VI. Monads and Algebras

In this chapter we will examine more closely the relation between universal algebra and adjoint functors. For each type τ of algebras (§ V.6), we have the category $\mathbf{Alg}_\tau$ of all algebras of the given type, the forgetful functor $G : \mathbf{Alg}_\tau \to \mathbf{Set}$, and its left adjoint F, which assigns to each set S the free algebra FS of type τ generated by elements of S. A trace of this adjunction $\langle F, G, \varphi\rangle : \mathbf{Set} \rightharpoonup \mathbf{Alg}_\tau$ resides in the category **Set**; indeed, the composite $T = GF$ is a functor $\mathbf{Set} \to \mathbf{Set}$, which assigns to each set S the set of all elements of its corresponding free algebra. Moreover, this functor T is equipped with certain natural transformations which give it a monoid-like structure, called a "monad". The remarkable part is then that the whole category $\mathbf{Alg}_\tau$ can be reconstructed from this monad in **Set**. Another principal result is a theorem due to Beck, which describes exactly those categories A with adjunctions $\langle F, G, \varphi\rangle : X \rightharpoonup A$ which can be so reconstructed from a monad T in the base category X. It then turns out that algebras in this last sense are so general as to include the compact Hausdorff spaces (§ 9).

1. Monads in a Category

Any endofunctor $T: X \to X$ has composites $T^2 = T \circ T: X \to X$ and $T^3 = T^2 \circ T: X \to X$. If $\mu : T^2 \dot{\to} T$ is a natural transformation, with components $\mu_x : T^2 x \to Tx$ for each $x \in X$, then $T\mu : T^3 \dot{\to} T^2$ denotes the natural transformation with components $(T\mu)_x = T(\mu_x) : T^3 x \to T^2 x$ while $\mu T : T^3 \dot{\to} T^2$ has components $(\mu T)_x = \mu_{Tx}$. Indeed, $T\mu$ and μT are "horizontal" composites in the sense of § II.5.

Definition. *A monad $T = \langle T, \eta, \mu\rangle$ in a category X consists of a functor $T: X \to X$ and two natural transformations*

$$\eta : I_X \dot{\to} T, \qquad \mu : T^2 \dot{\to} T \tag{1}$$

which make the following diagrams commute

$$\begin{array}{ccc} T^3 & \xrightarrow{T\mu} & T^2 \\ {\scriptstyle \mu T}\downarrow & & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow{\mu} & T, \end{array} \qquad \begin{array}{ccccc} IT & \xrightarrow{\eta T} & T^2 & \xleftarrow{T\eta} & TI \\ \| & & \downarrow{\scriptstyle \mu} & & \| \\ T & = & T & = & T. \end{array} \tag{2}$$

Formally, the definition of a monad is like that of a monoid M in sets, as described in the introduction. The set M of elements of the monoid is replaced by the endofunctor $T: X \to X$, while the cartesian product $\times$ of two sets is replaced by composite of two functors, the binary operation $\mu: M \times M \to M$ of multiplication by the transformation $\mu: T^2 \dot{\to} T$ and the unit (identity) element $\eta: 1 \to M$ by $\eta: I_X \dot{\to} T$. We shall thus call η the *unit* and μ the *multiplication* of the monad T; the first commutative diagram of (2) is then the *associative* law for the monad, while the second and third diagrams express the left and right *unit laws*, respectively. All told, a monad in X is just a monoid in the category of endofunctors of X, with product $\times$ replaced by composition of endofunctors and unit set by the identity endofunctor.

