III. Universals and Limits

Universal constructions appear throughout mathematics in various guises – as universal arrows to a given functor, as universal arrows from a given functor, or as universal elements of a set-valued functor. Each universal determines a representation of a corresponding set-valued functor as a hom-functor. Such representations, in turn, are analyzed by the Yoneda Lemma. Limits are an important example of universals – both the inverse limits (= projective limits = limits = left roots) and their duals, the direct limits (= inductive limits = colimits = right roots). In this chapter we define universals and limits and examine a few basic types of limits (products, pullbacks, and equalizers ...). Deeper properties will appear in Chapter IX on special limits, while the relation to adjoints will be treated in Chapter V.

1. Universal Arrows

Given the forgetful functor $U: \mathbf{Cat} \to \mathbf{Grph}$ and a graph G, we have constructed (§ II.7) the free category C on G and the morphism $P: G \to UC$ of graphs which embeds G in C, and we have shown that this arrow P is "universal" from G to U. A similar universality property holds for the morphisms embedding generators into free algebraic systems of other types, such as groups or rings. Here is the general concept.

Definition. *If $S: D \to C$ is a functor and c an object of C, a universal arrow from c to S is a pair $\langle r, u\rangle$ consisting of an object r of D and an arrow $u: c \to Sr$ of C, such that to every pair $\langle d, f\rangle$ with d an object of D and $f: c \to Sd$ an arrow of C, there is a unique arrow $f': r \to d$ of D with $Sf' \circ u = f$. In other words, every arrow f to S factors uniquely through the universal arrow u, as in the commutative diagram*

$$\begin{array}{ccc} c & \xrightarrow{u} & Sr \\ \| & & \downarrow{\scriptstyle Sf'} \\ c & \xrightarrow{f} & Sd, \end{array} \qquad \begin{array}{c} r \\ \downarrow{\scriptstyle f'} \\ d. \end{array} \tag{1}$$

Equivalently, $u: c \to Sr$ is universal from c to S when the pair $\langle r, u\rangle$ is an initial object in the comma category $(c \downarrow S)$, whose objects are the arrows $c \to Sd$. As with any initial object, it follows that $\langle r, u\rangle$ is unique up to isomorphism in $(c \downarrow S)$; in particular, the object r of D is unique up to isomorphism in D. This remark is typical of the use of comma categories.

This notion of a universal arrow has a great variety of examples; we list a few:

Bases of Vector Spaces. Let $\mathbf{Vct}_K$ denote the category of all vector spaces over a fixed field K, with arrows linear transformations, while $U: \mathbf{Vct}_K \to \mathbf{Set}$ is the forgetful functor, sending each vector space V to the set of its elements. For any set X there is a familiar vector space V_X with X as a set of basis vectors; it consists of all formal K-linear combinations of the elements of X. The function which sends each $x \in X$ into the same x regarded as a vector of V_X is an arrow $j: X \to U(V_X)$. For any other vector space W, it is a fact that each function $f: X \to U(W)$ can be extended to a unique linear transformation $f': V_X \to W$ with $Uf' \circ j = f$. This familiar fact states exactly that j is a universal arrow from X to U.

Free Categories from Graphs. Theorem II.7.1 for the free category C on a graph G states exactly that the functor $P: G \to UC$ is universal. The same observation applies to the free monoid on a given set of generators, the free group on a given set of generators, the free R-module (over a given ring R) on a given set of generators, the polynomial algebra over a given commutative ring in a given set of generators, and so on in many cases of free algebraic systems.

Fields of Quotients. To any integral domain D a familiar construction gives a field $Q(D)$ of quotients of D together with a monomorphism $j: D \to Q(D)$ (which is often formulated by making D a subdomain of $Q(D)$). This field of quotients is usually described as the smallest field containing D, in the sense that for each $D \subset K$ with K a field there is a monomorphism $f: Q(D) \to K$ of fields which is the identity on the common subdomain D. However, this inclusion $D \subset K$ may readily be replaced by any monomorphism $D \to K$ of domains. Hence our statement means that the pair $\langle Q(D), j\rangle$ is universal for the forgetful functor $\mathbf{Fld} \to \mathbf{Dom}_m$ from the category of fields to that of domains – provided we take arrows of $\mathbf{Dom}_m$ to be the monomorphisms of integral domains (note that a homomorphism of *fields* is necessarily a monomorphism). However, for the larger category **Dom** with arrows all homomorphisms of integral domains there does not exist a universal arrow from each domain to a field. For instance, for the domain $\mathbf{Z}$ of integers there is for each prime p a homomorphism $\mathbf{Z} \to \mathbf{Z}_p$; the reader should observe that this makes impossible the construction of a universal arrow from $\mathbf{Z}$ to the functor $\mathbf{Fld} \to \mathbf{Dom}$.

Complete Metric Spaces. Let **Met** be the category of all metric spaces $X, Y, \ldots$, with arrows $X \to Y$ those functions which preserve the metric (and which therefore are necessarily injections). The complete metric spaces form (the objects of) a full subcategory. The familiar completion $\bar{X}$ of a metric space X provides an arrow $X \to \bar{X}$ which is universal for the evident forgetful functor (from complete metric spaces to metric spaces).

In many other cases, the function embedding a mathematical object in a suitably completed object can be interpreted as a universal arrow. The general fact of the uniqueness of the universal arrow implies the uniqueness of the completed object, up to a unique isomorphism (who wants more?).

The idea of universality is sometimes expressed in terms of "universal elements". If D is a category and $H: D \to \mathbf{Set}$ a functor, a *universal element* of the functor H is a pair $\langle r, e \rangle$ consisting of an object $r \in D$ and an element $e \in Hr$ such that for every pair $\langle d, x \rangle$ with $x \in Hd$ there is a unique arrow $f: r \to d$ of D with $(Hf)e = x$.

Many familiar constructions are naturally examples of universal elements. For instance, consider an equivalence relation E on a set S, the corresponding quotient set S/E consisting of the equivalence classes of elements of S under E, and the projection $p: S \to S/E$ which sends each $s \in S$ to its E-equivalence class. Now S/E has the familiar property that any function f on S which respects the equivalence relation can be regarded as a function on S/E. More formally, this means that if $f: S \to X$ has $fs = fs'$ whenever sEs', then f can be written as a composite $f = f'p$ for a unique function $f': S/E \to X$:

$$\begin{array}{ccc} S & \xrightarrow{p} & S/E \\ \| & & \downarrow{\scriptstyle f'} \\ S & \xrightarrow{f} & X. \end{array}$$

This states exactly that $\langle S/E, p \rangle$ is a universal element for that functor $H: \mathbf{Set} \to \mathbf{Set}$ which assigns to each set X the set HX of all those functions $f: S \to X$ for which sEs' implies $fs = fs'$.

