

$$\varphi(k \circ f) = Gk \circ \varphi f, \quad \varphi(f \circ Fh) = \varphi f \circ h, \tag{3}$$

hold for all $f$ and all arrows $h: x' \to x$ and $k: a \to a'$. It is equivalent to require that $\varphi^{-1}$ be natural; i.e., that for every $h, k$ and $g: x \to Ga$ one has

$$\varphi^{-1}(g h) = \varphi^{-1} g \circ Fh, \quad \varphi^{-1}(Gk \circ g) = k \circ \varphi^{-1} g. \tag{4}$$

Given such an adjunction, the functor $F$ is said to be a *left-adjoint* for $G$, while $G$ is called a *right adjoint* for $F$. (Some authors write $F \dashv G$; others say that $F$ is the "adjoint" of $G$ and $G$ the "coadjoint" of $F$, but other authors say the opposite; therefore we shall stick to the language of "left" and "right" adjoints.)

Every adjunction yields a universal arrow. Specifically, set $a = Fx$ in (1). The left hand hom-set of (1) then contains the identity $1: Fx \to Fx$; call its $\varphi$-image $\eta_x$. By Yoneda's Proposition III.2.1, this $\eta_x$ is a universal arrow

$$\eta_x: x \to GFx, \quad \eta_x = \varphi(1_{Fx}),$$

from $x \in X$ to $G$. The adjunction gives such a universal arrow $\eta_x$ for *every* object $x$. Moreover, the function $x \mapsto \eta_x$ is a natural transformation $I_X \to GF$ because every diagram

$$\begin{array}{ccc} x' & \xrightarrow{\eta_{x'}} & GFx' \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle GFh} \\ x & \xrightarrow{\eta_x} & GFx \end{array}$$

is commutative. This one proves by the calculation

$$GFh \circ \varphi(1_{Fx'}) = \varphi(Fh \circ 1_{Fx'}) = \varphi(1_{Fx} \circ Fh) = \varphi(1_{Fx}) \circ h.$$

based on the Eq. (3) describing the naturality of $\varphi$. This calculation may also be visualized by the commutative diagram

$$\begin{array}{ccccc} A(Fx', Fx') & \xrightarrow{(Fh)_*} & A(Fx', Fx) & \xleftarrow{(Fh)^*} & A(Fx, Fx) \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi} \\ X(x', GFx') & \xrightarrow[(GFh)_*]{} & X(x', GFx) & \xleftarrow[h^*]{} & X(x, GFx), \end{array}$$

where $h^* = X(h, 1)$ and $h_* = X(1, h)$.

The bijection $\varphi$ can be expressed in terms of the arrows $\eta_x$ as

$$\varphi(f) = G(f)\eta_x \quad \text{for} \quad f: Fx \to a\,; \tag{5}$$

indeed, by the naturality (3) of $\varphi$ we may compute that

$$\varphi(f) = \varphi(f \circ 1_{Fx}) = Gf \circ \varphi 1_{Fx} = Gf \circ \eta_x\,.$$

This computation may be visualized by chasing 1 around the commutative square

$$\begin{array}{ccc} A(Fx, Fx) & \xrightarrow{\varphi} & X(x, GFx) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle (Gf)_*} \\ A(Fx, a) & \xrightarrow{\varphi} & X(x, Ga) \end{array} \qquad \begin{array}{ccc} 1 & \longmapsto & \eta_x \\ \downarrow & & \downarrow \\ f \circ 1 & \mapsto & \varphi f = Gf \circ \eta_x\,. \end{array}$$

Dually, the adjunction gives a universal arrow from $F$. Indeed, set $x = Ga$ in the adjunction (1). The identity arrow $1: Ga \to Ga$ is now present in the right-hand hom-set; its image under $\varphi^{-1}$ is called $\varepsilon_a$,

$$\varepsilon_a: FGa \to a\,, \quad \varepsilon_a = \varphi^{-1}(1_{Ga})\,, \quad a \in A\,,$$

and is a universal arrow from $F$ to $a$. As before, $\varepsilon$ is a natural transformation $\varepsilon: FG \dot{\to} I_A$, and

$$\varphi^{-1}(g) = \varepsilon_a \circ Fg \quad \text{for} \quad g: x \to Ga\,.$$

Finally, take $x = Ga$. Then $\varepsilon_a = \varphi^{-1}(1_{Ga})$ gives, by the formula (5) for $\varphi$,

$$1_{Ga} = \varphi(\varepsilon_a) = G(\varepsilon_a) \circ \eta_{Ga}\,.$$

This asserts that the composite natural transformation

$$G \xrightarrow{\eta G} GFG \xrightarrow{G\varepsilon} G$$

is the identity transformation.

To summarize, we have proved

**Theorem 1.** *An adjunction* $\langle F, G, \varphi \rangle : X \rightharpoonup A$ *determines*

(i) *A natural transformation* $\eta: I_X \dot{\to} GF$ *such that for each object* $x$ *the arrow* $\eta_x$ *is universal to* $G$ *from* $x$, *while the right adjunct of each* $f: Fx \to a$ *is*

$$\varphi f = Gf \circ \eta_x : x \to Ga\,; \tag{6}$$

(ii) *A natural transformation* $\varepsilon: FG \dot{\to} I_A$ *such that each arrow* $\varepsilon_a$ *is universal to* $a$ *from* $F$, *while each* $g: x \to Ga$ *has left adjunct*

$$\varphi^{-1} g = \varepsilon_a \circ Fg : Fx \to a\,. \tag{7}$$

*Moreover, both the following composites are the identities (of* $G$, *resp.* $F$*).*

$$G \xrightarrow{\eta G} GFG \xrightarrow{G\varepsilon} G\,, \qquad F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon F} F\,. \tag{8}$$

We call $\eta$ the *unit* and $\varepsilon$ the *counit* of the adjunction. (Formerly, we called $\eta$ a "front adjunction" and $\varepsilon$ a "back adjunction".)

The given adjunction is actually already determined by various portions of all these data, in the following sense.

**Theorem 2.** *Each adjunction $\langle F, G, \varphi\rangle : X \rightharpoonup A$ is completely determined by the items in any one of the following lists:*

(i) *Functors $F$, $G$, and a natural transformation $\eta : 1_X \dot{\rightarrow} GF$ such that each $\eta_x : x \to GFx$ is universal to $G$ from $x$. Then $\varphi$ is defined by* (6).

(ii) *The functor $G : A \to X$ and for each $x \in X$ an object $F_0 x \in A$ and a universal arrow $\eta_x : x \to GF_0 x$ from $x$ to $G$. Then the functor $F$ has object function $F_0$ and is defined on arrows $h : x \to x'$ by $GFh \circ \eta_x = \eta_{x'} \circ h$.*

(iii) *Functors $F$, $G$, and a natural transformation $\varepsilon : FG \dot{\rightarrow} I_A$ such that each $\varepsilon_a : FGa \to a$ is universal from $F$ to $a$. Here $\varphi^{-1}$ is defined by* (7).

(iv) *The functor $F : X \to A$ and for each $a \in A$ an object $G_0 a \in X$ and an arrow $\varepsilon_a : FG_0 a \to a$ universal from $F$ to $a$.*

(v) *Functors $F$, $G$ and natural transformations $\eta : I_X \dot{\rightarrow} GF$ and $\varepsilon : FG \dot{\rightarrow} I_A$ such that both composites* (8) *are the identity transformations. Here $\varphi$ is defined by* (6) *and $\varphi^{-1}$ by* (7).

