Category Theory and Functional Programming

Day 1

1 October 2009

Welcome

1. Why categories?
2. Why functional programming
3. Why the combination
4. This course

Why categories?

There's a tiresome young man in Bay Shore.
When his fiancée cried, 'I adore
The beautiful sea',
He replied, 'I agree,
It's pretty, but what is it for?'

Morris Bishop

Why categories?

What we are probably seeking is a "purer" view of functions: a theory of functions in themselves, not a theory of functions derived from sets. What, then, is a pure theory of functions? Answer: category theory.

Dana Scott

Why categories?

- Describe structure through their effect on other structure
- Internal (set theory) vs. external (category theory)
- “Abstract nonsense”
- General theory of *things* (“objects”) and their *relations* (“morphisms”)
- Applicable in a huge variety of contexts
- Organizing principle

Why functional programming

> SQL, Lisp, and Haskell are the only programming languages that I’ve seen where one spends more time thinking than typing.
>
> Philip Greenspun

Why the combination

- Category theory is a theory of functions
- and of functions on functions
- Functional programming treats functions as first-class objects
- Hence category theory and functional programming share a common mind-set
- (And advanced functional programming uses some advanced categorical concepts)

Organization

- Four days of lectures and exercises
- plus some self-study
- 1, 7, 21, 28 October
- Exercise sessions are too short to do all exercises
- so do some of them on your own (or in groups!)

People

Lecturers

René R. Hansen

Uli Fahrenberg

Organizers

René R. Hansen

Uli Fahrenberg

Hans Hüttel

How to pass this course

- Some of the exercises (marked with $*$) are for student presentation
- choose one, solve it, present solution to audience $\Rightarrow$ PASS
- Presentation lasts approx. 10 minutes
- Check your presentation with René or me before

Categories, Diagrams, and Morphisms

Categories

- Objects
- Arrows, AKA morphisms
- For each arrow f, a domain and a co-domain
- (hence write $f : A \rightarrow B$)
- Composition of compatible arrows: for $f : A \rightarrow B$ and $g : B \rightarrow C$, we have $f; g : A \rightarrow C$
- (usually write $g \circ f$ instead of $f; g$, bummer. . .)

- Composition is associative: $h \circ (g \circ f) = (h \circ g) \circ f$
- And for each object A there's an identity arrow id_A, such that $f \circ \mathrm{id}_A = f$ and $\mathrm{id}_B \circ f = f$ for all arrows $f : A \rightarrow B$

- *That's all folks*

Examples of categories

Objects	**Arrows**
Sets	Functions
Groups	Homomorphisms
Monoids	Homomorphisms
Posets	Monotone functions
CPOs	Continuous functions
Graphs	Homomorphisms

Diagrams

- A diagram:

$$\begin{tikzcd} X \arrow[r, "f'"] \arrow[d, "g'"'] & Z \arrow[d, "g"] \\ W \arrow[r, "f"'] & Y \end{tikzcd}$$

- so $f \circ g'$ and $g \circ f'$ exist
- The diagram commutes iff $f \circ g' = g \circ f'$

Comma categories

Given a category $\mathcal{C}$ and an object $A \in \mathcal{C}$, define the comma category $A \downarrow \mathcal{C}$ by:

- Objects: $\mathcal{C}(A, B)$ for all $B \in \mathcal{C}$
  – all morphisms $f : A \to B$ in $\mathcal{C}$ with domain A
- Arrows:

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ B \arrow[rr] & & C \end{tikzcd}$$

So the objects in $A \downarrow \mathcal{C}$ are arrows from $\mathcal{C}$, and the arrows in $A \downarrow \mathcal{C}$ are commuting triangles from $\mathcal{C}$!

- And composition of arrows in $A \downarrow \mathcal{C}$ is composition of commuting triangles in $\mathcal{C}$.

This is called the comma category, or co-slice of $\mathcal{C}$ under A.

Duality

Where there's a co-slice, there's also a slice (for any object $A \in \mathcal{C}$):

<table>
<tr><th></th><th>co-slice cat. $A \downarrow \mathcal{C}$</th><th>slice cat. $\mathcal{C} \downarrow A$</th></tr>
<tr><td>objects</td><td>$$\begin{tikzcd} A \arrow[d] \\ B \end{tikzcd}$$</td><td>$$\begin{tikzcd} B \arrow[d] \\ A \end{tikzcd}$$</td></tr>
<tr><td>arrows</td><td>$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ B \arrow[rr] & & C \end{tikzcd}$$</td><td>$$\begin{tikzcd} B \arrow[rr] \arrow[dr] & & C \arrow[dl] \\ & A & \end{tikzcd}$$</td></tr>
</table>

