Category Theory and Functional Programming

Day 2

7 October 2009

Categories, functors, natural transformations

Graphs

- Set of points V
- Set of edges E
- For each edge $e \in E$, a source $src(e) \in V$ and a target $tgt(e) \in V$
- (Write $e : x \to y$ if $src(e) = x$ and $tgt(e) = y$)

(These are directed multigraphs; to say $E \subseteq V \times V$ is fine as long as there's at most one edge between any two points.)

- *That's all folks:*
  $V, E, src : E \to V, tgt : E \to V$

Reflexive graphs

- Set of points V
- Set of edges E
- For each edge $e \in E$, a source $src(e) \in V$ and a target $tgt(e) \in V$
- (Write $e : x \to y$ if $src(e) = x$ and $tgt(e) = y$)
- For each point $x \in V$, a degenerate edge $\deg(v) \in E$

- *That's all folks:*
  $V, E, src : E \to V, tgt : E \to V, \deg : V \to E$

Categories

- Set of points V
- Set of edges E
- For each edge $e \in E$, a source $src(e) \in V$ and a target $tgt(e) \in V$
- (Write $e : x \to y$ if $src(e) = x$ and $tgt(e) = y$)
- For each point $x \in V$, a degenerate edge $\deg(v) \in E$
- For each $e_1 : x \to y$ and $e_2 : y \to z$, a composite $e_2 \circ e_1 : x \to z$,
- with associativity: $e_3 \circ (e_2 \circ e_1) = (e_3 \circ e_2) \circ e_1$ whenever these are defined,
- and identities: for all edges $e : x \to y$, $e \circ \deg(x) = e$ and $\deg(y) \circ e = e$.
- *That's all folks:* $V, E, src : E \to V, tgt : E \to V, \deg : V \to E, \circ : E \times_V E \to E$

Categories

- Set of objects $\mathcal{C}_0$
- Set of arrows $\mathcal{C}_1$
- For each arrow $f \in \mathcal{C}_1$, a domain $dom(f) \in \mathcal{C}_0$ and a co-domain $cod(f) \in \mathcal{C}_0$
- (Write $f : A \to B$ if $dom(f) = A$ and $cod(f) = B$)
- For each object $A \in \mathcal{C}_0$, an identity arrow $\mathrm{id}_A \in \mathcal{C}_0$
- For each $f_1 : A \to B$ and $f_2 : B \to C$, a composite $f_2 \circ f_1 : A \to C$,
- with associativity: $f_3 \circ (f_2 \circ f_1) = (f_3 \circ f_2) \circ f_1$ whenever these are defined,
- and identities: for all arrows $f : A \to B$, $f \circ \mathrm{id}_A = f$ and $\mathrm{id}_B \circ f = f$.
- *That's all folks:* $\mathcal{C}_0, \mathcal{C}_1, dom, cod : \mathcal{C}_1 \to \mathcal{C}_0, \mathrm{id} : \mathcal{C}_0 \to \mathcal{C}_1, \circ : \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \to \mathcal{C}_1$

Exercise P-1.1.20.2 (Petur)

A group $(G, *, e, ^{-1})$ is a set G equipped with a binary operation $*$, a distinguished element e, and a unary operation $^{-1}$ such that

(a) $(x * y) * z = x * (y * z)$ for all $x, y, z \in G$,

(b) $e * x = x = x * e$ for all $x \in G$, and

(c) $x * x^{-1} = e = x^{-1} * x$ for all $x \in G$.

Show how an arbitrary group can be considered as a category.

Exercise P-1.1.20.2 (Petur)

A monoid $(G, *, e)$ is a set G equipped with a binary operation $*$, a distinguished element e such that

(a) $(x * y) * z = x * (y * z)$ for all $x, y, z \in G$,

(b) $e * x = x = x * e$ for all $x \in G$, and

Show how an arbitrary monoid can be considered as a category.