Because of (v), we often denote the adjunction $\langle F, G, \varphi\rangle$ by $\langle F, G, \eta, \varepsilon\rangle : X \rightharpoonup A$.

*Proof*. Ad (i): The statement that $\eta_x$ is universal means that to each $f : x \to Ga$ there is exactly one $g$ as in the commutative diagram

$$\begin{array}{ccc} Fx & \qquad & x \xrightarrow{\ \eta_x\ } GFx \\ \Big\downarrow{\scriptstyle g} & & \quad{\scriptstyle f}\searrow \quad \Big\downarrow{\scriptstyle Gg} \\ a\,, & & \qquad\qquad Ga\,. \end{array}$$

This states precisely that $\theta(g) = Gg \circ \eta_x$ defines a bijection

$$\theta : A(Fx, a) \to X(x, Ga)\,.$$

This bijection $\theta$ is natural in $x$ because $\eta$ is natural, and natural in $a$ because $G$ is a functor, hence gives an adjunction $\langle F, G, \theta\rangle$. In case $\eta$ was the unit obtained from an adjunction $\langle F, G, \varphi\rangle$, then $\theta = \varphi$.

The data (ii) can be expanded to (i), and hence determine the adjunction. In (ii) we are given simply a universal arrow $\langle F_0 x, \eta_x\rangle$ for every object $x \in X$; we shall show that there is exactly one way to make $F_0$ the object function of a functor $F$ for which $\eta : I_X \dot{\rightarrow} GF$ will be natural. Specifically, for each $h : x \to x'$ the universality of $\eta_x$ states that there is exactly one arrow (dotted)

$$\begin{array}{ccc} F_0 x & \qquad & x \xrightarrow{\ \eta_x\ } GF_0 x \\ \Big\downarrow & & \Big\downarrow{\scriptstyle h} \qquad\quad \Big\downarrow \\ F_0 x' & & x' \xrightarrow{\ \eta_{x'}\ } GF_0 x' \end{array}$$

which can make the diagram commute. Choose this arrow as $Fh : F_0 x \to F_0 x'$; the commutativity states that $\eta$ is now natural, and it is easy to check that this choice of $Fh$ makes $F$ a functor.

The proofs of parts (iii) and (iv) are dual.

To prove part (v) we use $\eta$ and $\varepsilon$ to define functions

$$A(Fx, a) \underset{\theta}{\overset{\varphi}{\rightleftarrows}} X(x, Ga)$$

by $\varphi f = Gf \circ \eta_x$ for each $f: Fx \to a$ and $\theta g = \varepsilon_a \circ Fg$ for each $g: x \to Ga$. Then since $G$ is a functor and $\eta$ is natural

$$\varphi\theta g = G\varepsilon_a \circ GFg \circ \eta_x = G\varepsilon_a \circ \eta_{Ga} \circ g.$$

But our hypothesis (8) states that $G\varepsilon_a \circ \eta_{Ga} = 1$. Hence $\varphi\theta = \text{id}$. Dually $\theta\varphi = \text{id}$. Therefore $\varphi$ is a bijection (with inverse $\theta$). It is clearly natural, hence is an adjunction (and, if we started with an adjunction, it is the one from which we started).

This theorem is very useful. For example, parts (ii) and (iv) construct an adjunction whenever we have a universal arrow from (or to) every object of a given category. For example, the category $C$ has finite products when for each pair $\langle a, b\rangle \in C \times C$ there is a universal arrow from $\Delta: C \to C \times C$ to $\langle a, b\rangle$. By the theorem above we conclude that the function $\langle a, b\rangle \to a \times b$ giving the product object is actually a functor $C \times C \to C$, and that this functor is right adjoint to the diagonal functor $\Delta$:

$$\varphi : (C \times C)(\Delta c, \langle a, b\rangle) \cong C(c, a \times b).$$

Using the definition of the arrows in $C \times C$, this is

$$\varphi : C(c, a) \times C(c, b) \cong C(c, a \times b).$$

The counit of this adjunction (set $c = a \times b$ on the right) is an arrow $\langle a \times b, a \times b\rangle \to \langle a, b\rangle$; it is thus just a pair of arrows $a \leftarrow a \times b \to b$; namely, the projections $p: a \times b \to a$ and $q: a \times b \to b$ of the product. The adjunction $\varphi^{-1}$ sends each $f: c \to a \times b$ to the pair $\langle pf, qf\rangle$; this is the way in which $\varphi$ is determined by the counit $\varepsilon$.

Similarly, if the category $C$ has coproducts $\langle a, b\rangle \mapsto a \amalg b$, they define the coproduct functor $C \times C \to C$ which is a left adjoint to $\Delta$:

$$C(a \amalg b, c) \cong (C \times C)(\langle a, b\rangle, \Delta c).$$

All the other examples of limits (when they always exist) can be similarly read as examples of adjoints. In many further applications, it turns out that proving universality is an easy way of showing that adjoints are present.

On the other hand, part (v) of the theorem describes an adjunction by two simple identities

$$\begin{array}{ccc} F & \xrightarrow{F\eta} & FGF \\ & \searrow^{=} & \downarrow{\scriptstyle \varepsilon F} \\ & & F \end{array} \qquad \begin{array}{ccc} GFG & \xleftarrow{\eta G} & G \\ {\scriptstyle G\varepsilon}\downarrow & \swarrow_{=} & \\ G & & \end{array} \tag{9}$$

on the unit and counit of the adjunction. These *triangular identities* make no explicit use of the objects of the categories $A$ and $X$, and so are easy to manipulate. As we shall soon see, this is convenient for discussing properties of adjunctions. (For some authors, these identities are said to make $\eta$ a "quasi-inverse" to $\varepsilon$.)

**Corollary 1.** *Any two left-adjoints $F$ and $F'$ of a functor $G: A \to X$ are naturally isomorphic.*

The proof is just an application of the fact that a universal arrow, like an initial object, is unique up to isomorphism. Explicitly, adjunctions $\langle F, G, \varphi\rangle$ and $\langle F', G, \varphi'\rangle$ give to each $x$ two universal arrows $x \to GFx$ and $x \to GF'x$; hence there is a unique isomorphism $\theta_x: Fx \to F'x$ with $G\theta_x \circ \eta_x = \eta'_x$; it is easy to verify that $\theta: F \dot\to F'$ is natural.

**Corollary 2.** *A functor $G: A \to X$ has a left adjoint if and only if, for each $x \in X$, the functor $X(x, Ga)$ is representable as a functor of $a \in A$. If $\varphi: A(F_0 x, a) \cong X(x, Ga)$ is a representation of this functor, then $F_0$ is the object function of a left-adjoint of $G$ for which the bijection $\varphi$ is natural in $a$ and gives the adjunction.*

This is just a restatement of part (ii) of the theorem. Equivalently, $G$ has a left-adjoint if and only if there is a universal arrow to $G$ from every $x \in X$.

We leave the reader to state the duals.

Adjoints of additive functors are additive.