Duality

Where there's a co-slice, there's also a slice (for any object $A \in \mathcal{C}$):

	co-slice cat. $A \downarrow \mathcal{C}$	slice cat. $\mathcal{C} \downarrow A$	$(A \downarrow \mathcal{C})^{\mathrm{op}}$
objects	$\begin{tikzcd} A \arrow[d] \\ B \end{tikzcd}$	$\begin{tikzcd} B \arrow[d] \\ A \end{tikzcd}$	$\begin{tikzcd} A \\ B \arrow[u] \end{tikzcd}$
arrows	$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ B \arrow[rr] & & C \end{tikzcd}$	$\begin{tikzcd} B \arrow[rr] \arrow[dr] & & C \arrow[dl] \\ & A & \end{tikzcd}$	$\begin{tikzcd} & A & \\ B \arrow[ur] \arrow[rr] & & C \arrow[ul] \end{tikzcd}$

- So the slice is just the co-slice with all arrows turned around
- Definition: The dual of a category $\mathcal{C}$ is the category $\mathcal{C}^{\mathrm{op}}$, which has the same objects but all arrows turned around.

Monoids and pre-orders as categories

- A monoid is a set with an operation which is associative and has a unit.
- A monoid is a category with one object.

- A pre-order is a set with a relation which is reflexive and transitive.
- (A poset is a pre-order in which the relation is also antisymmetric.)
- A pre-order is a category with at most one morphism between any two objects.

Isomorphisms

Definition: An arrow $f : A \to B$ in a category $\mathcal{C}$ is an iso(morphism) if it has an inverse, *i.e.* an arrow $g : B \to A$ for which $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$.

$$\begin{tikzcd} A \arrow[r, shift left, "f"] & B \arrow[l, shift left, "g"] \end{tikzcd}$$

- One also writes $g = f^{-1}$.
- These are just the usual isomorphisms in your favourite categories.
- Definition: Objects $A, B \in \mathcal{C}$ are isomorphic if there is an isomorphism $f : A \to B$.
- Isomorphic objects are indistinguishable from the point of view of category theory
- (because their *external* properties are the same).

Monomorphisms

In the category of sets and functions,

- an arrow $f : B \to C$ is injective (one-to-one) if $f(x) = f(y)$ implies $x = y$ for all $x, y \in B$.
- Equivalent: $f : B \to C$ is injective if $f \circ g = f \circ h$ implies $g = h$ for all $g, h : A \to B$ and all A.

$$\begin{tikzcd} A \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & B \arrow[r, "f"'] & C \end{tikzcd}$$

- Arrow-only (*external*) property!

Definition: An arrow $f : B \to C$ in a category $\mathcal{C}$ is a mono(morphism) if $f \circ g = f \circ h$ implies $g = h$ for all $g, h : A \to B$ and all $A \in \mathcal{C}$.

Warning: In a lot of categories, “injective” does not make sense, and even if it does, it may not be the same as “mono”.

Epimorphisms

Again in the category of sets and functions,

- an arrow $f : A \to B$ is surjective (onto) if $\forall y \in B \exists x \in A : f(x) = y$.
- Equivalent: $f : A \to B$ is surjective if $g \circ f = h \circ f$ implies $g = h$ for all $g, h : B \to C$ and all C.

$$\begin{tikzcd} A \arrow[r, "f"'] & B \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & C \end{tikzcd}$$

- Arrow-only (*external*) property!

Definition: An arrow $f : A \to B$ in a category $\mathcal{C}$ is an epi(morphism) if $g \circ f = h \circ f$ implies $g = h$ for all $g, h : B \to C$ and all $C \in \mathcal{C}$.

Warning: In a lot of categories, "surjective" does not make sense, and even if it does, it may not be the same as "epi".

Example (Pierce 1.3.6)

In the category of monoids and homomorphisms, the inclusion function $i : \mathbb{N} \hookrightarrow \mathbb{Z}$ is

- injective,
- a mono,
- not surjective,
- but also an epi!

A category of transition systems

- A transition system is a tuple (S, i, L, Tr) with $Tr \subseteq S \times L \times S$.
- A morphism of transition systems $T = (S, i, L, Tr)$, $T' = (S', i', L', Tr')$ is a pair $f = (\sigma, \lambda) : T \to T'$ of functions $\sigma : S \to S'$, $\lambda : L \to L'$ for which $\sigma(i) = i'$ and

$$(s_1, a, s_2) \in Tr \quad \text{implies} \quad (\sigma(s_1), \lambda(a), \sigma(s_2)) \in Tr'$$

- (Almost like a graph homomorphism)
- But wait: We want to be able to map labels in L to "nothing" (so we can abstract away actions)
- So we need partial functions $\lambda : L \to L'_\perp$
- And if $\lambda(a) = \perp$ above, then we want the transition to disappear.