Terminology. These objects $\langle X, T, \eta, \mu \rangle$ have been variously called "dual standard construction", "triple", "monoid", and "triad". The frequent but unfortunate use of the word "triple" in this sense has achieved a maximum of needless confusion, what with the conflict with ordered triple, plus the use of associated terms such as "triple derived functors" for functors which are not three times derived from anything in the world. Hence the term *monad.*

Every adjunction $\langle F, G, \eta, \varepsilon \rangle : X \rightharpoonup A$ gives rise to a monad in the category X. Specifically, the two functors $F: X \to A$ and $G: A \to X$ have composite $T = GF$ an endofunctor, the unit η of the adjunction is a natural transformation $\eta: I \dot{\to} T$ and the counit $\varepsilon: FG \dot{\to} I_A$ of the adjunction yields by horizontal composition a natural transformation $\mu = G\varepsilon F: GFGF \dot{\to} GF = T$. The associative law of (2) above for this μ becomes the commutativity of the first diagram below

$$\begin{array}{ccc} GFGFGF & \xrightarrow{GFG\varepsilon F} & GFGF \\ {\scriptstyle G\varepsilon FGF}\downarrow & & \downarrow{\scriptstyle G\varepsilon F} \\ GFGF & \xrightarrow{G\varepsilon F} & GF, \end{array} \qquad \begin{array}{ccc} FGFG & \xrightarrow{FG\varepsilon} & FG \\ {\scriptstyle \varepsilon FG}\downarrow & & \downarrow{\scriptstyle \varepsilon} \\ FG & \xrightarrow{\varepsilon} & I_A. \end{array}$$

Dropping G in front and F behind, this amounts to the commutativity of the second diagram, which holds by the very definition (§ II.4) of the (horizontal) composite $\varepsilon\varepsilon = \varepsilon \cdot (FG\varepsilon) = \varepsilon \cdot (\varepsilon FG)$ (i.e., by the "interchange law" for functors and natural transformations). Similarly, the left and right unit laws of (2) reduce to the diagrams

$$\begin{array}{ccccc} I_X GF & \xrightarrow{\eta GF} & GFGF & \xleftarrow{GF\eta} & GFI_X \\ & {\scriptstyle =}\searrow & \downarrow{\scriptstyle G\varepsilon F} & \swarrow{\scriptstyle =} & \\ & & GF & & \end{array}$$

which are essentially just the two triangular identities

$$1 = G\varepsilon \cdot \eta G : G \dot{\to} G \qquad 1 = \varepsilon F \cdot F\eta : F \dot{\to} F$$

for an adjunction. Therefore $\langle GF, \eta, G\varepsilon F\rangle$ is indeed a monad in X. Call it the *monad defined by the adjunction* $\langle F, G, \eta, \varepsilon\rangle$.

For example, the *free group monad* in **Set** is the monad defined by the adjunction $\langle F, G, \varphi\rangle$: **Set**$\rightharpoonup$**Grp**, with G : **Grp**$\to$**Set** the usual forgetful functor.

Dually, a comonad in a category consists of a functor L and transformations

$$L : A \to A\,, \quad \varepsilon : L \dot{\to} I\,, \quad \delta : L \dot{\to} L^2 \qquad (1^{\mathrm{op}})$$

which render commutative the diagrams

$$\begin{array}{ccc} L & \xrightarrow{\delta} & L^2 \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle L\delta} \\ L^2 & \xrightarrow[\delta L]{} & L^3\,, \end{array} \qquad \begin{array}{ccccc} L & = & L & = & L \\ \| & & \downarrow{\scriptstyle\delta} & & \| \\ IL & \xleftarrow{\varepsilon L} & L^2 & \xrightarrow{L\varepsilon} & LI\,. \end{array}$$

Each adjunction $\langle F, G, \eta, \varepsilon\rangle : X \rightharpoonup A$ defines a comonad $\langle FG, \varepsilon, F\eta G\rangle$ in A.

What is a monad in a preorder P? A functor T: $P \to P$ is just a function T: $P \to P$ which is monotonic ($x \leqq y$ in P implies $Tx \leqq Ty$); there are natural transformations η and μ as in (1) precisely when

$$x \leqq Tx\,, \qquad T(Tx) \leqq Tx \qquad (3)$$

for all $x \in P$; the diagrams (2) then necessarily commute because in a preorder there is at most one arrow from here to yonder. The first equation of (3) gives $Tx \leqq T(Tx)$. Now suppose that the preorder P is a partial order ($x \leqq y \leqq x$ implies $x = y$). Then the Eqs. (3) imply that $T(Tx) = Tx$. Hence a monad T in a partial order P is just a *closure operation* t in P; that is, a monotonic function $t : P \to P$ with $x \leqq tx$ and $t(tx) = tx$ for all $x \in P$.