**Theorem 3.** *If the additive functor $G: A \to M$ between Ab-categories $A$ and $M$ has a left adjoint $F: M \to A$, then $F$ is additive and the adjunction bijections*

$$\varphi: A(Fm, a) \cong M(m, Ga)$$

*are isomorphisms of abelian groups (for all $m \in M$, $a \in A$).*

*Proof.* If $\eta: I \dot\to GF$ is the unit of the adjunction, then $\varphi$ may be written as $\varphi f = Gf \circ \eta_m$ for any $f: Fm \to a$. If also $f': Fm \to a$, the additivity of $G$ gives

$$\varphi(f + f') = G(f + f')\eta_m = (Gf + Gf')\eta_m = Gf \circ \eta_m + Gf' \circ \eta_m = \varphi f + \varphi f'.$$

Therefore $\varphi$ is a morphism of abelian groups. Next take $g, g' : m \to n$ in $M$. Since $\eta$ is natural,

$$GF(g+g') \circ \eta_m = \eta_n(g+g') = \eta_n g + \eta_n g' .$$

On the other hand, since $G$ is additive,

$$G(Fg+Fg') \circ \eta_m = (GFg+GFg')\eta_m = GFg \circ \eta_m + GFg' \circ \eta_m = \eta_n g + \eta_n g' .$$

The equality of these two results and the universal property of $\eta_m$ show that $F(g+g') = Fg+Fg'$. Hence $F$ is additive.

Dually, any right adjoint of an additive functor is additive.

### Exercises

1. Show that Theorem 2 can have an added clause (and its dual):
   (vi) A functor $G : A \to X$ and for each $x \in X$ a representation $\varphi_x$ of the functor $X(x, G-) : A \to \mathbf{Set}$.
2. (Lawvere.) Given functors $G : A \to X$ and $F : X \to A$, show that each adjunction $\langle F, G, \varphi\rangle$ can be described as an isomorphism $\theta$ of comma categories such that the following diagram commutes

$$\begin{array}{ccc} \theta : (F \downarrow I_A) & \cong & (I_X \downarrow G) \\ \downarrow & & \downarrow \\ X \times A & = & X \times A \, . \end{array}$$

   Here the vertical maps have components the projection functors $P$ and $Q$ of II.6(5).
3. For the adjunction $\langle \Delta, \times, \varphi\rangle$ – product right adjoint to diagonal – show that the unit $\delta_c : c \to c \times c$ for each object $c \in C$ is the unique arrow such that the diagram

$$\begin{array}{ccccc} & & c & & \\ & {}^{1}\swarrow & \vdots\,\delta_c & \searrow^{1} & \\ c & \xleftarrow[p]{} & c \times c & \xrightarrow[q]{} & c \end{array}$$

   commutes. (This arrow $\delta_c$ is often called the *diagonal arrow* of $c$.) If $C = \mathbf{Set}$, show that $\delta_c x = \langle x, x\rangle$ for $x \in c$.
4. (Paré.) Given functors $G : A \to X$ and $K : X \to A$ and natural transformations $\varepsilon : KG \dot{\to} \mathrm{id}_A$, $\varrho : \mathrm{id}_X \dot{\to} GK$ such that $G\varepsilon \cdot \varrho G = 1_G : G \dot{\to} GKG \dot{\to} G$, prove that $\varepsilon K \cdot K\varrho : K \dot{\to} K$ is an idempotent in $A^X$ and that $G$ has a left adjoint if and only if this idempotent splits; explicitly if $\varepsilon K \cdot K\varrho$ splits as $\alpha \cdot \beta$ with $\beta \cdot \alpha = 1$ and $\beta : K \dot{\to} F$, then $F$ is a left adjoint of $G$ with unit $G\beta \cdot \varrho$ and counit $\varepsilon \cdot \alpha G$.

## 2. Examples of Adjoints

We now summarize a number of examples of adjoints, beginning with a table of left-adjoints of typical forgetful functors.

| *Forgetful functor* | *Left adjoint F* | *Unit of adjunction* |
|---|---|---|
| $U: R$-**Mod**$\to$**Set** | $X \mapsto FX$<br>Free $R$-module, basis $X$ | $j: X \to UFX$ (cf. § III.1)<br>"insertion of generators" |
| $U:$ **Cat**$\to$**Grph** | $G \mapsto CG$<br>Free category on graph $G$ | $G \to UCG$<br>"insertion of generators" |
| $U:$ **Grp**$\to$**Set** | $X \mapsto FX$<br>Free group, generators $x \in X$ | $X \to UFX$<br>"insertion of generators" |
| $U:$ **Ab**$\to$**Set** | $X \mapsto F_a X$<br>Free abelian group on $X$ | "insertion of generators" |
| $U:$ **Ab**$\to$**Grp** | $G \mapsto G/[G,G]$<br>Factor commutator group | $G \to G/[G,G]$<br>projection on the quotient |
| $U: R$-**Mod**$\to$**Ab** | $A \mapsto R \otimes A$ | $A \to U(R \otimes A)$<br>$a \mapsto 1 \otimes a$ |
| $U: R$-**Mod**-$S \to R$-**Mod** | $A \mapsto A \otimes S$ | $A \to U(A \otimes S)$<br>$a \mapsto a \otimes 1$ |
| $U:$ **Rng**$\to$**Mon**<br>(cf. Exercise III.1.1) | $M \mapsto \mathbf{Z}(M)$<br>(integral) monoid ring | $M \to U\mathbf{Z}M$<br>$m \mapsto m$ |
| $U: K$-**Alg**$\to K$-**Mod** | $V \mapsto TV$<br>Tensor algebra on $V$ | $V \subset TV$<br>"insertion of generators" |
| $U:$ **Fld**$\to$**Dom**$_m$<br>(cf. § III.1) | $D \mapsto QD$<br>Field of quotients | $D \subset UQD$<br>"insertion of $D: a \mapsto a/1$" |
| $U:$ **Compmet**$\to$**Met** | Completion of metric space | (§ III.1) |

There is a similar description of counits. For example, in the free $R$-module $FX$ generated by elements $jx = \langle x \rangle$ for $x \in X$, the elements may be written as finite sums $\Sigma r_i \langle x_i \rangle$ with scalars $r_i \in R$. Then for any $R$-module $A$ the counit $\varepsilon_A: FUA \to A$ is $\Sigma r_i \langle a_i \rangle \mapsto \Sigma r_i a_i$ (linear combinations in $A$). In other words $\varepsilon_A$ is the epimorphism appearing in the standard representation of an arbitrary $R$-module as a quotient of a free module (the free module on its own elements as generators).

Next, we list some left and right adjoints (which need not exist in every category $C$) for diagonal functors; with the unit when $C$ is **Set**.

| *Diagonal functor* | *Adjoint* | *Unit* | *Counit* |
|---|---|---|---|
| $\Delta: C \to C \times C$ | Left: Coproduct<br>$\amalg: C \times C \to C$<br>$\langle a, b \rangle \mapsto a \amalg b$ | (pair of) injections<br>$i: a \to a \amalg b$<br>$j: b \to a \amalg b$ | "folding" map<br>$c \amalg c \to c$<br>$ix \mapsto x,\ jx \mapsto x$ |
| | Right: Product<br>$\Pi: C \times C \to C$<br>$\langle a, b \rangle \mapsto a \times b$ | Diagonal arrow<br>$\delta_c: c \to c \times c$<br>$x \mapsto \langle x, x \rangle$ | (pair of) projections<br>$p: a \times b \mapsto a$<br>$q: a \times b \mapsto b$ |

| *Diagonal functor* | *Adjoint* | *Unit* | *Counit* |
|---|---|---|---|
| $C \to \mathbf{1}$ | Left: Initial object $s$ | | $s \to c$ |
| | Right: Terminal object $t$ | $c \to t$ | |
| $\Delta: C \to C^{\downdownarrows}$ (III.3.6) | Left: Coequalizer $\langle f, g\rangle \mapsto$ coeq. object $e$ | Coequalizing arrow $\langle f, g\rangle \xrightarrow{\langle uf, u\rangle} \langle e, e\rangle$ | Identity $1: c \to c$ |
| (III.4.7) | Right: Equalizer $d$ $\langle f, g\rangle \mapsto$ equal. object | Identity | Equalizing $\langle d, d\rangle \to \langle f$ |
| $\Delta: C \to C^{\to \cdot \leftarrow}$ | Left: (Vertex of) pushout (III.3.7) | | |
| | Right: (Vertex of) pullback (III.4.8) | | |
| $\Delta: C \to C^J$ | Left: Colimit object | Universal cone | |
| | Right: Limit object | | Universal c |