Again, let N be a normal subgroup of a group G. The usual projection $p: G \to G/N$ which sends each $g \in G$ to its coset $pg = gN$ in the quotient group G/N is a universal element for that functor $H: \mathbf{Grp} \to \mathbf{Set}$ which assigns to each group G' the set HG' of all those homomorphisms $f: G \to G'$ which kill N (have $fN = 1$). Indeed, every such homomorphism factors as $f = f'p$, for a unique $f': G/N \to G'$. Now the quotient group is usually described as a group whose elements are cosets. However, once the cosets are used to prove this *one* "universal" property of $p: G \to G/N$, *all* other properties of quotient groups – for example, the isomorphism theorems – can be proved with no further mention of cosets (see Mac Lane-Birkhoff [1967]). All that is needed is the existence of a universal element

p of the functor H. For that matter, even this existence could be proved without using cosets (see the adjoint functor theorem stated in § V.6).

Tensor products provide another example of universal elements. Given two vector spaces V and V' over the field K, the function H which assigns to each vector space W the set $HW = \mathrm{Bilin}(V, V'; W)$ of all bilinear functions $V \times V' \to W$ is the object function of a functor $H : \mathbf{Vect}_K \to \mathbf{Set}$, and the usual construction of the tensor product provides both a vector space $V \otimes V'$ and a bilinear function $\otimes : V \times V' \to V \otimes V'$, usually written $\langle v, v' \rangle \mapsto v \otimes v'$, so that the pair $\langle V \otimes V', \otimes \rangle$ is a universal element for the functor $H = \mathrm{Bilin}(V, V'; -)$. This applies equally well when the field K is replaced by a commutative ring (and vector spaces by K-modules).

The notion "universal element" is a special case of the notion "universal arrow". Indeed, if $*$ is the set with one point, then any element $e \in Hr$ can be regarded as an arrow $e : * \to Hr$ in **Ens**. Thus a universal element $\langle r, e \rangle$ for H is exactly a universal arrow from $*$ to H. Conversely, if C has small hom-sets, the notion "universal arrow" is a special case of the notion "universal element". Indeed, if $S : D \to C$ is a functor and $c \in C$ is an object, then $\langle r, u : c \to Sr \rangle$ is a universal arrow from c to S if and only if the pair $\langle r, u \in C(c, Sr) \rangle$ is a universal element of the functor $H = C(c, S-)$. This is the functor which acts on objects d and arrows h of D by

$$d \mapsto C(c, Sd), \qquad h \mapsto C(c, Sh).$$

Hitherto we have treated universal arrows from an object $c \in C$ to a functor $S : D \to C$. The dual concept is also useful. A *universal arrow* from S to c is a pair $\langle r, v \rangle$ consisting of an object $r \in D$ and an arrow $v : Sr \to c$ with codomain c such that to every pair $\langle d, f \rangle$ with $f : Sd \to c$ there is a unique $f' : d \to r$ with $f = v \circ Sf'$, as in the commutative diagram

$$\begin{array}{ccccc} d & \qquad & Sd & \xrightarrow{f} & c \\ {\scriptstyle f'}\downarrow & & {\scriptstyle Sf'}\downarrow & & \| \\ r, & & Sr & \xrightarrow{v} & c. \end{array}$$

The projections $p : a \times b \to a$, $q : a \times b \to b$ of a product in C (for $C = \mathbf{Grp}$, $\mathbf{Set}$, $\mathbf{Cat}$, ...) are examples of such a universal. Indeed, given any other pair of arrows $f : c \to a$, $g : c \to b$ to a and b, there is a unique $h : c \to a \times b$ with $ph = f, qh = g$. Therefore $\langle p, q \rangle$ is a "universal pair". To make it a universal arrow, introduce the diagonal functor $\Delta : C \to C \times C$, with $\Delta c = \langle c, c \rangle$. Then the pair f, g above becomes an arrow $\langle f, g \rangle : \Delta c \to \langle a, b \rangle$ in $C \times C$, and $\langle p, q \rangle$ is a universal arrow from Δ to the object $\langle a, b \rangle$.

Similarly, the kernel of a homomorphism (in **Ab**, **Grp**, **Rng**, R-**Mod**, ...) is a universal, more exactly, a universal for a suitable contravariant functor.

Note that we say "universal arrow to S" and "universal arrow from S" rather than "universal" and "couniversal".

Exercises

1. Show how each of the following familiar constructions can be interpreted as a universal arrow:
   (a) The integral group ring of a group (better, of a monoid).
   (b) The tensor algebra of a vector space.
   (c) The exterior algebra of a vector space.
2. Find a universal element for the contravariant power set functor $\mathscr{P}:\mathbf{Set}^{\mathrm{op}}\to\mathbf{Set}$.
3. Find (from any given object) universal arrows to the following forgetful functors: $\mathbf{Ab}\to\mathbf{Grp}$, $\mathbf{Rng}\to\mathbf{Ab}$ (forget the multiplication), $\mathbf{Top}\to\mathbf{Set}$, $\mathbf{Set}_*\to\mathbf{Set}$.
4. Use only universality (of projections) to prove the following isomorphisms of group theory:
   (a) For normal subgroups M, N of G with $M\subset N$, $(G/M)/(N/M)\cong G/M$.
   (b) For subgroups S and N of G, N normal, with join SN, $SN/N\cong S/S\cap N$.
5. Show that the quotient K-module A/S (S a submodule of A) has a description by universality. Derive isomorphism theorems.
6. Describe quotients of a ring by a two-sided ideal by universality.
7. Show that the construction of the polynomial ring $K[x]$ in an indeterminate x over a commutative ring K is a universal construction.