Transition systems revisited

- A transition system is a tuple (S, i, L, Tr) with $Tr \subseteq S \times L \times S$. Goal: Externalize this
- A transition system is a graph (S, Tr) with an initial state $i : * \to S$ and a labeling $\lambda : Tr \to L$
- $*$: the one-element set; $i : * \to S$ picks out one element of S
- The category of pointed sets: *comma category* $* \downarrow$ **Set**
- objects: $\begin{array}{c} * \\ \downarrow \\ A \end{array}$ arrows: $\begin{array}{ccc} & * & \\ \swarrow & & \searrow \\ A & \longrightarrow & B \end{array}$

⇒ objects: sets with a basepoint
arrows: functions which preserve the basepoint

Transition systems revisited

- Transition system without labels = pointed graph

⇒ want comma category $* \downarrow$ **Graph**

- Turn one-element set $*$ into graph: add degenerate edge

⇒ the “terminal” reflexive graph:

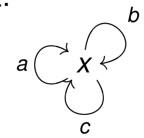
$* =$ 

- The comma category of pointed reflexive graphs $* \downarrow$ **RGraph**:
- objects: reflexive graphs with initial state
arrows: graph homomorphisms which preserve the initial state

= unlabeled transition systems (and functional simulations)

Transition systems revisited

- A transition system is a pointed reflexive graph $* \xrightarrow{i} (S, Tr)$ together with a labeling $\ell : Tr \to L$.
Need more externalization
- Idea: a set is a graph with one point:

$\{a, b, c\} =$ 

⇒ A transition system is a diagram $* \xrightarrow{i} (S, Tr) \xrightarrow{\ell} (*, L)$ in the category of reflexive graphs.

- Forget about internal structure: $* \xrightarrow{i} T \xrightarrow{\ell} G_L$ (externalization!)

Transition systems revisited

- A morphism of transition systems $T = (S, i, L, Tr)$, $T' = (S', i', L', Tr')$ is a pair $f = (\sigma, \lambda) : T \to T'$ of functions $\sigma : S \to S'$, $\lambda : L \to L'_\perp$ for which $\sigma(i) = i'$ and

$$(s_1, a, s_2) \in Tr \quad \text{implies} \quad (\sigma(s_1), \lambda(a), \sigma(s_2)) \in Tr'_\perp$$

- Now looks like $\begin{array}{ccc} & * & \\ {\scriptstyle i}\swarrow & & \searrow{\scriptstyle i'} \\ T & \xrightarrow{\sigma} & T' \\ {\scriptstyle \ell}\downarrow & & \downarrow{\scriptstyle \ell'} \\ G_L & \longrightarrow & G'_L \end{array}$

⇒ a diagram in the category of reflexive graphs

- (“Pointed arrow category”)

Functors

- A functor from a category $\mathcal{C}$ to a category $\mathcal{D}$ consists of a function F on objects and a function F on arrows

$$\begin{array}{ccc} \mathcal{C} & & \mathcal{D} \\ A & \overset{F}{\longmapsto} & F(A) \\ f\downarrow & \longmapsto^{F} & \downarrow F(f) \\ B & \underset{F}{\longmapsto} & F(B) \end{array}$$

- for which $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$
- and $F(g \circ f) = F(g) \circ F(f)$.
- Structure-preserving function between categories.
- F is full $\Leftrightarrow$ surjective on arrows
- F is faithful $\Leftrightarrow$ injective on arrows

Exercise P-2.1.10.3

Let M, N be two monoids (groups; preorders) considered as one-object categories. What are the functors from M to N?

Exercise ML-1.3.4

Prove that there is no functor from groups to Abelian groups which maps each group to its center.