In the case of limits, the form of the unit depends on the numbe of connected components of $J$. Here a category $J$ is called *connecte* when to any two objects $j, k \in J$ there is a finite sequence of arrow

$$j = j_0 \to j_1 \leftarrow j_2 \to \cdots \to j_{2n-1} \leftarrow j_{2n} = k \qquad \text{(both directions possible)}$$

joining $j$ to $k$ (see Exercises 7, 8).

Duality functors provide further examples. For vector spaces $V$, $V$ over a field $K$, the dual $\bar{D}$ is a contravariant functor on **Vct** to **Vc** given on objects by $\bar{D}V = \mathbf{Vct}(V, K)$ with the usual vector space structur and on arrows $h: V \to W$ as $\bar{D}h: \bar{D}W \to \bar{D}V$, where $(\bar{D}h)f = fh$ for eac $f: W \to K$. A function

$$\varphi = \varphi_{V,W}: \mathbf{Vct}(V, \mathbf{Vct}(W, K)) \to \mathbf{Vct}(W, \mathbf{Vct}(V, K)) \qquad (1$$

is defined for $h: V \to \bar{D}W$ by $[(\varphi h)w]v = (hv)w$ for all $v \in V$, $w \in W$ Since $\varphi_{W,V}\varphi_{V,W}$ is the identity, each $\varphi$ is a bijection. This bijection can b made into an adjuction as follows. The contravariant functor $\bar{D}$ leads t two different (covariant!) functors with the same object function,

$$D: \mathbf{Vct}^{\mathrm{op}} \to \mathbf{Vct}, \qquad D^{\mathrm{op}}: \mathbf{Vct} \to \mathbf{Vct}^{\mathrm{op}},$$

defined (as usual) for arrows $h^{\mathrm{op}}: W \to V$ and $h: V \to W$ by

$$Dh^{\mathrm{op}} = \bar{D}h: \bar{D}W \to \bar{D}V; \qquad D^{\mathrm{op}}h = (\bar{D}h)^{\mathrm{op}}: \bar{D}V \to \bar{D}W.$$

The bijection $\varphi$ of (1) above may now be written as

$$\mathbf{Vct}^{\mathrm{op}}(D^{\mathrm{op}}W, V) \cong \mathbf{Vct}(W, DV), \qquad (2$$

natural in $V$ and $W$. Therefore $D^{\mathrm{op}}$ is the left adjoint of $D$. (Warning It is not a right adjoint of $D$, see § V.5, Exercise 2.) If $\kappa_W: W \to \bar{D}\bar{D}W$ is th usual canonical map to the double dual, the unit of the adjunction (set $V = D^{op} W$ in (2)) is this map $\eta_W = \kappa_W : W \to D D^{op} W$, and the counit is an arrow $\varepsilon_V : D^{op} D V \to V$ in $\mathbf{Vct}^{op}$ which turns out to be $\varepsilon_V = (\kappa_V)^{op}$ for the *same* $\kappa$.

This example illustrates the way in which adjunctions may replace isomorphisms of categories. For finite dimensional vector spaces, $D$ and $D^{op}$ are isomorphisms; for the general case, this is not true, but $D$ is the right adjoint of $D^{op}$.

This example also bears on adjoints for other contravariant functors. Two contravariant functors $\bar{S}$ from $A$ to $X$ and $\bar{T}$ from $X$ to $A$ are "adjoint on the right" (Freyd) when there is a bijection $A(a, \bar{T}x) \cong X(x, \bar{S}a)$, natural in $a$ and $x$. We shall not need this terminology, because we can replace $\bar{S}$ and $\bar{T}$ by the covariant functors $S : A^{op} \to X$ and $T : X^{op} \to A$ and form the dual $S^{op} : A \to X^{op}$, also covariant: thus the natural bijection above becomes $X^{op}(S^{op} a, x) \cong A(a, Tx)$, and so states that $S^{op}$ is left adjoint (in our usual sense) to $T$ – or, equivalently, that $T^{op}$ is left adjoint to $S$. It is not necessarily equivalent to say that $\bar{T}$ and $\bar{S}$ are adjoint "on the left".

The next three sections will be concerned with three other types of adjoints: A left adjoint to an inclusion functor (of a full subcategory) is called a *reflection*; certain other special sorts of adjoints are "equivalences" of categories. Some other amusing examples of adjoints are given in the exercises to follow, some of which require knowledge of the subject matter involved. Goguen [1971] shows for finite state machines that the functor "minimal realization" is left adjoint to the functor "behavior". The reader is urged to find his own examples as well.

## Exercises

1. For $K$ a field and $V$ a vector space over $K$, there is an "exterior algebra" $E(V)$, which is a graded, anticommutative algebra. Show that $E$ is the left adjoint of a suitable forgetful functor (one which is not faithful).
2. Show that the functor $U : R\text{-}\mathbf{Mod} \to \mathbf{Ab}$ has not only a left adjoint $A \mapsto R \otimes A$, but also a right adjoint $A \mapsto \hom_{\mathbf{Z}}(R, A)$.
3. For $K$ a field, let $\mathbf{Lie}_K$ be the category of all (small) Lie algebras $L$ over $K$, with arrows the morphisms of $K$-modules which also preserve the Lie bracket operation $\langle a, b \rangle \mapsto [a, b]$. Let $V : \mathbf{Alg}_K \to \mathbf{Lie}_K$ be the functor which assigns to each (associative) algebra $A$ the Lie algebra $VA$ on the same vector space, with bracket $[a, b] = ab - ba$ for $a, b \in A$. Using the Poincaré-Birkhoff-Witt Theorem show that the functor $E$, where $EL$ is the enveloping associative algebra of $L$, is a left adjoint for $V$.
4. Let $\mathbf{Rng}'$ denote the category of rings $R$ which do not necessarily have an identity element for multiplication. Show that the standard process of adding an identity to $R$ provides a left adjoint for the forgetful functor $\mathbf{Rng} \to \mathbf{Rng}'$ (forget the presence of the identity).