2. The Yoneda Lemma

Next we consider some conceptual properties of universality. First, universality can be formulated with hom-sets, as follows:

Proposition 1. *For a functor $S:D\to C$ a pair $\langle r, u:c\to Sr\rangle$ is universal from c to S if and only if the function sending each $f':r\to d$ into $Sf'\circ u:c\to Sd$ is a bijection of hom-sets*

$$D(r,d)\cong C(c,Sd). \qquad (1)$$

This bijection is natural in d. Conversely, given r and c, any natural isomorphism (1) is determined in this way by a unique arrow $u:c\to Sr$ such that $\langle r,u\rangle$ is universal from c to S.

Proof. The statement that $\langle r,u\rangle$ is universal is exactly the statement that $f'\mapsto Sf'\circ u=f$ is a bijection. This bijection is natural in d, for if $g':d\to d'$, then $S(g'f')\circ u=Sg'\circ(Sf'\circ u)$.

Conversely, a natural isomorphism (1) gives for each object d of D a bijection $\varphi_d:D(r,d)\to C(c,Sd)$. In particular, choose the object d to be r;

the identity $1_r \in D(r, r)$ then goes by φ_r to an arrow $u: c \to Sr$ in C. For any $f': r \to d$ the diagram

$$\begin{array}{ccc} D(r, r) & \xrightarrow{\varphi_r} & C(c, Sr) \\ {\scriptstyle D(r, f')}\downarrow & & \downarrow{\scriptstyle C(c, Sf')} \\ D(r, d) & \xrightarrow{\varphi_d} & C(c, Sd) \end{array} \qquad (2)$$

commutes because φ is natural. But in this diagram, $1_r \in D(r, r)$ is mapped (top and right) to $Sf' \circ u$ and (left and bottom) to $\varphi_d(f')$. Since φ_d is a bijection, this states precisely that each $f: c \to Sd$ has the form $f = Sf' \circ u$ for a unique f'. This is precisely the statement that $\langle r, u \rangle$ is universal.

If C and D have small hom-sets, this result (1) states that the functor $C(c, S -)$ to **Set** is naturally isomorphic to a covariant hom-functor $D(r, -)$. Such isomorphisms are called representations:

Definition. *Let D have small hom-sets. A representation of a functor $K: D \to$ **Set** is a pair $\langle r, \psi \rangle$, with r an object of D and*

$$\psi : D(r, -) \cong K \qquad (3)$$

a natural isomorphism. The object r is called the representing object. The functor K is said to be representable when such a representation exists.

Up to isomorphism, a representable functor is thus just a covariant hom-functor $D(r, -)$. This notion can be related to universal arrows as follows.

Proposition 2. *Let $*$ denote any one-point set and let D have small hom-sets. If $\langle r, u: * \to Kr \rangle$ is a universal arrow from $*$ to $K: D \to$ **Set**, then the function ψ which for each object d of D sends the arrow $f': r \to d$ to $K(f')(u*) \in Kd$ is a representation of K. Every representation of K is obtained in this way from exactly one such universal arrow.*

Proof. For any set X, a function $f: * \to X$ from the one-point set $*$ to X is determined by the element $f(*) \in X$. This correspondence $f \mapsto f(*)$ is a bijection **Set**$(*, X) \xrightarrow{\sim} X$, natural in $X \in$ **Set**. Composing with K yields a natural isomorphism **Set**$(*, K -) \xrightarrow{\sim} K$. This plus the representation ψ of (3) gives

$$\mathbf{Set}(*, K -) \cong K \cong D(r, -).$$

Therefore a representation of K amounts to a natural isomorphism **Set**$(*, K -) \cong D(r, -)$. The proposition thus follows from the previous one.

A direct proof is equally easy: Given the universal arrow u, the correspondence $f' \mapsto K(f')(u(*))$ is a representation; given a representation ψ as in (3), ψ_r maps $1: r \to r$ to an element of Kr, which is a universal element, hence also a universal arrow $* \to Kr$.

Observe that each of the notions "universal arrow", "universal element", and "representable functor" subsumes the other two. Thus, a universal arrow from c to $S: D \to C$ amounts (Proposition 1) to a natural isomorphism $D(r, d) \cong C(c, Sd)$ and hence to a representation of the functor $C(c, S-): D \to \mathbf{Set}$ or equally well to a universal element for the same functor.

The argument for Proposition 1 rested on the observation that each natural transformation $\varphi: D(r, -) \xrightarrow{\cdot} K$ is completely determined by the image under φ_r of the identity $1: r \to r$. This fact may be stated as follows:

Lemma *(Yoneda). If $K: D \to \mathbf{Set}$ is a functor from D and r an object in D (for D a category with small hom-sets), there is a bijection*

$$y: \mathrm{Nat}(D(r, -), K) \cong Kr \qquad (4)$$

which sends each natural transformation $\alpha: D(r, -) \xrightarrow{\cdot} K$ to $\alpha_r 1_r$, the image of the identity $r \to r$.

The proof is indicated by the following commutative diagram:

$$\begin{array}{ccccc} D(r, r) & \xrightarrow{\alpha_r} & K(r) & & r \\ {\scriptstyle f_* = D(r, f)} \downarrow & & \downarrow {\scriptstyle K(f)} & & \downarrow {\scriptstyle f} \\ D(r, d) & \xrightarrow{\alpha_d} & K(d), & & d. \end{array} \qquad (5)$$

Corollary. *For objects $r, s \in D$, each natural transformation $D(r, -) \xrightarrow{\cdot} D(s, -)$ has the form $D(h, -)$ for a unique arrow $h: s \to r$.*

The Yoneda map y of (4) is natural in K and r. To state this fact formally, we must consider K as an object in the functor category $\mathbf{Set}^D$, regard both domain and codomain of the map y as functors of the pair $\langle K, r \rangle$, and consider this pair as an object in the category $\mathbf{Set}^D \times D$. The codomain for y is then the evaluation functor E, which maps each pair $\langle K, r \rangle$ to the value Kr of the functor K at the object r; the domain is the functor N which maps the object $\langle K, r \rangle$ to the set $\mathrm{Nat}(D(r, -), K)$ of all natural transformations and which maps a pair of arrows $F: K \to K'$, $f: r \to r'$ to $\mathrm{Nat}(D(f, -), F)$. With these observations we may at once prove an addendum to the *Yoneda Lemma:*

Lemma. *The bijection of (4) is a natural isomorphism $y: N \xrightarrow{\cdot} E$ between the functors $E, N: \mathbf{Set}^D \times D \to \mathbf{Set}$.*