We leave the reader to describe a morphism $\langle T, \mu, \eta\rangle \to \langle T', \mu', \eta'\rangle$ of monads (a suitable natural transformation $T \dot{\to} T'$) and the category of all monads in a given category X.

2. Algebras for a Monad

The natural question, "Can every monad be defined by a suitable pair of adjoint functors?" has a positive answer, in fact there are two positive answers provided by two suitable pairs of adjoint functors. The first answer (due to Eilenberg-Moore [1965]) constructs from a monad $\langle T, \eta, \mu\rangle$ in X a category of X^T of "T-algebras" and an adjunction $X \rightharpoonup X^T$ which defines $\langle T, \eta, \mu\rangle$ in X. Formally, the definition of a T-algebra is that of a set on which the "monoid" T acts (cf. the introduction).

Definition. *If* $T = \langle T, \eta, \mu \rangle$ *is a monad in* X, *a T-algebra* $\langle x, h \rangle$ *a pair consisting of an object* $x \in X$ *(the underlying object of the algebra and an arrow* $h: Tx \to x$ *of* X *(called the structure map of the algebra which makes both the diagrams*

$$\begin{array}{ccc} T^2x & \xrightarrow{Th} & Tx \\ {\scriptstyle \mu_x}\downarrow & & \downarrow{\scriptstyle h} \\ Tx & \xrightarrow{h} & x \end{array} \qquad \begin{array}{ccc} x & \xrightarrow{\eta_x} & Tx \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle h} \\ & & x \end{array} \qquad (1$$

commute. (The first diagram is the associative law, the second the unit law A morphism $f: \langle x, h \rangle \to \langle x', h' \rangle$ *of T-algebras is an arrow* $f: x \to x'$ *o X which renders commutative the diagram*

$$\begin{array}{ccc} x & \xleftarrow{h} & Tx \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Tf} \\ x' & \xleftarrow{h'} & Tx' \end{array} . \qquad (2$$

Theorem 1 *(Every monad is defined by its T-algebras). If* $\langle T, \eta, \mu \rangle$ *is a monad in* X, *then the set of all T-algebras and their morphisms forr a category* X^T. *There is an adjunction*

$$\langle F^T, G^T; \eta^T, \varepsilon^T \rangle : X \rightharpoonup X^T$$

in which the functors G^T *and* F^T *are given by the respective assignment*

$$G^T: \begin{array}{ccc} \langle x, h \rangle & \longmapsto & x \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ \langle x', h' \rangle & \longmapsto & x' , \end{array} \qquad F^T: \begin{array}{ccc} x & \longmapsto & \langle Tx, \mu_x \rangle \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle Tf} \\ x' & \longmapsto & \langle Tx', \mu_{x'} \rangle , \end{array} \qquad (3$$

while $\eta^T = \eta$ *and* $\varepsilon^T \langle x, h \rangle = h$ *for each T-algebra* $\langle x, h \rangle$. *The monac defined in* X *by this adjunction is the given monad* $\langle T, \eta, \mu \rangle$.

The proof is straightforward verification. If $f: \langle x, h \rangle \to \langle x', h' \rangle$ and $g: \langle x', h' \rangle \to \langle x'', h'' \rangle$ are morphisms of T-algebras, so is theii composite gf; with this composition of arrows, the T-algebras evidently form a category X^T, as asserted. The functor $G^T: X^T \to X$ is the evident functor which simply forgets the structure map of each T-algebra. On the other hand, for each $x \in X$ the pair $\langle Tx, \mu_x : T(Tx) \to Tx \rangle$ is a T-algebra (the *free T-algebra* on x), in view of the associative and (left) unit laws for the monad T. Hence $x \mapsto \langle Tx, \mu_x \rangle$ does indeed define a functor $F^T: X \to X^T$, as asserted. Then $G^T F^T x = G^T \langle Tx, \mu_x \rangle = Tx$, so the unit η of the given monad is a natural transformation $\eta = \eta^T : I_X \dot{\to} G^T F^T$. On the other hand, $F^T G^T \langle x, h \rangle = \langle Tx, \mu_x \rangle$, while the first square in the definition (1) of a T-algebra $\langle x, h \rangle$ states that the structure map $h: Tx \to x$