5. If a monoid $M$ is regarded as a discrete category, with objects the elements $x \in M$, then the multiplication of $M$ is a bifunctor $\mu: M \times M \to M$. If $M$ is a group, show that the group inverse provides right adjoints for the functors $\mu(x, -)$ and $\mu(-, y): M \to M$. Conversely, does the presence of such adjoints make a monoid into a group?
6. Describe units and counits for pushout and pullback.
7. If the category $J$ is a disjoint union (coproduct) $\amalg J_k$ of categories $J_k$, for index $k$ in some set $K$, with $I_k: J_k \to J$ the injections of the coproduct, then each functor $F: J \to C$ determines functors $F_k = F I_k: J_k \to C$.
   (a) Prove that $\operatorname{Lim} F \cong \Pi_k \operatorname{Lim} F_k$, if the limits on the right exist.
   (b) Show that every category $J$ is a disjoint union of connected categories (called the *connected components* of $J$).
   (c) Conclude that all limits can be obtained from products and limits over connected categories.
8. (a) If the category $J$ is connected, prove for any $c \in C$ that $\operatorname{Lim} \Delta c = c$ and $\operatorname{Colim} \Delta c = c$, where $\Delta c: J \to C$ is the constant functor.
   (b) Describe the unit for the right adjoint to $\Delta: C \to C^J$.
9. (Smythe.) Show that the functor $O$ : **Cat** → **Set** assigning to each category $C$ the set of its objects has a left adjoint $D$ which assigns to each set $X$ the discrete category on $X$, and that $D$ in turn has a left adjoint assigning to each category the set of its connected components. Also show that $O$ has a right adjoint which assigns to each set $X$ a category with objects $X$ and exactly one arrow in every hom-set.
10. If a category $C$ has both cokernel pairs and equalizers, show that the functor $K: C^2 \to C^{\downarrow\downarrow}$ which assigns to each arrow of $C$ its cokernel pair has as right adjoint the functor which assigns to each parallel pair of arrows its equalizing arrow.
11. If $C$ has finite coproducts and $a \in C$, prove that the projection $Q: (a \downarrow C) \to C$ of the comma category $(Q(a \to c) = c)$ has a left adjoint, with $c \mapsto (a \to a \amalg c)$.
12. If $X$ is a set and $C$ a category with powers and copowers, prove that the copower $c \mapsto X \cdot c$ is left adjoint to the power $c \mapsto c^X$.

## 3. Reflective Subcategories

For many of the forgetful functors $U: A \to X$ listed in § 2, the counit $\varepsilon: FU \dot{\to} I_A$ of the adjunction assigns to each $a \in A$ the epimorphism $\varepsilon_a: F(Ua) \to a$ which gives the standard representation of $a$ as a quotient of a free object. This is a general fact: Whenever a right adjoint $G$ is faithful, every counit $\varepsilon_a$ of the adjunction is epi.

**Theorem 1.** *For an adjunction* $\langle F, G, \eta, \varepsilon \rangle: X \rightharpoonup A$: (i) *$G$ is faithful if and only if every component $\varepsilon_a$ of the counit $\varepsilon$ is epi,* (ii) *$G$ is full if and only if every $\varepsilon_a$ is a split monic. Hence $G$ is full and faithful if and only if each $\varepsilon_a$ is an isomorphism $FGa \cong a$.*

The proof depends on a lemma.

**Lemma.** *Let $f^*: A(a, -) \dot\to A(b, -)$ be the natural transformation induced by an arrow $f: b \to a$ of $A$. Then $f^*$ is monic if and only if $f$ is epi, while $f^*$ is epi if and only if $f$ is a split monic (i.e., if and only if $f$ has a left inverse).*

Note that $f^* \mapsto f$ is the bijection $\mathrm{Nat}(A(a, -), A(b, -)) \cong A(b, a)$ given by the Yoneda lemma.

Observe, also, that for functors $S, T: C \to B$, a natural transformation $\tau: S \dot\to T$ is epi (respectively, monic) in $B^C$ if and only if every component $\tau_c: S_c \to T_c$ is epi (respectively, monic) in $B$ for $B = \mathbf{Set}$; this follows by Exercise III.4.4, computing the pushout pointwise as in Exercise III.5.5.

*Proof*. For $h \in A(a, c)$, $f^*h = hf$. Hence the first result is just the definition of an epi $f$. If $f^*$ is epi, there is an $h_0: a \to b$ with $f^*h_0 = h_0 f = 1: b \to b$, so $f$ has a left inverse. The converse is immediate.

Now we prove the theorem. Apply the Yoneda Lemma to the natural transformation (arrow function of $G$ followed by the adjunction)

$$A(a, c) \xrightarrow{G_{a,c}} X(Ga, Gc) \xrightarrow{\varphi^{-1}} A(FGa, c).$$

It is determined (set $c = a$) by the image of $1: a \to a$, which is exactly the definition of the counit $\varepsilon_a: FGa \to a$. But $\varphi^{-1}$ is an isomorphism, hence this natural transformation is monic or epi, respectively, when every $G_{a,c}$ is injective or surjective, respectively; that is, when $G$ is faithful or full, respectively. The result now follows by the lemma.

A subcategory $A$ of $B$ is called *reflective* in $B$ when the inclusion functor $K: A \to B$ has a left adjoint $F: B \to A$. This functor $F$ may be called a *reflector* and the adjunction $\langle F, K, \varphi \rangle = \langle F, \varphi \rangle : B \rightharpoonup A$ a *reflection* of $B$ in its subcategory $A$. Since the inclusion functor $K$ is always faithful, the counit $\varepsilon$ of a reflection is always epi. A reflection can be described in terms of the composite functor $R = KF: B \to B$; indeed, $A \subset B$ is reflective in $B$ if and only if there is a functor $R: B \to B$ with values in the subcategory $A$ and a bijection of sets

$$A(Rb, a) \cong B(b, a)$$

natural in $b \in B$ and $a \in A$. A reflection may be described in terms of universal arrows: $A \subset B$ is reflective if and only if to each $b \in B$ there is an object $Rb$ of the subcategory $A$ and an arrow $\eta_b: b \to Rb$ such that every arrow $g: b \to a \in A$ has the form $g = f \circ \eta_b$ for a unique arrow $f: Rb \to a$ of $A$. As usual, $R$ is then (the object function of) a functor $B \to B$ (with values in $A$).

If a full subcategory $A \subset B$ is reflective in $B$, then by Theorem 1 each object $a \in A$ is isomorphic to $FKa$, and hence $Ra \cong a$ for all $a$.

Dually, $A \subset B$ is *coreflective* in $B$ when the inclusion functor $A \to B$ has a right adjoint. (Warning: Mitchell [1965] has interchanged the meanings of "reflection" and "coreflection".)

Here are some examples. **Ab** is reflective in **Grp**. For, if $G/[G, G]$ is the usual factor-commutator group of a group $G$, then $\hom(G/[G, G], A) \cong \hom(G, A)$ for $A$ abelian, and **Ab** is full in **Grp**. Or consider the category of all metric spaces $X$, with arrows uniformly continuous functions. The (full) subcategory of complete metric spaces is reflective; the reflector sends each metric space to its completion. Again, consider the category of all completely regular Hausdorff spaces (with arrows all continuous functions). The (full) subcategory of all compact Hausdorff spaces is reflective; the reflector sends each completely regular space to its Stone-Čech compactification.

A coreflective subcategory of **Ab** is the full subcategory of all torsion abelian groups (a group is torsion if all elements have finite order); the coreflector sends each abelian group $A$ to the subgroup $TA$ of all elements of finite order in $A$.