The object function $r \mapsto D(r, -)$ and the arrow function

$$(f: s \to r) \mapsto D(f, -): D(r, -) \xrightarrow{\cdot} D(s, -)$$

for f an arrow of D together define a full and faithful functor

$$Y: D^{\mathrm{op}} \to \mathbf{Set}^D \qquad (6)$$

called the *Yoneda functor*. Its dual is another such functor

$$Y' : D \to \mathbf{Set}^{D^{\mathrm{op}}} \tag{7}$$

(also faithful) which sends $f: s \to r$ to the natural transformation

$$D(-, f) : D(-, s) \dot{\to} D(-, r) : D^{\mathrm{op}} \to \mathbf{Set}.$$

D must have small hom-sets if these functors are to be defined (because **Set** is the category of all *small* sets). For larger D, the Yoneda lemmas remain valid if **Set** is replaced by any category **Ens** whose objects are sets $X, Y, \dots$, and for which $\mathbf{Ens}(X, Y)$ is the set of all functions from X to Y, provided of course that D has hom-sets which are objects in **Ens**. (The meaning of naturality is not altered by further enlargement of **Ens**; see Exercise 4.)

Exercises

1. Let functors $K, K' : D \to \mathbf{Set}$ have representations $\langle r, \psi \rangle$ and $\langle r', \psi' \rangle$, respectively. Prove that to each natural transformation $\tau : K \dot{\to} K'$, there is a unique morphism $h : r' \to r$ of D such that

$$\tau \circ \psi = \psi' \circ D(h, -) : D(r, -) \dot{\to} K'.$$

2. State the dual of the Yoneda Lemma (D replaced by D^{op}).
3. (Kan; the coyoneda lemma.) For $K : D \to \mathbf{Set}$, $(* \downarrow K)$ is the category of elements $x \in Kd$, $Q : (* \downarrow K) \to D$ is the projection $x \in Kd \mapsto d$ and for each $a \in D$, $a : (* \downarrow K) \to D$ is the diagonal functor sending everything to the constant value a. Establish a natural isomorphism

$$\mathrm{Nat}(K, D(a, -)) \cong \mathrm{Nat}(a, Q).$$

4. (Naturality is not changed by enlarging the codomain category.) Let E be a full subcategory of E'. For functors $K, L : D \to E$, with $J : E \to E'$ the inclusion, prove that $\mathrm{Nat}(K, L) \cong \mathrm{Nat}(JK, JL)$.

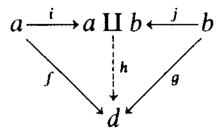
3. Coproducts and Colimits

We introduce colimits by a variety of special cases, each of which is a universal.

Coproducts. For any category C, the diagonal functor $\Delta : C \to C \times C$ is defined on objects by $\Delta(c) = \langle c, c \rangle$, on arrows by $\Delta(f) = \langle f, f \rangle$. A universal arrow from an object $\langle a, b \rangle$ of $C \times C$ to the functor Δ is called a *coproduct diagram*. It consists of an object c of C and an arrow $\langle a, b \rangle \to \langle c, c \rangle$ of $C \times C$; that is, a pair of arrows $i : a \to c, j : b \to c$ from a and b to a common codomain c. This pair has the familiar universal property: For any pair of arrows $f : a \to d$, $g : b \to d$ there is a unique $h : c \to d$ with $f = h \circ i$, $g = h \circ j$. When such a coproduct diagram exists, the object c is necessarily unique (up to isomorphism in C); it is written $c = a \amalg b$ or $c = a + b$ and is called a *coproduct object*. The coproduct diagram then is

$$a \xrightarrow{i} a \amalg b \xleftarrow{j} b \;;$$

the arrows i and j are called the *injections* of the coproduct $a \amalg b$ (though they are not required to be injective as functions). The universality of this diagram states that any diagram of the following form can be filled in uniquely (at h) so as to be commutative:


(1)

Hence the assignment $\langle f, g \rangle \mapsto h$ is a bijection

$$C(a, d) \times C(b, d) \cong C(a \amalg b, d) \qquad (2)$$

natural in d, with inverse $h \mapsto \langle hi, hj \rangle$. If every pair of objects a, b in C has a coproduct then, choosing a coproduct diagram for each pair, the coproduct $\amalg : C \times C \to C$ is a bifunctor, with $h \amalg k$ defined for arrows $h : a \to a'$, $k : b \to b'$ as the unique arrow $h \amalg k : a \amalg b \to a' \amalg b'$ with $(h \amalg k) i = i' h$, $(h \amalg k) j = j' k$ (draw the diagram!).

The diagram (1) is more familiar in other guises. For example, in **Set** take $a \amalg b$ to be a disjoint union of the sets a and b (i.e., a union of disjoint copies of a and b), while i and j are the inclusion maps $a \subset a \amalg b$, $b \subset a \amalg b$. Now a function h on a disjoint union is uniquely determined by independently giving its values on a and on b; i.e., by giving the composites hi and hj. This says exactly that diagram (1) can be filled in uniquely at h. To be sure, a disjoint union is not unique, but it is unique up to a bijection, as befits a universal.

The coproduct of any two objects exists in many of the familiar categories, where it has a variety of names as indicated in the following list:

Set	disjoint union of sets,
Top	disjoint union of spaces,
$\mathbf{Top}_*$	wedge product (join two spaces at the base points),
Ab, *R*-**Mod**	direct sum $A \oplus B$,
Grp	free product,
CRng	tensor product $R \otimes S$.

In a preorder P, a least upper bound $a \cup b$ of two elements a and b, if it exists, is an element $a \cup b$ with the properties (i) $a \leqq a \cup b$, $b \leqq a \cup b$; and (ii) if $a \leqq c$ and $b \leqq c$, then $a \cup b \leqq c$. These properties state exactly that $a \cup b$ is a coproduct of a and b in P, regarded as a category.

Infinite Coproducts. In the description of the coproduct, replace $C \times C = C^2$ by C^X for any set X. Here the set X is regarded as a discrete category, so the functor category C^X has as its objects the X-indexed families $a = \{a_x \mid x \in X\}$ of objects of C. The corresponding diagonal functor $\Delta : C \to C^X$ sends each c to the constant family (all $c_x = c$). A universal arrow from a to Δ is an X-fold coproduct diagram; it consists of a coproduct object $\amalg_x a_x \in C$ and arrows (coproduct injections) $i_x : a_x \to \amalg_x a_x$ of C with the requisite universal property. This universal property states that the assignment $f \mapsto \{f i_x \mid x \in X\}$ is a bijection

$$C(\amalg_x a_x, c) \cong \prod_{x \in X} C(a_x, c), \tag{3}$$

natural in c. In **Set**, a coproduct is an X-fold disjoint union.