- A group G is Abelian if its operation $*$ is commutative; $x * y = y * x$ for all $x \in G$.
- The center $Z(G)$ of a group G is the set of all elements which commute with all others;

$$Z(G) = \{x \in G \mid \forall y \in G : x * y = y * x\}$$

Natural transformations

A natural transformation $\eta : F \stackrel{\cdot}{\to} G$ between functors $F, G : \mathcal{C} \to \mathcal{D}$ is a function from $\mathcal{C}$-objects to $\mathcal{D}$-arrows, $A \quad \mapsto \quad \eta_A : F(A) \to G(A)$ such that the diagrams

$$\begin{array}{ccc} F(A) & \xrightarrow{\eta_A} & G(A) \\ F(f)\downarrow & & \downarrow G(f) \\ F(B) & \xrightarrow[\eta_B]{} & G(B) \end{array}$$

commute for all arrows $f : A \to B$ in $\mathcal{C}$.

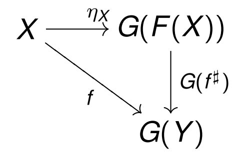
Exercise P-2.3.11.2 (Mikkel)

Let $\mathcal{P}$ be a preorder (regarded as a category) and $\mathcal{C}$ a category. Let $S, T : \mathcal{C} \to \mathcal{P}$ be functors. Show that there is a unique natural transformation $\tau : S \dot{\to} T$ if and only if $S(C) \leq T(C)$ for all $C \in \mathcal{C}$.

Adjoint functors

Adjoint functors

Definition: Functors $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ are adjoint if there is a natural transformation $\eta : I_{\mathcal{C}} \dot{\to} G \circ F$ such that for each arrow $f : X \to G(Y) \in \mathcal{C}$, there is a *unique* arrow $f^\sharp : F(X) \to Y \in \mathcal{D}$ for which the diagram

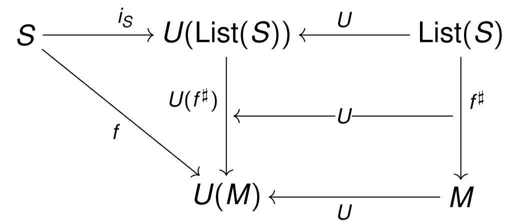


commutes. This is called the *universal property*.

- $F =$ left adjoint, $G =$ right adjoint
- $\eta =$ unit – transformation from identity functor to $G \circ F$

Example (Pierce 2.4.1-2)

The functor List : **Set** $\to$ **Mon** is left adjoint to the forgetful functor U : **Mon** $\to$ **Set**, with unit $i : I_{\mathbf{Set}} \dot{\to} U \circ \mathrm{List}$ given by $i_S(s) = [s]$:



- Example 2.4.2: *length* $= 1^\sharp$
- Left adjoints to forgetful functors are called free functors
- So List(S) is the free monoid on S

Example: free groups

For a set S, define the free group $F(S)$ on S as follows:

- Let W be the set of finite words $w = w_1 w_2 \dots w_n$, with each $w_i \in S$, or $w_i = s^{-1}$ for some $s \in S$. That's just *syntax*.
- A word w can be reduced if it contains a subword ss^{-1} or $s^{-1}s$. Then w is equivalent to w-with-the-subword-removed.
- This defines an equivalence relation on W. The free group $F(S)$ is the set of equivalence classes of W.
- Example: $F(\{a,b\}) = \{\varepsilon, a, b, ab, ab^{-1}, a^{-1}b, a^{-1}b^{-1}, ba, ba^{-1}, \dots\}$

This is functorial, and F is left adjoint to the forgetful functor U : **Group** $\to$ **Set**.