### Exercises

1. Show that the table of dual statements (§ II.1) extends as follows:

| *Statement* | ***Dual statement*** |
|---|---|
| $S, T: C \to B$ are functors | $S, T: C \to B$ are functors |
| $T$ is full | $T$ is full |
| $T$ is faithful | $T$ is faithful |
| $\eta: S \dot{\to} T$ is a natural transformation. | $\eta: T \dot{\to} S$ is a natural transformation. |
| $\langle F, G, \varphi \rangle : X \rightharpoonup A$ is an adjunction | $\langle G, F, \varphi^{-1} \rangle : A \rightharpoonup X$ is an adjunction |
| $\eta$ is the unit of $\langle F, G, \varphi \rangle$. | $\eta$ is the counit of $\langle G, F, \varphi^{-1} \rangle$. |

2. Show that the torsion-free abelian groups form a full reflective subcategory of **Ab**.
3. If $\langle G, F, \varphi \rangle : X \rightharpoonup A$ is an adjunction with $G$ full and every unit $\eta_x$ a monic, then every $\eta_x$ is also epi.
4. Show the following subcategories to be reflective:
   (a) The full subcategory of all partial orders in the category **Preord** of all preorders, with arrows all monotone functions.
   (b) The full subcategory of $T_0$-spaces in **Top**.
5. Given an adjunction $\langle F, G, \varphi \rangle : X \rightharpoonup A$, prove that $G$ is faithful if and only if $\varphi^{-1}$ carries epis to epis.
6. Given an adjunction $\langle F, G, \eta, \varepsilon \rangle$ with either $F$ or $G$ full, prove that $G\varepsilon : GFG \to G$ is invertible with inverse $\eta G : G \to GFG$.
7. If $A$ is a full and reflective subcategory of $B$, prove that every functor $S: J \to A$ with a limit in $B$ has a limit in $A$.

## 4. Equivalence of Categories

A functor $S: A \to C$ is an *isomorphism* of categories when there is a functor $T: C \to A$ (backwards) such that $ST = I: C \to C$ and $TS = I: A \to A$. In this case, the identity natural transformations

$\eta : I \dot{\to} ST$ and $\varepsilon : TS \dot{\to} I$ make $\langle T, S; \eta, \varepsilon\rangle : C \rightharpoonup A$ an adjunction. In other words, a two-sided inverse $T$ of a functor $S$ is a left-adjoint of $S$ – and for that matter, $T$ is also a right-adjoint of $S$.

There is a more general (and more useful) notion:

A functor $S : A \to C$ is an *equivalence of categories* (and the categories $A$ and $C$ are *equivalent*) when there is a functor $T: C \to A$ (backwards) and natural isomorphisms $ST \cong I : C \to C$ and $TS \cong I : A \to A$. In this case $T: C \to A$ is also an equivalence of categories. We shall soon see that $T$ is then both a left adjoint and a right adjoint of $S$.

Here is an example. In any category $C$ a *skeleton* of $C$ is any full subcategory $A$ such that each object of $C$ is isomorphic (in $C$) to exactly one object of $A$. Then $A$ is equivalent to $C$ and the inclusion $K : A \to C$ is an equivalence of categories. For, select to each $c \in C$ an isomorphism $\theta_c : c \cong Tc$ with $Tc$ an object of $A$. Then we can make $T$ a functor $T: C \to A$ in exactly one way so that $\theta$ will become a natural isomorphism $\theta: I \cong KT$. Moreover $TK \cong I$, so $K$ is indeed an equivalence: *A category is equivalent to (any one of) its skeletons.* For example, the category of all finite sets has as a skeleton the full subcategory with objects all finite ordinal numbers 0, 1, 2, ..., $n$, .... (Here 0 is the empty set and each $n = \{0, 1, \ldots, n-1\}$.)

A category is called *skeletal* when any two isomorphic objects are identical; i.e., when the category is its own skeleton.

An *adjoint equivalence* of categories is an adjunction $\langle T, S; \eta, \varepsilon\rangle : C \rightharpoonup A$ in which both the unit $\eta : I \dot{\to} ST$ and the counit $\varepsilon : TS \dot{\to} I$ are natural isomorphisms: $I \cong ST$, $TS \cong I$. Then $\eta^{-1}$ and $\varepsilon^{-1}$ are also natural isomorphisms, and the triangular identities $\varepsilon T \cdot T\eta = 1$, $S\varepsilon \cdot \eta S = 1$ can be written as $T\eta^{-1} \cdot \varepsilon^{-1} T = 1$, $\eta^{-1} S \cdot S\varepsilon^{-1} = 1$, respectively. These identities then state that $\langle S, T, \varepsilon^{-1}, \eta^{-1}\rangle : A \rightharpoonup C$ is an adjunction with $\varepsilon^{-1} : I \dot{\to} TS$ as unit and $\eta^{-1} : ST \dot{\to} I$ as counit. Thus in an adjoint equivalence $\langle T, S, -, -\rangle$ the functor $T: C \to A$ is the left adjoint of $S: A \to C$ with unit $\eta$ and at the same time $T$ is the right adjoint of $S$, with unit $\varepsilon^{-1}$.

We can now state the main facts about equivalence.

**Theorem 1.** *The following properties of a functor $S: A \to C$ are logically equivalent:*

(i) *$S$ is an equivalence of categories,*

(ii) *$S$ is part of an adjoint equivalence $\langle T, S; \eta, \varepsilon\rangle : C \rightharpoonup A$,*

(iii) *$S$ is full and faithful, and each object $c \in C$ is isomorphic to $Sa$ for some object $a \in A$.*

*Proof.* Trivially, (ii) implies (i). To prove that (i) implies (iii), note that $ST \cong I$ shows that each $c \in C$ has the form $c \cong S(Tc)$ for an $a = Tc \in A$. The natural isomorphism $\theta: TS \cong I$ gives for each $f: a \to a'$ the commutative square

$$\begin{array}{ccc} TSa & \xrightarrow{\theta_a} & a \\ {\scriptstyle TSf}\downarrow & & \downarrow{\scriptstyle f} \\ TSa' & \xrightarrow[\theta_{a'}]{} & a' . \end{array}$$

Hence $f = \theta_{a'} \circ TSf \circ \theta_a^{-1}$; it follows that $S$ is faithful. Symmetrically, $ST \cong I$ proves $T$ faithful. To show $S$ full, consider any $h: Sa \to Sa'$ and set $f = \theta_{a'} \circ Th \circ \theta_a^{-1}$. Then the square above commutes also with $Sf$ replaced by $h$, so $TSf = Th$. Since $T$ is faithful, $Sf = h$, which means that $S$ is full.

To prove that (iii) implies (ii) we must construct from $S$ a (left) adjoint $T$. For each $c \in C$ we can choose some object $a_0 = T_0 c \in A$ and an isomorphism $\eta_c$:

$$\begin{array}{ccc} \eta_c : c & \cong & S(T_0 c) \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle Sg} \\ & & Sa \end{array} \qquad g: T_0 c \to a .$$

For every arrow $f: c \to Sa$, the composite $f \circ \eta_c^{-1}$ has the form $Sg$ for some $g$ because $S$ is full; this $g$ is unique because $S$ is faithful. In other words, $f = Sg \circ \eta_c$ for a unique $g$, so $\eta_c$ is universal from $c$ to $S$. Therefore $T_0$ can be made a functor $T: C \to A$ in exactly one way so that $\eta: I \dot{\to} ST$ is natural, and then $T$ is the left adjoint of $S$ with unit the isomorphism $\eta$. As with any adjunction, $S\varepsilon_a \cdot \eta_{Sa} = 1$ (put $c = Sa$, $f = 1$ in the diagram above). Thus $S\varepsilon_a = (\eta_{Sa})^{-1}$ is invertible. Since $S$ is full and faithful, the counit $\varepsilon_a$ is also invertible. Therefore $\langle T, S; \eta, \varepsilon \rangle : C \rightharpoonup A$ is an adjoint equivalence, and the proof is complete.