is a morphism $\langle Tx, \mu_x \rangle \rightarrow \langle x, h \rangle$ of T-algebras. The resulting transformation

$$\varepsilon^T_{\langle x, h \rangle} = h : F^T G^T \langle x, h \rangle \rightarrow \langle x, h \rangle$$

is natural, by the definition (above) of a morphism of T-algebras. The triangular identities for an adjunction read

$$\begin{array}{ccc} Tx & \xrightarrow{T\eta_x} & TTx \\ & \searrow^{=} & \downarrow{\scriptstyle \mu_x} \\ & & Tx \end{array} \qquad \qquad \begin{array}{ccc} x & \xrightarrow{\eta_x} & Tx \\ & \searrow^{=} & \downarrow{\scriptstyle h} \\ & & x \end{array}$$

The first holds by the (right) unit law for T, the second by the unit law (see (1)) for a T-algebra. Therefore η^T and ε^T define an adjunction, as stated.

This adjunction thus determines a monad in X. The endofunctor $G^T F^T$ is the original T, its unit η^T is the original unit, and its multiplication $\mu^T = G^T \varepsilon^T F^T$ has $\mu^T x = G^T \varepsilon^T \langle Tx, \mu_x \rangle = G^T \mu_x = \mu_x$, so is the original multiplication of T. The proof is complete.

We now give several examples which show that the T-algebras for familiar monads are the familiar algebras.

Closure. A closure operation T on a preorder P is a monad in P (see § 1); a T-algebra is then an $x \in P$ with $Tx \leqq x$ (the structure map). Since $x \leqq Tx$ for all x, a T-algebra is simply an element $x \in P$ with $x \leqq Tx \leqq x$. If P is a partial order, this means that $x = Tx$, so that a T-algebra is simply an element x of the partial order which is *closed*, in the usual sense.

Group actions. If G is a (small) group, then for every (small) set X the definitions

$$\begin{array}{lll} TX = G \times X\,, & X \xrightarrow{\eta_X} G \times X\,, & G \times (G \times X) \longrightarrow G \times X\,, \\ & x \longmapsto \langle u, x \rangle\,, & \langle g_1, \langle g_2, x \rangle \rangle \longmapsto \langle g_1 g_2, x \rangle \end{array}$$

for $x \in X$, $g_1, g_2 \in G$ and u the unit element of G, define a monad $\langle T, \eta, \mu \rangle$ on **Set**. A T-algebra is then a set X together with a function $h: G \times X \rightarrow X$ (the structure map) such that always

$$h(g_1 g_2, x) = h(g_1, h(g_2, x)), \qquad h(u, x) = x\,.$$

If we write $g \cdot x$ for $h(g, x)$, these are just the usual conditions that $\langle g, x \rangle \mapsto g \cdot x$ defines an action of the group G on the set X. That T-algebras for the monad T are just the group actions is not a surprise, since our definition of T-algebras was constructed on the model of group actions.

Modules. If R is a (small) ring, then for each (small) abelian group A the definitions

$$TA = R \otimes A, \quad A \to R \otimes A, \quad R \otimes (R \otimes A) \to R \otimes A,$$
$$a \mapsto 1 \otimes a, \quad r_1 \otimes (r_2 \otimes a) \mapsto r_1 r_2 \otimes a,$$

for $a \in A$, $r_1, r_2 \in R$, define a monad on **Ab**. Much as in the previous case, the T-algebras are exactly the left R-modules.

