

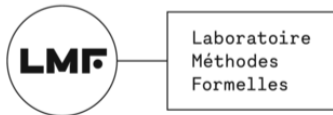
Discrete and Continuous Models for Concurrent Systems

1. The Geometry of Concurrency

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ISIMM, April 2026



Introduction

¿ Discrete and Continuous Models for Concurrent Systems ?



Introduction

Discrete and Continuous Models for Concurrent Systems

Introduction

Discrete and Continuous Models for **Concurrent Systems**

- plane ticket reservation system
- fleet of robots
- your computer!



Introduction

Discrete and Continuous Models for Concurrent Systems



Introduction

Discrete and Continuous Models for Concurrent Systems

- networks of automata
- Petri nets
- event structures
- higher-dimensional automata



Introduction

Discrete and **Continuous Models** for Concurrent Systems

Introduction

Discrete and **Continuous Models** for Concurrent Systems

- Variants of directed topological spaces

① Introduction

② Geometric Semantics

③ Directed Algebraic Topology

④ Exercises

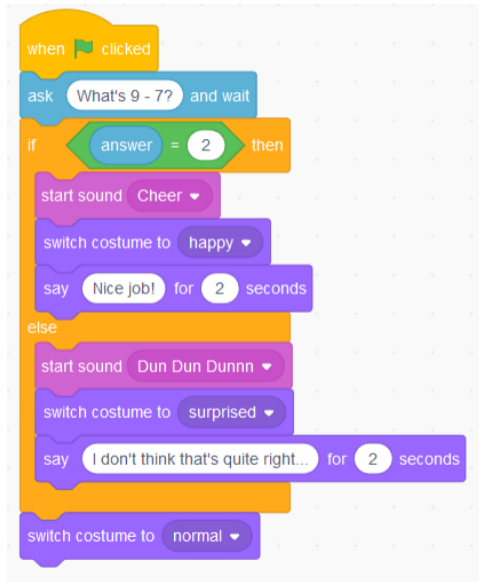
Algebraic View

A program is

- a sequence of instructions
- plus branches
- and loops

Kleene algebra: set S with

- concatenation \otimes
- choice \oplus
- repetition $*$
- idempotent semiring with unary $*$ which computes fixed points



Geometric View

A program is a sequence of instructions

- ignoring branches and loops for now



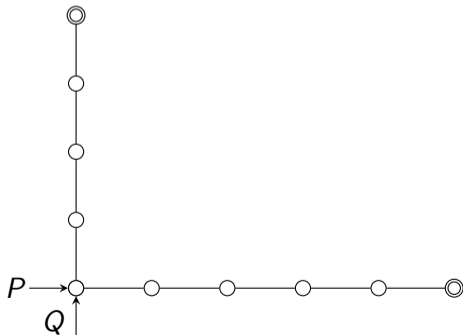
Geometric View

A program is a sequence of instructions

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Now, a second program in parallel:



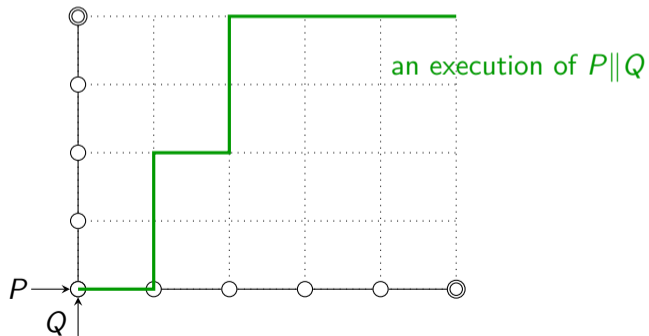
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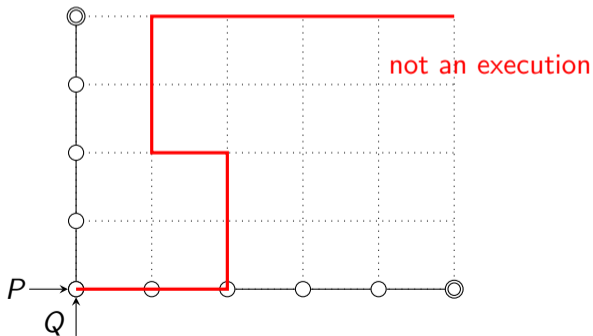
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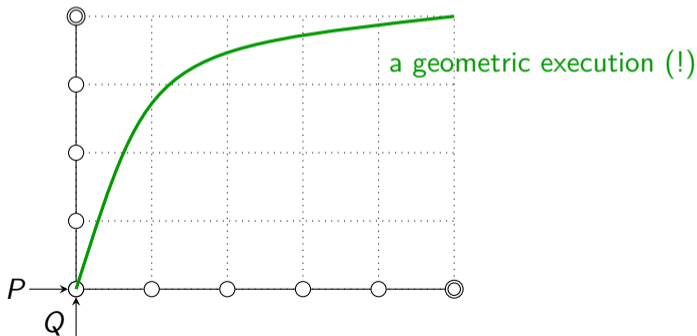
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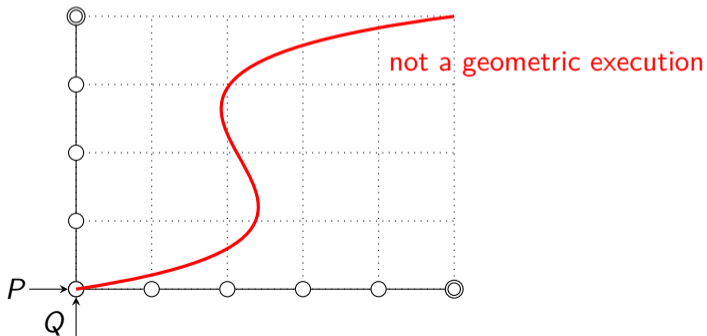
Geometric View