In this proof, suppose that $A$ is a full subcategory of $C$ and that $S = K: A \to C$ is the insertion. For objects $a \in A \subset C$ we can then choose $a_0 = a = Ka$ and $\eta_{Ka}$ the identity. Then $K\varepsilon_a = 1$, hence $\varepsilon_a = 1$ for all $a$. This proves

**Proposition 2.** *If $A$ is a full subcategory of $C$ and every $c \in C$ is isomorphic (in $C$) to some object of $A$, then the insertion $K: A \to C$ is an equivalence and is part of an adjoint equivalence $\langle T, K; \eta, 1 \rangle : C \rightharpoonup A$ with counit the identity. Therefore $A$ is reflective in $C$.*

This includes in particular the case already noted, when $A$ is a skeleton of $C$.

A functor $F: X \to A$ is said to be a *left-adjoint-left-inverse* of $G: A \to X$ when there is an adjunction $\langle F, G; \eta, 1 \rangle : X \rightharpoonup A$ with counit the identity. This means (Exercise 4) that $G$ is an isomorphism of $A$ to a reflective subcategory of $X$. In the case of the Proposition 2 just above, we have shown that the insertion $A \to C$ has a left-adjoint-left-inverse.

Duality theorems in functional analysis are often instances of equivalences. For example, let **CAb** be the category of compact topological abelian groups, and let $P$ assign to each such group $G$ its character group $PG$, consisting of all continuous homomorphisms $G \to \mathbf{R}/\mathbf{Z}$. The Pontrjagin duality theorem asserts that $P : \mathbf{CAb} \to \mathbf{Ab}^{\text{op}}$ is an equivalence of categories. Similarly, the Gelfand-Naimark theorem states that the functor $C$ which assigns to each compact Hausdorff space $X$ its abelian $C^*$-algebra of continuous complex-valued functions is an equivalence of categories (see Negrepontis [1971]).

### Exercises

1. Prove: (a) Any two skeletons of a category $C$ are isomorphic.
   (b) If $A_0$ is a skeleton of $A$ and $C_0$ a skeleton of $C$, then $A$ and $C$ are equivalent if and only if $A_0$ and $C_0$ are isomorphic.
2. (a) Prove: the composite of two equivalences $D \to C$, $C \to A$ is an equivalence.
   (b) State and prove the corresponding fact for adjoint equivalences.
3. If $S : A \to C$ is full, faithful, and surjective on objects (each $c \in C$ is $c = Sa$ for some $a \in A$), prove that there is an adjoint equivalence $\langle T, S; 1, \varepsilon \rangle : C \rightharpoonup A$ with unit the identity (and thence that $T$ is a left-adjoint-right-inverse of $S$).
4. Given a functor $G : A \to X$, prove the three following conditions logically equivalent:
   (a) $G$ has a left-adjoint-left-inverse.
   (b) $G$ has a left adjoint, and is full, faithful, and injective on objects.
   (c) There is a full reflective subcategory $Y$ of $X$ and an isomorphism $H : A \cong Y$ such that $G = KH$, where $K : Y \to X$ is the insertion.
5. If $J$ is a connected category and $\Delta : C \to C^J$ has a left adjoint (colimit), show that this left adjoint can be chosen to be a left-adjoint-left-inverse.

## 5. Adjoints for Preorders

Recall that a preorder $P$ is a set $P = \{p, p', \dots\}$ equipped with a reflexive and transitive binary relation $p \leqq p'$, and that preorders may be regarded as categories so that order-preserving functions become functors. An order-reversing function $\bar{L}$ on $P$ to $Q$ is then a functor $L : P \to Q^{\text{op}}$.

**Theorem 1** *(Galois connections are adjoint pairs). Let $P$, $Q$ be two preorders and $L : P \to Q^{\text{op}}$, $R : Q^{\text{op}} \to P$ two order-preserving functions. Then $L$ (regarded as a functor) is a left adjoint to $R$ if and only if, for all $p \in P$ and $q \in Q$,*

$$Lp \geqq q \ \text{ in } Q \ \text{ if and only if } \ p \leqq Rq \ \text{ in } P. \tag{1}$$

*When this is the case, there is exactly one adjunction $\varphi$ making $L$ the left adjoint of $R$. For all $p$ and $q$, $p \leqq RLp$ and $LRq \geqq q$; hence also*

$$Lp \geqq LRLp \geqq Lp, \qquad Rq \leqq RLRq \leqq Rq. \tag{2}$$

*Proof*. Recall that $P$ becomes a category in which there is (exactly) one arrow $p \to p'$ whenever $p \leqq p'$. Thus the condition (1) states precisely that there is a bijection $\hom_{Q^{op}}(Lp, q) \cong \hom_P(p, Rq)$; since each hom-set has at most one element, this bijection is automatically natural. The unit of the adjunction is the inequality $p \leqq RLp$ for all $p$, while the counit is $LRq \geqq q$ for all $q$. The two Eqs. (2) are the triangular identities connecting unit and counit. In the convenient case when both $P$ and $Q$ are posets (i.e., when both the relations $\leqq$ are antisymmetric) these conditions become $L = LRL$, and $R = RLR$ (each three passages reduce to one!).

A pair of order-preserving functions $L$ and $R$ which satisfy (1) is called a *Galois connection* from $P$ to $Q$. Here is the fundamental example, for a group $G$ acting on a set $U$, by $\langle \sigma, x\rangle \mapsto \sigma \cdot x$ for $\sigma \in G$, $x \in U$. Take $P = \mathscr{P}(U)$, the set of all subsets $X \subset U$, ordered by inclusion, while $Q = \mathscr{P}(G)$ is the set of subsets $S \subset G$ also ordered by inclusion ($S \leqq S'$ if and only if $S \subset S'$). Let $LX = \{\sigma \mid x \in X \text{ implies } \sigma \cdot x = x\}$, $RS = \{x \mid \sigma \in S \text{ implies } \sigma \cdot x = x\}$; in other words, $LX$ is the subgroup of $G$ which fixes all points $x \in X$ and $RS$ is the set of fixed points of the automorphisms of $S$. Then $LX \geqq S$ in $Q$ if and only if $\sigma \cdot x = x$ for all $\sigma \in S$ and all $x \in X$, which in turn holds if and only if $X \leqq RS$ in $P$. Therefore, $L$ and $R$ form an adjoint pair (a Galois connection). The original instance is that with $G$ a group of automorphisms of a field $U$, as in the classical Galois theory.

If $U$ and $V$ are sets, the set $\mathscr{P}(U)$ of all subsets of $U$ is a preorder under inclusion. For each function $f: U \to V$ the direct image $f_*$, defined by $f_*(X) = \{f(x) \mid x \in X\}$ is an order-preserving function and hence a functor $f_*: \mathscr{P}(U) \to \mathscr{P}(V)$. The inverse image $f^*(Y) = \{x \mid fx = y$ for some $y \in Y\}$ defines a functor $f^*: \mathscr{P}(V) \to \mathscr{P}(U)$ in the opposite direction. Since $f_* X \subset Y$ if and only if $X \subset f^* Y$, the direct image functor $f_*$ is left adjoint to the inverse image functor $f^*$.