Copowers. If the factors in a coproduct are all equal ($a_x = b$ for all x), the coproduct $\amalg_x b$ is called a *copower* and is written $X \cdot b$, so that

$$C(X \cdot b, c) \cong C(b, c)^X, \tag{4}$$

natural in c. For example, in **Set**, with $b = Y$ a set, the copower $X \cdot Y = X \times Y$ is the cartesian product of the sets X and Y.

Cokernels. Suppose that C has a null object z, so that for any two objects $b, c \in C$ there is a zero arrow $0 : b \to z \to c$. The cokernel of $f : a \to b$ is then an arrow $u : b \to e$ such that (i) $uf = 0 : a \to e$; (ii) if $h : b \to c$ has $hf = 0$, then $h = h'u$ for a unique arrow $h' : e \to c$. The picture is

$$\begin{array}{ccccl} a & \xrightarrow{f} & b & \xrightarrow{u} & e \qquad uf = 0, \\ & & & \searrow^{h} & \vdots\, h' \\ & & & & c, \qquad hf = 0. \end{array} \tag{5}$$

In **Ab**, the cokernel of $f : A \to B$ is the projection $B \to B/fA$ to a quotient group of B, and in many other such categories a cokernel is essentially a suitable quotient object. However, in categories without a null object cokernels are not available. Hence we consider more generally certain "coequalizers".

Coequalizers. Given in C a pair $f, g : a \to b$ of arrows with the same domain a and the same codomain b, a *coequalizer* of $\langle f, g \rangle$ is an arrow $u : b \to e$ (or, a pair $\langle e, u \rangle$) such that (i) $uf = ug$; (ii) if $h : b \to c$ has $hf = hg$, then $h = h'u$ for a unique arrow $h' : e \to c$. The picture is

$$\begin{array}{ccccl} a & \overset{f}{\underset{g}{\rightrightarrows}} & b & \xrightarrow{u} & e \qquad uf = ug, \\ & & & \searrow^{h} & \vdots\, h' \\ & & & & c, \qquad hf = hg. \end{array} \tag{6}$$

A coequalizer u can be interpreted as a universal arrow as follows. Let $\downarrow\downarrow$ denote the category which has precisely two objects and two non-identity arrows from the first object to the second; thus the category is $\cdot \rightrightarrows \cdot$. Form the functor category $C^{\downarrow\downarrow}$. An object in $C^{\downarrow\downarrow}$ is then a functor from $\cdot \rightrightarrows \cdot$ to C; that is, a pair $\langle f, g\rangle : a \to b$ of parallel arrows $a \rightrightarrows b$ in C. An arrow in $C^{\downarrow\downarrow}$ from one such pair $\langle f, g\rangle$ to another $\langle f', g'\rangle$ is a natural transformation between the corresponding functors; this means that it is a pair $\langle h, k\rangle$ of arrows $h : a \to a'$ and $k : b \to b'$ in C

$$\begin{array}{ccc} a & \overset{f}{\underset{g}{\rightrightarrows}} & b \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ a' & \overset{f'}{\underset{g'}{\rightrightarrows}} & b', \end{array} \qquad \begin{array}{c} kg = g'h, \\ \\ kf = f'h, \end{array}$$

which make the f-square *and* the g-square commute. There is also a diagonal functor $\Delta : C \to C^{\downarrow\downarrow}$, defined on objects c and arrows r of C as

$$\begin{array}{c} c \\ \downarrow{\scriptstyle r} \\ c' \end{array} \quad \mapsto \quad \begin{array}{ccc} c & \overset{1}{\underset{1}{\rightrightarrows}} & c \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle r} \\ c' & \overset{1}{\underset{1}{\rightrightarrows}} & c'; \end{array}$$

in symbols, $\Delta c = \langle 1_c, 1_c\rangle$ and $\Delta r = \langle r, r\rangle$. Now given the pair $\langle f, g\rangle : a \to b$, an arrow $h : b \to c$ with $hf = hg$ is the same thing as an arrow $\langle hf = hg, h\rangle : \langle f, g\rangle \to \langle 1_c, 1_c\rangle$ in the functor category $C^{\downarrow\downarrow}$:

$$\begin{array}{ccc} a & \overset{f}{\underset{g}{\rightrightarrows}} & b \\ {\scriptstyle hf}\downarrow & & \downarrow{\scriptstyle h} \\ c & \overset{1}{\underset{1}{\rightrightarrows}} & c, \end{array} \qquad hf = hg.$$

In other words, the arrows h which "coequalize" f and g are the arrows from $\langle f, g\rangle$ to Δ. Therefore a coequalizer $\langle e, u\rangle$ of the pair $\langle f, g\rangle$ is just a universal arrow from $\langle f, g\rangle$ to the functor Δ.

Coequalizers of any set of maps from a to b are defined in the same way.

In **Ab**, the coequalizer of two homomorphisms $f, g : A \to B$ is the projection $B \to B/(f-g)A$ on a quotient group of B (by the image of the difference homomorphism). In **Set**, the coequalizer of two functions $f, g : X \to Y$ is the projection $p : Y \to Y/E$ on the quotient set of Y by the least equivalence relation $E \subset Y \times Y$ which contains all pairs $\langle fx, gx\rangle$ for $x \in X$. The same construction, using the quotient topology, gives coequalizers in **Top**.