- Universal property:
$$\begin{array}{ccc} S & \xrightarrow{i} & U(F(S)) \\ & \searrow^{f} & \downarrow{\scriptstyle U(f^\sharp)} \\ & & U(G) \end{array}$$

Alternative characterizations of adjoints

Functors $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ are adjoint if

there is a unit $\eta : I_{\mathcal{C}} \dot{\to} G \circ F$ such that for each arrow $f : X \to G(Y) \in \mathcal{C}$, there is a *unique* arrow $f^\sharp : F(X) \to Y \in \mathcal{D}$ for which the diagram
$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & G(F(X)) \\ & \searrow^{f} & \downarrow{\scriptstyle G(f^\sharp)} \\ & & G(Y) \end{array}$$
commutes.

there is a co-unit $\varepsilon : F \circ G \dot{\to} I_{\mathcal{D}}$ such that for each arrow $g : F(X) \to Y \in \mathcal{D}$, there is a *unique* arrow $g^* : X \to G(Y) \in \mathcal{C}$ for which the diagram
$$\begin{array}{ccc} F(G(Y)) & \xrightarrow{\varepsilon_Y} & Y \\ \uparrow{\scriptstyle F(g^*)} & \nearrow_{g} & \\ F(X) & & \end{array}$$
commutes.

- For F : **Set** $\leftrightarrows$ **Mon** : G and F : **Set** $\leftrightarrows$ **Group** : G, $\varepsilon_Y([s_1, s_2, \dots, s_n]) = s_1 * s_2 * \dots * s_n$.

Example RB-6.3.1: floor and ceiling

- $\mathbb{Z}$ and $\mathbb{R}$ are partial orders $\Rightarrow$ categories
- The forgetful functor $U : \mathbb{Z} \to \mathbb{R}$ has
- left adjoint Ceil : $\mathbb{R} \to \mathbb{Z}$ ("smallest integer not smaller than") and
- right adjoint Floor : $\mathbb{R} \to \mathbb{Z}$ ("greatest integer not greater than")
- Adjunction Ceil : $\mathbb{R} \leftrightarrows \mathbb{Z}$: U has unit $\eta_X = (X \leq \text{Ceil}(X))$ and co-unit $\varepsilon_Y = (\textit{Ceil}(Y) = Y)$ (an *iso*!)
- Adjunction $U : \mathbb{Z} \leftrightarrows \mathbb{R}$: Floor has unit $\eta_X = (X = \text{Floor}(X))$ (an *iso*!) and co-unit $\varepsilon_Y = (\text{Floor}(Y) \leq Y)$

Example RB-6.3.4(1): free category on a graph

The free category $F(G)$ on a graph $G = (V, E)$ has

- as objects all points in V
- as arrows all paths in G: all sequences $(e_1, e_2, \dots, e_n)$ of edges in E with $tgt(e_i) = src(e_{i+1})$
- and composition of arrows is concatenation of paths

$\Rightarrow$ left adjoint to the forgetful functor: F : **Graph** $\leftrightarrows$ **Cat** : U

– Like the adjunction **Set** $\leftrightarrows$ **Mon**, but *many-object*!

Special types of adjoints

An adjunction $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ is

- a reflection if G is fully faithful
  $\Leftrightarrow$ all arrows $\varepsilon_Y : F(G(Y)) \to Y$ are isos
- a co-reflection if F is fully faithful
  $\Leftrightarrow$ all arrows $\eta_X : X \to G(F(X))$ are isos
- an adjoint equivalence if it is both a reflection and a co-reflection

Transition systems, synchronization trees, languages

Synchronization trees

- Recall: A transition system is a tuple (S, i, L, Tr) with $Tr \subseteq S \times L \times S$. (Back to the old notation!)
- A synchronization tree is a transition system in which there is precisely one path from i to any state $s \in S$:
  - every state is reachable
  - acyclic
  - no joins
- Recall: A morphism of transition systems is a pair $(\sigma, \lambda) : (S, i, L, Tr) \to (S', i', L', Tr')$ of functions $\sigma : S \to S'$, $\lambda : L \to L'_\perp$ for which $\sigma(i) = i'$ and

$$(s_1, a, s_2) \in Tr \quad \text{implies} \quad (\sigma(s_1), \lambda(a), \sigma(s_2)) \in Tr'_\perp$$

- **T**: category of transition systems
- **S**: fully faithful subcategory of synchronization trees