Exercises

1. Complete semi-lattices (E. Manes; thesis). Recall that a complete semi-lattice is a partial order Q in which every subset $S \subset Q$ has a supremum (least upper bound) in Q. Let $\mathscr{P}$ be the *covariant power set functor* on **Set** so that $\mathscr{P}X$ is the set of all subsets $S \subset X$, while for each function $f: X \to Y$, $(\mathscr{P}f)S$ is the direct image of S under f. For each set X, let $\eta_X : X \to \mathscr{P}X$ send each $x \in X$ to the one point set $\{x\}$, while $\mu_X : \mathscr{P}\mathscr{P}X \to \mathscr{P}X$ sends each set of sets into its union.
   (a) Prove that $\langle \mathscr{P}, \eta, \mu \rangle$ is a monad $\mathscr{P}$ on **Set**.
   (b) Prove that each $\mathscr{P}$-algebra $\langle X, h \rangle$ is a complete semi-lattice when $x \leq y$ is defined by $h\{x, y\} = y$, and $\sup S = hS$ for each $S \subset X$.
   (c) Prove conversely that every (small) complete semi-lattice is a $\mathscr{P}$-algebra in this way.
   (d) Conclude that the category of $\mathscr{P}$-algebras is the category of all (small) complete semi-lattices, with morphisms the order and sup-preserving functions.
2. Show that $G^T : X^T \to X$ creates limits.
3. (a) For monads $\langle T, \eta, \mu \rangle$ and $\langle T', \eta', \mu' \rangle$ on X, define a morphism θ of monads as a suitable natural transformation $\theta : T \dot{\to} T'$, and construct the category of all monads in X.
   (b) From θ construct a functor $\theta^* : X^{T'} \to X^T$ such that $G^T \circ \theta^* = G^{T'}$ and a natural transformation $F^T \dot{\to} \theta^* \circ F^{T'}$.

3. The Comparison with Algebras

Suppose we start with an adjunction $X \rightharpoonup A$, construct the monad T defined in X by the adjunction and then the category X^T of T-algebras; we then ask: How is this related to the original category A? A full answer will relate not only the categories, but the adjunctions, and is provided by the following comparison theorem.

Theorem 1 *(Comparison of adjunctions with algebras). Let*

$$\langle F, G, \eta, \varepsilon \rangle : X \rightharpoonup A$$

be an adjunction, $T = \langle GF, \eta, G\varepsilon F \rangle$ the monad it defines in X. Then there is a unique functor $K : A \to X^T$ with $G^T K = G$ and $KF = F^T$.

Proof. The conclusion asserts that we can fill in the arrow K in the following diagram so that both the F-square and the G-square commute

$$\begin{array}{ccc} A & \overset{K}{\dashrightarrow} & X^T \\ F \uparrow\downarrow G & & F^T \uparrow\downarrow G^T \\ X & = & X. \end{array} \tag{1}$$

Now the counit ε of the given adjunction defines for each $a \in A$ an arrow $G\varepsilon_a : GFGa \to Ga$. This arrow may be considered as a structure map h for a T-algebra structure on the object $Ga = x$, for the requisite diagrams (cases of (2.1)) are

$$\begin{array}{ccc} GFGFGa & \xrightarrow{GFG\varepsilon_a} & GFGa \\ {\scriptstyle \mu_{Ga} = G\varepsilon FGa}\downarrow & & \downarrow {\scriptstyle G\varepsilon_a} \\ GFGa & \xrightarrow[G\varepsilon_a]{} & Ga \ , \end{array} \qquad \begin{array}{ccc} Ga & \xrightarrow{\eta Ga} & GFGa \\ & {\scriptstyle 1}\searrow & \downarrow {\scriptstyle G\varepsilon_a} \\ & & Ga \ . \end{array}$$