Pushouts. Given in C a pair $f : a \to b$, $g : a \to c$ of arrows with a common domain a, a *pushout* of $\langle f, g\rangle$ is a commutative square, such as that on the left below

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle u} \\ c & \xrightarrow[v]{} & r\,, \end{array} \qquad \begin{array}{ccc} a & \xrightarrow{f} & b \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ c & \xrightarrow[k]{} & s\,, \end{array} \qquad (7)$$

such that to every other commutative square (right above) built on f, g there is a unique $t: r \to s$ with $tu = h$ and $tv = k$. In other words, the pushout is the universal way of filling out a commutative square on the sides f, g. It may be interpreted as a universal arrow. Let $\cdot \leftarrow \cdot \rightarrow \cdot$ denote the category which looks just like that. An object in the functor category $C^{\cdot \leftarrow \cdot \rightarrow \cdot}$ is then a pair of arrows $\langle f, g \rangle$ in C with a common domain, while $\Delta(c) = \langle 1_c, 1_c \rangle$ is the object function of an evident "diagonal" functor $\Delta: C \to C^{\leftarrow \cdot \rightarrow}$. A commutative square $hf = kg$ as on the right above can then be read as an arrow

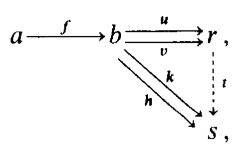
$$\begin{array}{c} \langle f, g \rangle \\ \downarrow \\ \Delta(s) \end{array} \qquad \begin{array}{ccccc} b & \xleftarrow{f} & a & \xrightarrow{g} & c \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle hf = kg} & & \downarrow{\scriptstyle k} \\ s & \xleftarrow[1]{} & s & \xrightarrow[1]{} & s \end{array}$$

in $C^{\leftarrow \cdot \rightarrow}$ from $\langle f, g \rangle$ to Δs. The pushout is a universal such arrow. Its vertex r, which is uniquely determined up to (a unique) isomorphism, is often written as a coproduct "over a"

$$r = b \amalg_a c = b \amalg_{\langle f, g \rangle} c\,,$$

and called a "fibered sum" or (the vertex of) a "cocartesian square". In **Set**, the pushout of $\langle f, g \rangle$ always exists; it is the disjoint union $b \amalg c$ with the elements fx and gx identified for each $x \in a$. A similar construction gives pushouts in **Top** – they include such useful constructions as adjunction spaces. Pushouts exist in **Grp**; in particular, if f and g above are monic in **Grp**, the arrows u and v of the pushout square are also monic, and the vertex r is called the "amalgamated product" of b with c.

Cokernel Pair. Given an arrow $f: a \to b$ in C, the pushout of f with f is called the cokernel pair of f. Thus the cokernel pair of f consists of an object r and a parallel pair of arrows $u, v: b \to r$, with domain b, such that $uf = vf$ and such that to any parallel pair $h, k: b \to s$ with $hf = kf$ there is a unique $t: r \to s$ with $tu = h$ and $tv = k$:



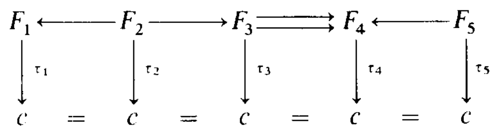
$uf = vf,$

$hf = kf.$

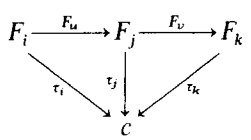
Colimits. The preceding cases all deal with particular functor categories and have the following pattern. Let C and J be categories (J for index category, usually small and often finite). The *diagonal functor*

$$\Delta : C \to C^J$$

sends each object c to the constant functor Δc – the functor which has the value c at each object $i \in J$ and the value 1_c at each arrow of J. If $f : c \to c'$ is an arrow of C, Δf is the natural transformation $\Delta f : \Delta c \dot{\to} \Delta c'$ which has the same value f at each object i of J. Each functor $F : J \to C$ is an object of C^J. A universal arrow $\langle r, u \rangle$ from F to Δ is called a *colimit* (a "direct limit" or "inductive limit") diagram for the functor F. It consists of an object r of C, usually written $r = \varinjlim F$ or $r = \operatorname{Colim} F$, together with a natural transformation $u : F \dot{\to} \Delta r$ which is universal among natural transformations $\tau : F \dot{\to} \Delta c$. Since Δc is the constant functor, the natural transformation τ consists of arrows $\tau_i : F_i \to c$ of C, one for each object i of J, with $\tau_j \ F u = \tau_i$ for each arrow $u : i \to j$ of J. Pictorially, all the squares in the following schematic diagram (for a special choice of J)



must commute. It is convenient to visualize these diagrams with all the "bottom" objects identified. For this reason, a natural transformation $\tau : F \dot{\to} \Delta c$, often written as $\tau : F \dot{\to} c$, omitting Δ, is called a *cone* from the base F to the vertex c, as in the figure



(all triangles commutative). In this language, a colimit of $F : J \to C$ consists of an object $\varinjlim F \in C$ and a cone $\mu : F \dot{\to} \Delta(\varinjlim F)$ from the base F to the vertex $\varinjlim F$ which is universal: For any cone $\tau : F \dot{\to} \Delta c$ from the base F there is a unique arrow $t' : \varinjlim F \to c$ with $\tau_i = t' \mu_i$ for every index $i \in J$. We call μ the *limiting cone* or the *universal cone* (from F).

For example, let $J = \omega = \{0 \to 1 \to 2 \to 3 \to \cdots\}$ and consider a functor $F : \omega \to \mathbf{Set}$ which maps every arrow of ω to an inclusion (subset in set). Such a functor F is simply a nested sequence of sets $F_0 \subset F_1 \subset F_2 \subset \cdots$. The union U of all sets F_n, with the cone given by the inclusion maps

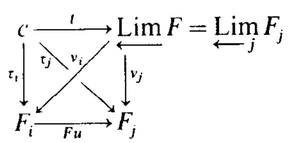
$F_n \to U$, is $\varinjlim F$. The same interpretation of unions as special colimits applies in **Grp**, **Ab**, and other familiar categories. The reader may wish to convince himself now of what we shall soon prove (Exercise V.1.8): For J small, any $F : J \to$ **Set** has a colimit.

Exercises

1. In the category of commutative rings, show that $R \to R \otimes S \leftarrow S$, with maps $r \mapsto r \otimes 1$, $1 \otimes s \leftarrow s$, is a coproduct diagram.
2. If a category has (binary) coproducts and coequalizers, prove that it also has pushouts. Apply to **Set**, **Grp**, and **Top**.
3. In the category $\mathbf{Matr}_K$ of § I.2, describe the coequalizer of two $m \times n$ matrices A, B (i.e., of two arrows $n \to m$ in $\mathbf{Matr}_K$).
4. Describe coproducts (and show that they exist) in **Cat**, in **Mon**, and in **Grph**.
5. If E is an equivalence relation on a set X, show that the usual set X/E of equivalence classes can be described by a coequalizer in **Set**.
6. Show that a and b have a coproduct in C if and only if the following functor is representable: $C(a, -) \times C(b, -) : C \to \mathbf{Set}$, by $c \mapsto C(a, c) \times C(b, c)$.
7. (Every abelian group is a colimit of its finitely generated subgroups.) If A is an abelian group, and J_A the preorder with objects all finitely generated subgroups $S \subset A$ ordered by inclusion, show that A is the colimit of the evident functor $J_A \to \mathbf{Ab}$. Generalize.