Categories Examples Diagrams and commutativity Examples Monos, epis, isos A category of transition systems

Idle transitions

(A transition system is a tuple (S, i, L, Tr) with $Tr \subseteq S \times L \times S$.)

- Second try: Introduce idle transitions:

$$Tr_\perp = Tr \cup \{(s, \perp, s) \mid s \in S\}$$

- *Now it works:* A morphism of transition systems $T = (S, i, L, Tr)$, $T' = (S', i', L', Tr')$ is a pair $f = (\sigma, \lambda) : T \to T'$ of functions $\sigma : S \to S'$, $\lambda : L \to L'_\perp$ for which $\sigma(i) = i'$ and

$$(s_1, a, s_2) \in Tr \quad \text{implies} \quad (\sigma(s_1), \lambda(a), \sigma(s_2)) \in Tr'_\perp$$

- Together these form a category.

And we shall have to say much more about this category later.

Functors

Functors

Going up one level: We've seen lots of different categories now. What about a category of categories?

- Objects: categories
- Arrows: functors

Definition: A functor from a category $\mathcal{C}$ to a category $\mathcal{D}$ consists of a function F on objects and a function F on arrows

$$
\begin{tikzcd}
\mathcal{C} & & \mathcal{D} \\
A \arrow[rr, mapsto, "F"] \arrow[d, "f"'] & & F(A) \arrow[d, "F(f)"] \\
B \arrow[rr, mapsto, "F"'] & & F(B)
\end{tikzcd}
$$

for which $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ and $F(g \circ f) = F(g) \circ F(f)$.

- A bit like graph homomorphisms!

Example (Pierce 2.1.2)

- The Kleene star (or List) function from sets to sets:

$$S \mapsto S^* = \mathrm{List}(S) = \{\text{words } s_1 s_2 \ldots s_n \mid n \in \mathbb{N}, \text{all } s_i \in S\}$$

- Turn this into a functor from the category of sets and functions to itself:

$$f : S \to T \quad \mapsto \quad f^* : S^* \to T^*$$
$$f^*(s_1 s_2 \ldots s_n) = f(s_1)f(s_2)\ldots f(s_n)$$

- Or, in other words,

$$\mathrm{List}(f) = \lambda s_1 s_2 \ldots s_n \,.\, f(s_1)f(s_2)\ldots f(s_n)$$

Example (Pierce 2.1.3)

- Actually, S^* is a monoid for all sets S:
  - Strings can be concatenated,
  - concatenation is associative
  - and has unit ε (empty string).
- Is Kleene star a functor from sets to monoids?
- Yes, for f^* is a monoid homomorphism for all functions f.

The category of categories

Recall the category of categories:

- Objects: categories
- Arrows: functors
- What about composition of arrows?

Definition: For functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{E}$, the composite functor $G \circ F : \mathcal{C} \to \mathcal{E}$ is defined by

$$\begin{aligned} (G \circ F)(A) &= G(F(A)) && \text{on objects} \\ (G \circ F)(f) &= G(F(f)) && \text{on arrows} \end{aligned}$$

- (Nothing surprising here)
- Associativity $\checkmark$
- Identity functors $\checkmark$

Natural transformations

Going up another level:

1. Categories
2. Functors: arrows between categories
3. What about arrows between functors?

The functor category $\mathcal{D}^{\mathcal{C}}$ (for $\mathcal{C}, \mathcal{D}$ categories) has

- objects: functors
- arrows: natural transformations

Natural transformations

Definition: A natural transformation $\eta : F \stackrel{\cdot}{\to} G$ between functors $F, G : \mathcal{C} \to \mathcal{D}$ is a function from $\mathcal{C}$-objects to $\mathcal{D}$-arrows,
$A \quad \mapsto \quad \eta_A : F(A) \to G(A)$ such that the diagrams

$$\begin{tikzcd} F(A) \arrow[r, "\eta_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\eta_B"'] & G(B) \end{tikzcd}$$

commute for all arrows $f : A \to B$ in $\mathcal{C}$.

Example (Pierce 2.3.3)

rev : the function which reverses lists

- *Polymorphic:* input is list of any type
- So for any set S, we have a function $rev_S : S^* \to S^*$
- (Remember the Kleene star functor List from sets to monoids.)
- So *rev* is a function from sets to monoid homorphisms,

$$rev : S \quad \mapsto \quad rev_S : S^* \to S^*$$

- A natural transformation $rev : \mathrm{List} \stackrel{\cdot}{\to} \mathrm{List}$?
- Yes indeed $\checkmark$