They commute (the first is the definition of $G\varepsilon\varepsilon$, the second is one of the triangular identities for the given adjunction). Therefore for any $f: a \to a'$ in A we define K by

$$Ka = \langle Ga, G\varepsilon_a \rangle, \qquad Kf = Gf : \langle Ga, G\varepsilon_a \rangle \to \langle Ga', G\varepsilon_{a'} \rangle; \tag{2}$$

since ε is natural, the proposed arrow Kf commutes with $G\varepsilon$ and so is a morphism of T-algebras. It is routine to verify that K is a functor with

$$KF = F^T, \qquad G^T K = G. \tag{3}$$

It remains to show K unique. First, each Ka must be a T-algebra, and the commutativity requirement $G^T K = G$ means that the underlying X-object of this T-algebra Ka is Ga. Therefore Ka must have the form $Ka = \langle Ga, h \rangle$ for some structure map h; moreover $G^T K = G$ means that the value of K on an arrow f in A must be $Kf = Gf$, exactly as in (2) above. It remains only to determine the structure map h. Now (1) commutes, and the two adjunctions $\langle F, G, \ldots \rangle$ and $\langle F^T, G^T, \ldots \rangle$ have the *same* unit η, so the two functors $K : A \to X^T$ and the identity $I : X \to X$ define a map of the first adjunction to the second, in the sense considered in § IV.7. Proposition IV.7.1 for this map then states that $K\varepsilon = \varepsilon^T K$. But K on arrows is G, so $K\varepsilon_a = G\varepsilon_a$ for each $a \in A$, while the definition of the counit ε^T of an algebra gives $\varepsilon^T Ka = \varepsilon^T \langle Ga, h \rangle = h$. Thus $K\varepsilon = \varepsilon^T K$ implies $G\varepsilon_a = h$, so the structure map h is determined and K is unique.

For many familiar adjunctions $\langle F, G, \ldots \rangle$ this comparison functor K will be an isomorphism; we then say that G is *monadic* (tripleable). For other authors (Barr-Wells [1985]), "triplable" means only that K be an equivalence of categories. However, here is an easy example when K is not an isomorphism, and not even an equivalence. The forgetful functor $G : \mathbf{Top} \to \mathbf{Set}$ has a left adjoint D which assigns to each set X the discrete topological space (all subsets open in X), for the identity arrow $\eta_X : X \to GDX$ is trivially universal from the object X to the functor G. This adjunction $\langle D, G, \eta, \dots \rangle : \mathbf{Set} \rightharpoonup \mathbf{Top}$ defines on **Set** the monad $I = \langle I, 1, 1 \rangle$ which is the identity (identity functor, identity as unit and as multiplication). The I-algebras in **Set** are just the sets, so the comparison functor $\mathbf{Top} \to \mathbf{Top}^I = \mathbf{Set}$ is in this case the given forgetful functor G.

4. Words and Free Semigroups

The comparison functor can be illustrated explicitly in the case of semigroups. A semigroup is a set S equipped with an associative binary operation $\nu : S \times S \to S$. The free semigroup WX on a set X is like the free monoid on X (§ II.7). It consists of all words $\langle x_1 \rangle \dots \langle x_n \rangle$ of positive length n spelled in letters $x_i \in X$, where we write $\langle x \rangle$ to distinguish the word $\langle x \rangle$ in WX from the element $x \in X$. Words are multiplied by juxtaposition,

$$(\langle x_1 \rangle \dots \langle x_n \rangle)(\langle y_1 \rangle \dots \langle y_m \rangle) = \langle x_1 \rangle \dots \langle x_n \rangle \langle y_1 \rangle \dots \langle y_m \rangle ;$$

this multiplication ν is associative, so makes $FX = \langle WX, \nu \rangle$ a semigroup, with the set WX the disjoint union $\amalg X^n$, $n = 1, 2, \dots$. If $G : \mathbf{Smgrp} \to \mathbf{Set}$ is the forgetful functor from the category of all small semigroups (forget the multiplication), then the arrow $\eta_X : X \to GFX$ defined by $x \mapsto \langle x \rangle$ (send each x to the one-letter word in x) is universal from X to G. Therefore F is a functor, left adjoint to G, and η defines an adjunction

$$\langle F, G, \eta, \varepsilon \rangle : \mathbf{Set} \rightharpoonup \mathbf{Smgrp} .$$

If S is any semigroup (set S with an associative binary operation $S \times S \to S$, written as multiplication) the counit ε_S of this adjunction is by definition that morphism $\varepsilon_S : FGS \to S$ of semigroups for which the composite $G\varepsilon_S \ \eta_{GS} : GS \to GFGS \to GS$ is the identity; in other words, ε_S is the unique morphism of semi-groups which sends each generator $\langle s \rangle$ to s. This means that

$$\varepsilon_S(\langle s_1 \rangle \dots \langle s_n \rangle) = s_1 \dots s_n \quad \text{(product in } S) \tag{1}$$

for all $s_i \in S$: The counit ε removes the "pointy bracket" $\langle \ \rangle$.