4. Products and Limits

The limit notion is dual to that of a colimit. Given categories C, J, and the diagonal functor $\Delta : C \to C^J$, a *limit* for a functor $F : J \to C$ is a universal arrow $\langle r, v \rangle$ from Δ to F. It consists of an object r of C, usually written $r = \varprojlim F$ or $\operatorname{Lim} F$ and called the *limit* object (the "inverse limit" or "projective limit") of the functor F, together with a natural transformation $v : \Delta r \dot{\to} F$ which is universal among natural transformations $\tau : \Delta c \dot{\to} F$, for objects c of C. Since $\Delta c : J \to C$ is the functor constantly c, this natural transformation τ consists of one arrow $\tau_i : c \to F_i$ of C for each object i of J such that for every arrow $u : i \to j$ of J one has $\tau_j = Fu \circ \tau_i$. We may call $\tau : c \dot{\to} F$ a *cone* to the base F from the vertex c. (We say "cone to the base F" rather than "cocone"). The universal property of v is this: It is a cone to the base F from the vertex $\varprojlim F$; for any cone τ to F from an object c, there is a unique arrow $t : c \to \varprojlim F$ such that $\tau_i = v_i t$ for all i. The situation may be pictured as



$v =$ limiting cone ,

each cone is represented by a commuting triangle (just one of many), with vertex up; there is a unique arrow t which makes all the added (vertex down) triangles commute. As with any universal, the object $\varprojlim F$ and its *limiting cone* $v : \varprojlim F \to F$ are determined uniquely by the functor F, up to isomorphism in C.

The properties of $\varprojlim$ and $\varinjlim$ are summarized in the diagram

$$\begin{array}{ccccc} \operatorname{Lim} F = \varprojlim F & \xrightarrow{\;\nu\;} & F & \xrightarrow{\;\mu\;} & \varinjlim F = \operatorname{Colim} F \\ \uparrow \vdots & & \| & & \vdots \downarrow \\ c & \xrightarrow[\tau]{} & F & \xrightarrow[\sigma]{} & c\,, \end{array} \tag{1}$$

where the horizontal arrows are cones, the vertical arrows are arrows in C. When the limits exist, there are natural isomorphisms

$$C(c, \varprojlim F) \cong \operatorname{Nat}(\Delta c, F) = \operatorname{Cone}(c, F)\,, \tag{2}$$

$$\operatorname{Cone}(F, c) = \operatorname{Nat}(F, \Delta c) \cong C(\varinjlim F, c)\,. \tag{3}$$

There are familiar names for various special limits, dual to those for colimits:

Products. If J is the discrete category $\{1, 2\}$, a functor $F : \{1, 2\} \to C$ is a pair of objects $\langle a, b\rangle$ of C. The limit object is called a *product* of a and b, and is written $a \times b$ or $a \amalg b$; the limit diagram consists of $a \times b$ and two arrows p, q (or sometimes pr_1, pr_2),

$$a \xleftarrow{p} a \times b \xrightarrow{q} b\,,$$

called the *projections* of the product. They constitute a cone from the vertex $a \times b$, so by the definition above of a limit, there is a bijection of sets

$$C(c, a \times b) \cong C(c, a) \times C(c, b) \tag{4}$$

natural in c, which sends each $h : c \to a \times b$ to the pair of composites $\langle ph, qh\rangle$. Conversely, given arrows $f : c \to a$ and $g : c \to b$, there is a unique $h : c \to a \times b$ with $ph = f$ and $qh = g$. We write

$$h = (f, g) : c \to a \times b$$

and call h the arrow with *components* f and g. We have already observed (in § II.3) that the product of any two objects exists in **Cat**, in **Grp**, in **Top**, and in **Mon**; in these cases (and in many others) it is called the *direct product*. In a preorder, a product is a greatest lower bound.

Infinite products. If J is a set (= discrete category = category with all arrows identities), then a functor $F : J \to C$ is simply a J-indexed family of objects $a_j \in C$, while a cone with vertex c and base a_j is just a J-indexed family of arrows $f_j: c \to a_j$. A universal cone $p_j: \Pi_j a_j \to a_j$ thus consists of an object $\Pi_j a_j$, called the *product* of the factors a_j, and of arrows p_j, called the *projections* of the product, with the following universal property: To each J-indexed family (=cone) $f_j: c \to a_j$ there is a unique f

$$f: c \to \Pi_j a_j, \quad \text{with} \quad p_j f = f_j, \quad j \in J.$$

The arrow f uniquely determined by this property is called the map (to the product) with *components* f_j, $j \in J$. Also $\{f_j \mid j \in J\} \mapsto f$ is a bijection

$$\Pi_j C(c, a_j) \cong C(c, \Pi_j a_j), \tag{5}$$

natural in c. Here the right hand product is that in C, while the left-hand product is taken in **Set** (where we assume that C has small hom-sets). Observe that the hom-functor $C(c, -)$ carries products in C to products in **Set** (see § V.4). Products over any small set J exist in **Set**, in **Top**, and in **Grp**; in each case they are just the familiar cartesian products.

Powers. If the factors in a product are all equal ($a_j = b \in C$ for all j) the product $\Pi_j a_j = \Pi_j b$ is called a *power* and is written $\Pi_j b = b^J$, so the

$$C(c, b)^J \cong C(c, b^J), \tag{6}$$

natural in c. The power on the left is that in **Set**, where every small power X^J exists (and is the set of all functions $J \to X$).