Certain adjoints for Boolean algebras are closely related to the basic connectives in logic. We again regard $\mathscr{P}(U)$ as a preorder, and hence as a category. The diagonal functor $\Delta: \mathscr{P}(U) \to \mathscr{P}(U) \times \mathscr{P}(U)$ has (as we have already noted) a right adjoint $\cap$, sending subsets $X$, $Y$ to their intersection $X \cap Y$, and a left adjoint $\cup$, with $\langle X, Y\rangle \mapsto X \cup Y$, the union. If $X$ is a fixed subset of $U$, then intersection with $X$ is a functor $X \cap - : \mathscr{P}(U) \to \mathscr{P}(U)$. Since $X \cap Y \leqq Z$ if and only if $Y \leqq X' \cup Z$, where $X'$ is the complement of $X$ in $U$, the right adjoint of $X \cap -$ is $X' \cup -$. Thus the construction of suitable adjoints yields the Boolean operations $\cap$, $\cup$, and $'$ corresponding to "and", "or", and "not". Now consider the first projection $P: U \times V \to U$ from the product of two sets $U$ and $V$. Each subset $S \subset U \times V$ defines two corresponding subsets of $U$ by

$$P_* S = \{x \mid \exists y, y \in V \quad \text{and} \quad \langle x, y\rangle \in S\},$$
$$P_\# S = \{x \mid \forall y, y \in V \quad \text{implies} \quad \langle x, y\rangle \in S\};$$

they arise from $\langle x, y\rangle \in S$ by applying the existential quantifier $\exists y$, "there exists a $y$" and the universal quantifier $\forall y$, "for all $y$", respectively to $\langle x,y\rangle \in S$. Also $P_* S$ is the direct image of $S$ under the projection $P$. Now for all subsets $X \subset U$ one has

$$S \leqq P^* X \Leftrightarrow P_* S \leqq X \,; \qquad P^* X \leqq S \Leftrightarrow X \leqq P_{\#} S \,,$$

where "$\Leftrightarrow$" means "if and only if". These state that $P^*$, which is the inverse image operation, has both a left adjoint $P_*$ and a right adjoint $P_{\#}$. In this sense, both quantifiers $\exists$ and $\forall$ can be interpreted as adjoints.

There is also a geometric interpretation: $P^* X$ is the cylinder $X \times V \subset U \times V$ over the base $X \subset U$, $P_* S$ is the projection of $S \subset U \times V$ on the base $U$, and $P_{\#} S$ is the largest subset $X$ of $U$ such that the cylinder on $X$ is wholly contained in $S$. This analysis has revealed several basic concepts of logic (and, or, not, $\forall y$, $\exists y$) to be adjoints. This illustrates the slogan "adjoints are everywhere".

**Exercises**

1. Let $H$ be a space with an inner product (e.g., Hilbert space). If $P = Q$ is the set of all subsets $S$ of $H$, ordered by inclusion, show that $LS = RS =$ the orthogonal complement of $S$ gives a Galois connection.
2. In a Galois connection between posets, show that the subset $\{p \mid p = RLp\}$ of $P$ equals $\{p \mid p = Rq$ for some $q\}$ and give a bijection from this set to the subset $\{q \mid q = LRq\}$ of $Q$. What are these sets in the case of a group of automorphisms of a field? Does this generalize to an arbitrary adjunction?
3. For $C$ a category with pullbacks, each arrow $f: a \to a'$ defines a functor $(C \downarrow f) = f_* : (C \downarrow a) \to (C \downarrow a')$ which carries each object $x \to a$ of $(C \downarrow a)$ to the composite $x \to a \to a'$. Show that $f_*$ has a right adjoint $f^*$ with $f^*(x' \to a') = y \to a$, where $y$ is the vertex of the pullback of $a \to a' \leftarrow x'$.

## 6. Cartesian Closed Categories

Much of the force of category theory will be seen to reside in using categories with specified additional structures. One basic example will be the closed categories (§ VII. 7); at present we can define readily one useful special case, "cartesian closed".

To assert that a category $C$ has all finite products and coproducts is to assert that products, terminal, initial and coproducts exist, thus the functors $C \to \mathbf{1}$ and $\Delta : C \to C \times C$ have both left and right adjoints. Indeed, the left adjoints give initial object and coproduct, respectively, while the right adjoints give terminal object and product, respectively.

Using just adjoints we will now define "cartesian closed category". A category $C$ with all finite products specifically given is called *cartesian closed* when each of the following functors

$$C \to \mathbf{1}, \qquad C \to C \times C, \qquad C \xrightarrow{- \times b} C,$$
$$c \mapsto 0, \qquad c \mapsto \langle c, c\rangle, \qquad a \mapsto a \times b,$$

has a *specified* right adjoint (with a specified adjunction). These adjoint are written as follows

$$t \leftarrow\!\!\dashv 0\,, \qquad a \times b \leftarrow\!\!\dashv \langle a, b\rangle\,, \qquad c^b \leftarrow\!\!\dashv c\,.$$

Thus to specify the first is to specify a terminal object $t$ in $C$, and specifyin the second is specifying for each pair of objects $a, b \in C$ a product objec $a \times b$ together with its projections $a \leftarrow a \times b \rightarrow b$. These projection determine the adjunction (they constitute the counit of the adjunction as already noted, $\times$ is then a bifunctor. The third required adjoir specifies for each functor $- \times b : C \rightarrow C$ a right adjoint, with the corre sponding bijection

$$\hom(a \times b, c) \cong \hom(a, c^b)$$

natural in $a$ and in $c$. By the parameter theorem (to be proved in the ne section), $\langle b, c\rangle \mapsto c^b$ is then (the object function of) a bifunctor $C^{\mathrm{op}} \times C \rightarrow ($ Specifying the adjunction amounts to specifying for each $c$ and $b$ an arrow

$$e : c^b \times b \rightarrow c$$

which is natural in $c$ and universal from $- \times b$ to $c$. We call this $e = e_{b,}$ the *evaluation* map. It amounts to the ordinary evaluation $\langle f, x\rangle \mapsto f\,x$ o a function $f$ at an argument $x$ in both of the following cases:

**Set** is a cartesian closed category, with $c^b = \hom(b, c)$.

**Cat** is cartesian closed, with exponent $C^B$ the functor category.

A closely related example of adjoints is the functor

$$- \otimes_K B : K\text{-}\mathbf{Mod} \rightarrow K\text{-}\mathbf{Mod}$$

which has a right adjoint $\hom_K(B, -)$; the adjunction is determined b a counit $\hom_K(B, A) \otimes_K B \rightarrow A$ given by evaluation.

## Exercises

1. (a) If $U$ is any set, show that the preorder $\mathscr{P}(U)$ of all subsets of $U$ is a cartesia closed category.

   (b) Show that any Boolean algebra, regarded as a preorder, is cartesian closec
2. In some elementary theory $T$, consider the set $S = \{p, q, \dots\}$ of sentences of as a preorder, with $p \leqq q$ meaning "$p$ entails $q$" (i.e., $q$ is a consequence of on the basis of the axioms of $T$). Prove that $S$ is a cartesian closed categor with product given by conjunction and exponential $q^p$ given by "$p$ implies $q$
3. In any cartesian closed category, prove $c^1 \cong c$ and $c^{b \times b'} \cong (c^b)^{b'}$.
4. In any cartesian closed category obtain a natural transformation $c^b \times b^a \rightarrow ($ which agrees in **Set** with composition of functions. Prove it (like compositior associative.
5. Show that $A$ cartesian closed need not imply $A^J$ cartesian closed.