Synchronization trees

- $i : \mathbf{S} \to \mathbf{T}$ is fully faithful
- Right adjoint: unfolding:
- Given transition system $T = (S, i, L, Tr)$, define synchronization tree $ts(T) = (S', i', L, Tr')$ (same labels) by
  - S' = set of all paths in T
  - $i' = ()$ (empty path)
  - Tr' = one-step continuations of paths:

$$Tr' = \{((s_1, \dots, s_k), a, (s_1, \dots, s_k, s_{k+1}) \mid (s_k, a, s_{k+1}) \in Tr\}$$

- Co-unit morphisms $\varepsilon_T : i(ts(T)) \to T$ given as $\varepsilon_T = (\varphi, \mathrm{id}_L)$, with

$$\varphi(()) = i \qquad \varphi(s_1, \dots, s_n) = s_n$$

- Universal property:

$$\begin{array}{ccc} i(ts(T)) & \xrightarrow{\varepsilon_T} & T \\ \uparrow{\scriptstyle i(f^*)} & \nearrow{\scriptstyle f} & \\ i(Y) & & \end{array}$$

Synchronization trees

$\Rightarrow$ co-reflection $i : \mathbf{S} \leftrightarrows \mathbf{T} : ts$

$\Rightarrow$ all unit morphisms are isos.

- That is, for all synchronization trees Y, the morphism $\eta_Y : Y \to ts(i(Y))$ is an iso.
- Any synchronization tree is isomorphic to its unfolding.

Languages

- A language over a labeling L is a pair (H, L) with $H \subseteq L^*$ prefix-closed: $\forall s \in L^* \forall a \in L : sa \in H \Rightarrow s \in H$
- Morphisms of languages $(H, L) \to (H', L')$: partial functions $\lambda : L \to L'_\perp$ for which $\lambda^*(w) \in H'$ for all $w \in H$

$\Rightarrow$ category of languages $\mathbf{L}$

- The language of a transition system $T = (S, i, L, Tr)$: usual stuff: $tl(T) = (H, L)$ with

$$H = \{a_1 a_2 \dots a_n \mid \exists \text{ path } i \xrightarrow{a_1} s_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} s_n \text{ in } T\}$$

- Extend to functor $tl : \mathbf{T} \to \mathbf{L}$ by $sl(\sigma, \lambda) = \lambda$
- Composition gives functor $sl = tl \circ i : \mathbf{S} \to \mathbf{T} \to \mathbf{L}$

Languages

- Languages as synchronization trees: Given language (H, L), define $ls(H, L) = (H, \varepsilon, L, Tr)$ with $Tr = \{(h, a, ha) \mid ha \in H\}$
- Extend to functor $ls : \mathbf{L} \to \mathbf{S}$ by $ls(\lambda) = (\lambda^*_{1H}, \lambda)$ (restriction of λ^* to H)

- $sl : \mathbf{S} \leftrightarrows \mathbf{L} : ls$ is an adjunction:
- Co-unit morphisms $\varepsilon_{(H,L)} : sl(ls(H, L)) \to (H, L)$ are identities
- Universal property:

$$\begin{array}{ccc} sl(ls(H,L)) & \xrightarrow{\text{id}} & (H,L) \\ \uparrow{\scriptstyle sl(\lambda^\sharp)} & \nearrow{\scriptstyle \lambda} & \\ sl(Y) & & \end{array}$$

$\Rightarrow$ $sl : \mathbf{S} \leftrightarrows \mathbf{L} : ls$ is a reflection

Conclusion

- Co-reflection $i : \mathbf{S} \leftrightarrows \mathbf{T} : ts$
- Reflection $sl : \mathbf{S} \leftrightarrows \mathbf{L} : ls$
- But the composed functors $i \circ ls : \mathbf{L} \to \mathbf{T}$, $\mathbf{L} \leftarrow \mathbf{T} : sl \circ ts$ are not even adjoint!