Proposition 1. *The monad on* **Set** *determined by the adjunction* **Set** $\rightharpoonup$ **Smgrp** *is*

$$W = \langle W : \mathbf{Set} \to \mathbf{Set},\ \eta : I \dot\to W,\ \mu : W^2 \dot\to W \rangle$$

where $WX = \coprod_{n=1}^{\infty} X^n$, $\eta_X x = \langle x \rangle$ *for each* $x \in X$, *while* μ_X *is*

$$\mu_X(\langle\langle x_{11}\rangle \dots \langle x_{1n_1}\rangle\rangle \dots \langle\langle x_{k1}\rangle \dots \langle x_{kn_k}\rangle\rangle) = \langle x_{11}\rangle \dots \langle x_{1n_1}\rangle \dots \langle x_{k1}\rangle \dots \langle x_{kn_k}\rangle$$

for all positive integers k, all k-tuples $n_1, \dots, n_k$ *of positive integers, and all* $x_{ij} \in X$.

Proof. By definition, $\eta x = \langle x \rangle$, while $\mu = G\varepsilon F : W^2 \dot{\to} W$ is determined by the formula above for ε_S, where we have written each element of $W^2 X$ as a word (of length k) in k words of the respective lengths $n_1, \dots, n_k$. More briefly, μ_X applied to a word of words removes the *outer* pointy brackets.

Note that this description allows direct verification of the unit and associative laws for the monad W, without overt reference to the notion of a semi-group. For example, the associative law for μ amounts to an observation on *three* layers of pointy brackets, that removing first the middle brackets and then the outer brackets gives the same result as removing first the outer brackets and then the (newly) outer brackets.

Proposition 2. *For the above word-monad W in* **Set**, *the W-algebras have the form* $\langle S, \nu_1, \nu_2, \dots \rangle$: *A set S equipped with one n-ary operation* $\nu_n : S^n \to S$ *for each positive integer n, such that* $\nu_1 = 1$ *while for every positive k and every k-tuple of positive integers* $n_1, \dots, n_k$ *one has the identity*

$$\nu_k(\nu_{n_1} \times \cdots \times \nu_{n_k}) = \nu_{n_1 + \cdots + n_k} : S^{n_1 + \cdots + n_k} \to S . \tag{2}$$

A morphism $f : \langle S, \nu_1, \dots \rangle \to \langle S', \nu'_1, \dots \rangle$ *of W-algebras is a function* $f : S \to S'$ *which commutes with each* ν_n, *so that* $f\nu_n = \nu'_n f^n : S^n \to S'$.

Proof. Consider a W-algebra $\langle S, h : WS \to S \rangle$. Since $WS = \coprod S^n$, the structure map h is a list of *n-ary* operations $\nu_n : S^n \to S$, one for each n. The unit law for the algebra requires that $h\eta_X = 1$, hence that ν_1 be the identity. On the other hand, since the product of sets is distributive over the coproducts of sets,

$$W(WX) = \coprod_k \Big(\coprod_n X^n\Big)^k \cong \coprod_k \coprod_n (X^{n_1} \times \cdots \times X^{n_k}) \cong \coprod_k \coprod_n X^{n_1 + \cdots + n_k},$$

where n at the middle and the right runs over all k-tuples $\langle n_1, \dots, n_k \rangle$. With this notation, the associative law for the structure map h takes the stated form (2).