Equalizers. If $J = \downarrow\downarrow$, a functor $F: \downarrow\downarrow \to C$ is a pair $f, g: b \to a$ of parallel arrows of C. A limit object d of F, when it exists, is called an *equalizer* (or, a "difference kernel") of f and g. The limit diagram is

$$d \xrightarrow{e} b \underset{g}{\overset{f}{\rightrightarrows}} a, \qquad fe = ge \tag{7}$$

(the limit arrow e amounts to a cone $a \leftarrow d \to b$ from the vertex d). The limit arrow is often called the equalizer of f and g; its universal property reads: To any $h: c \to b$ with $fh = gh$ there is a unique $h': c \to d$ with $eh' = h$.

In **Set**, the equalizer always exists; d is the set $\{x \in b \mid fx = gx\}$ and $e: d \to b$ is the injection of this subset of b into b. In **Top**, the equalizer has the same description (d has the subspace topology). In **Ab** the equalizer d of f and g is the usual kernel of the difference homomorphism $f - g: b \to a$.

Equalizers for any set of arrows from b to a are described similarly. Any equalizer e is necessarily a monic.

Pullbacks. If $J = (\to \cdot \leftarrow)$, a functor $F : (\to \cdot \leftarrow) \to C$ is a pair of arrows $b \xrightarrow{f} a \xleftarrow{g} d$ of C with a common codomain a. A cone over such a functor is a pair of arrows from a vertex c such that the square (on the left)

$$\begin{array}{ccc} c & \xrightarrow{k} & d \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ b & \xrightarrow[f]{} & a, \end{array} \qquad \begin{array}{ccc} b \times_a d & \xrightarrow{q} & d \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle g} \\ b & \xrightarrow{f} & a \end{array} \tag{8}$$

commutes. A universal cone is then a commutative square of this form, with new vertex written $b \times_a d$ and arrows p, q as shown on the right, such that for any square with vertex c there is a unique $r : c \to b \times_a d$ with $k = qr$, $h = pr$. The square formed by this universal cone is called a *pullback square* or a "cartesian square" and the vertex $b \times_a d$ of the universal cone is called a pullback, a "fibered product", or a product over (the object) a. This construction, possible in many categories, first became prominent in the category **Top**. If $g : d \to a$ is a "fiber map" (of some type) with "base" a and f is a continuous map into the base, then the projection p of the pullback is the "induced fiber map" (of the type considered).

The pullback of a pair of equal arrows $f : b \to a \leftarrow b : f$, when it exists, is called the *kernel pair* of f. It is an object d and a pair of arrows $p, q : d \to b$ such that $fp = fq : d \to a$ and such that any pair $h, k : c \to a$ with $fh = fk$ can be written as $h = pr$, $k = qr$ for a unique $r : c \to d$.

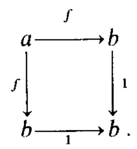
If $J = \mathbf{0}$ is the empty category, there is exactly one functor $\mathbf{0} \to C$; namely, the empty functor; a cone over this functor is just an object $c \in C$ (i.e., just a vertex). Hence a universal cone on $\mathbf{0}$ is an object t of C such that each object $c \in C$ has a unique arrow $c \dashrightarrow t$. In other words, a limit of the empty functor to C is a terminal object of C.

Limits are sometimes defined for diagrams rather than for functors. In detail, let C be a category, UC the underlying graph of C, and G any graph. Then a *diagram* in C of shape G is a morphism $D : G \to UC$ of graphs. Now define a *cone* $\mu : c \dot{\to} D$ to be a function assigning to each object $i \in G$ an arrow $\mu_i : c \to D_i$ of C such that $Dh \circ \mu_i = \mu_j$ for every arrow $h : i \to j$ of the graph G. This is just the previous definition of a cone (a natural transformation $\mu : \Delta c \dot{\to} D$), coupled with the observation that this definition uses the composition of arrows in C but *not* in the domain G of D. A limit for the diagram D is now a universal cone $\lambda : c \dot{\to} D$.

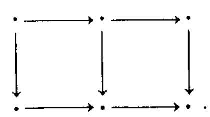
This variation on the definition of a limit yields no essentially new information. For, let FG be a free category generated by the graph G, and $P : G \to U(FG)$ the corresponding universal diagram. Then each diagram $D : G \to UC$ can be written uniquely as $D = UD' \circ P$ for a (unique) functor $D' : FG \to C$, and one readily observes that limits (and limiting cones) for D' correspond exactly to those for D.

Exercises

1. In **Set**, show that the pullback of $f: X \to Z$ and $g: Y \to Z$ is given by the set of pairs $\{\langle x, y \rangle \mid x \in X,\ y \in Y,\ fx = gy\}$. Describe pullbacks in **Top**.
2. Show that the usual cartesian product over an index set J, with its projections, is a (categorical) product in **Set** and in **Top**.
3. If the category J has an initial object s, prove that every functor $F: J \to C$ to any category C has a limit, namely $F(s)$. Dualize.
4. In any category, prove that $f: a \to b$ is epi if and only if the following square is a pushout:



5. In a pullback square (8), show that f monic implies q monic.
6. In **Set**, show that the kernel pair of $f: X \to Y$ is given by the equivalence relation $E = \{\langle x, x' \rangle \mid x, x' \in X \text{ and } fx = fx'\}$, with suitable maps $E \rightrightarrows X$.
7. (Kernel pairs via products and equalizers.) If C has finite products and equalizers, show that the kernel pair of $f: a \to b$ may be expressed in terms of the projections $p_1, p_2: a \times a \to a$ as $p_1 e, p_2 e$, where e is the equalizer of $fp_1, fp_2: a \times a \to b$ (cf. Exercise 6). Dualize.
8. Consider the following commutative diagram



   (a) If both squares are pullbacks, prove that the outside rectangle (with top and bottom edges the evident composites) is a pullback.
   (b) If the outside rectangle and the right-hand square are pullbacks, so is the left-hand square.
9. (Equalizers via products and pullbacks.) Show that the equalizer of $f, g: b \to a$ may be constructed as the pullback of

$$(1_b, f): b \to b \times a \leftarrow b: (1_b, g).$$

10. If C has pullbacks and a terminal object, prove that C has all finite products and equalizers.

5. Categories with Finite Products

A category C is said to *have finite products* if to any finite number of objects $c_1, \ldots, c_n$ of C there exists a product diagram, consisting of a product object $c_1 \times \cdots \times c_n$ and n projections $p_i: c_1 \times \cdots \times c_n \to c_i$, for $i = 1, \ldots, n$, with the usual universal property. In particular, C then has a product of no objects, which is simply a terminal object t in C, as well