The simplest case of this identity (2), for $3 = 2 + 1 = 1 + 2$ and ν_1 the identity, is

$$\nu_2(\nu_2 \times 1) = \nu_3 = \nu_2(1 \times \nu_2) : S \times S \times S \to S .$$

If we write the binary operation ν_2 as multiplication, this states that the ternary operation ν_3 satisfies, for all elements $x, y, z \in S$,

$$(xy)z = \nu_3(x, y, z) = x(yz) .$$

Similarly, ν_n must be the n-fold product. An easy induction proves

Corollary. *The system $\langle S, \nu_1, \nu_2, \ldots\rangle$ is a W-algebra, as above, if and only if $\nu_1 = 1$, $\nu_2 : S \times S \to S$ is an associative binary operation on S, and for all $n \geqq 2$, $\nu_{n+1} = \nu_n(\nu_2 \times 1) : S^{n+1} \to S$.*

Thus, if we start with semigroups, regarded as sets $\langle S, \nu\rangle$ with one associative binary operation, define the resulting monad W on **Set**, and then construct the category of W-algebras, we get the same semigroups, now regarded as algebraic system $\langle S, \nu_1, \nu_2, \ldots\rangle$, where $\nu_1 = 1$, $\nu_2 = \nu$, and ν_{n+1} is ν_2 iterated. The comparison functor $K : \mathbf{Smgrp} \to \mathbf{Set}^W$ is the evident map $\langle S, \nu\rangle \mapsto \langle S, 1, \nu_2, \ldots, \nu_n, \ldots\rangle$ where ν_n is the iterate of the binary ν. In other words, K is an isomorphism, but it replaces the algebraic system $\langle S, \nu\rangle$ with one associative binary operation by the same set with all the iterated operations derived from this binary operation.

A similar description applies to algebras over other familiar monads (Exercises 1, 2).

Exercises

1. Let W_0 be the monad in **Set** defined by the forgetful functor **Mon**$\to$**Set**. Show that a W_0-algebra is a set M with a string $\nu_0, \nu_1, \ldots$ of n-ary operations ν_n, where $\nu_0 : * \to M$ is the unit of the monoid M and ν_n is the n-fold product.
2. For any ring R with identity, the forgetful functor $G : R$-**Mod**$\to$**Set** from the category of left R-modules has a left adjoint and so defines a monad $\langle T_R, \eta, \mu\rangle$ in **Set**.
   (a) Prove that this monad may be described as follows: For each set X, $T_R X$ is the set of all those functions $f : X \to R$ with only a finite number of non-zero values; for each function $t : X \to Y$ and each $y \in Y$, $[(T_R t)f]_y = \Sigma' f_x$, with sum taken over all $x \in X$ with $tx = y$; for each $x \in X$, $\eta_x x : X \to R$ is defined by $(\eta_x x)x = 1$, $(\eta_x x)x' = 0$; for each $k \in T_R(T_R X)$, $\mu_x k : X \to R$ is defined for $x \in X$ by $(\mu_x k)_x = \Sigma_f k_f f_x$, the sum taken over all $f \in T_R X$.
   (b) From this description, verify directly that $\langle T_R, \eta, \mu\rangle$ is a monad.
   (c) Show that the $\langle T_R, \eta, \mu\rangle$-algebras are the usual R-modules, described not via addition and scalar multiple, but via all operations of linear combination (The structure map h assigns to each f the "linear combination with coefficients f_x for each $x \in X$".)
3. Give a similar complete description of the adjunction defined by the forgetful functor **CRng**$\to$**Set**, noting that TX is the ring of all polynomials with integral coefficients in letters (i.e., indeterminates) $x \in X$.
4. The adjunction $\langle F, G, \varphi\rangle : \mathbf{Ab} \to \mathbf{Rng}$ with G the functor "forget the multiplication in a ring" defines a monad T in **Ab**.
   (a) Give a direct description of this monad, like that in the text for W, with X^n replaced by the n-fold tensor power and coproduct $\amalg$ by the (infinite) direct sum of abelian groups.
   (b) Give the corresponding description of T-algebras and show that the comparison functor from rings to T-algebras is an isomorphism.