A program is a sequence of instructions

- ignoring branches and loops for now

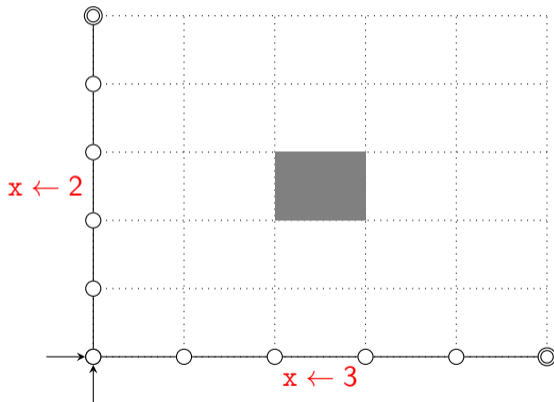


Now, a second program in parallel:



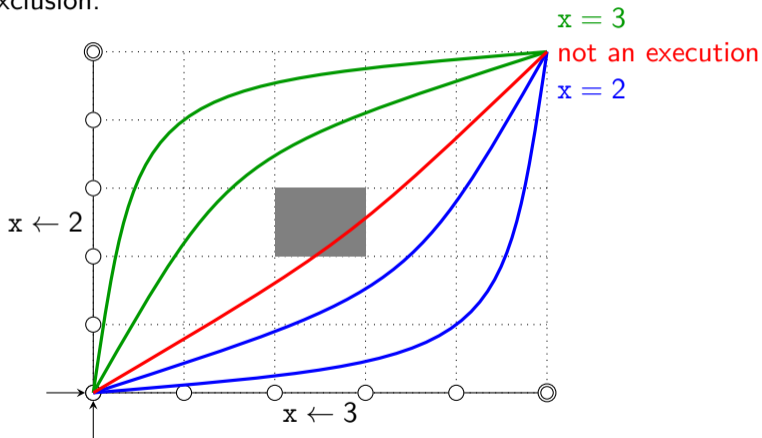
Holes

Adding mutual exclusion:



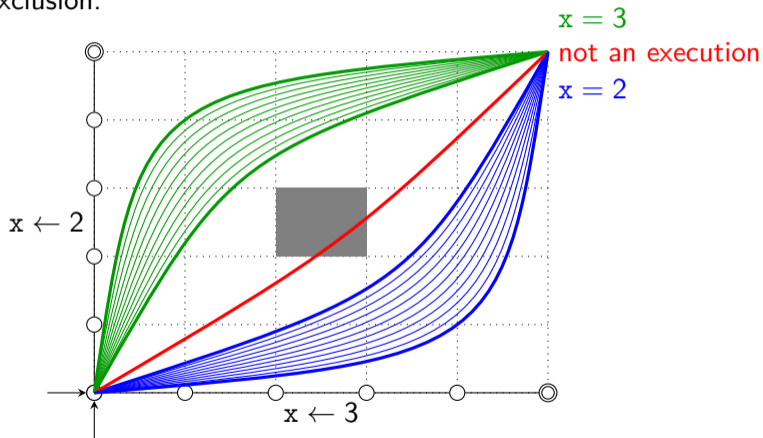
Holes

Adding mutual exclusion:



Holes

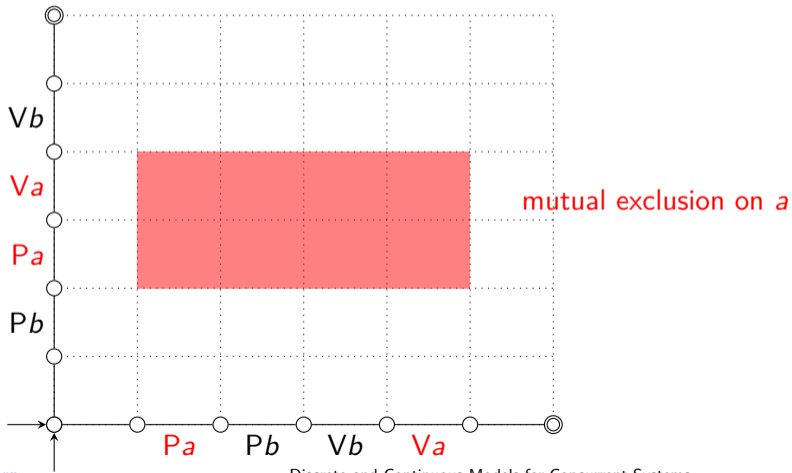
Adding mutual exclusion:



- **homotopic** paths $\hat{=}$ **equivalent** executions

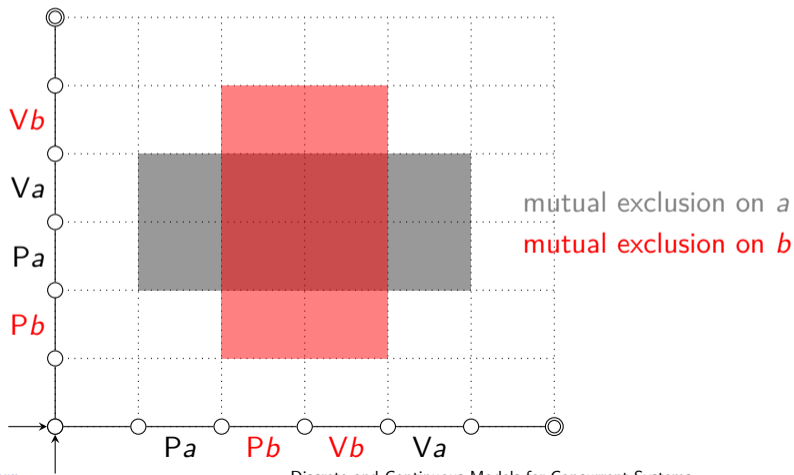
More Holes

Semaphores à la Dijkstra ($P \hat{=}$ acquire; $V \hat{=}$ release):



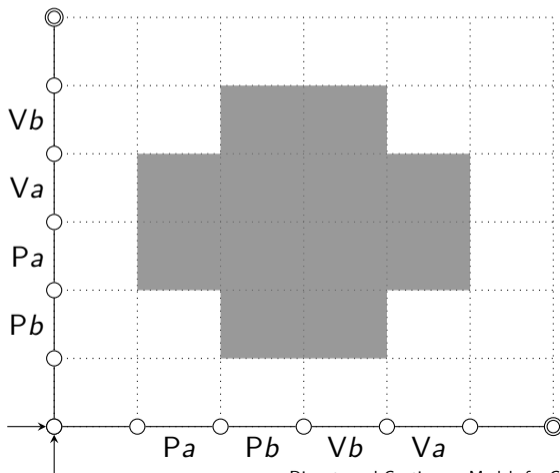
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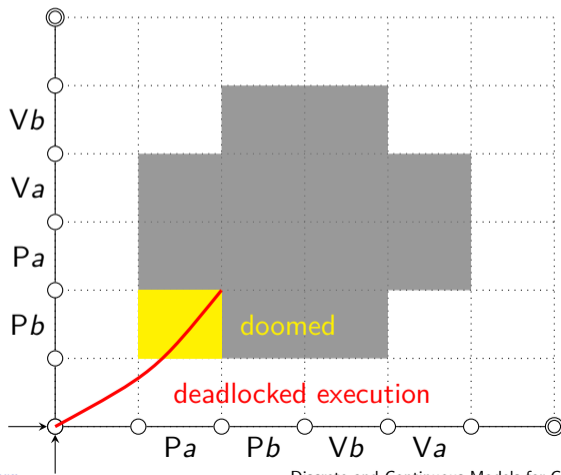
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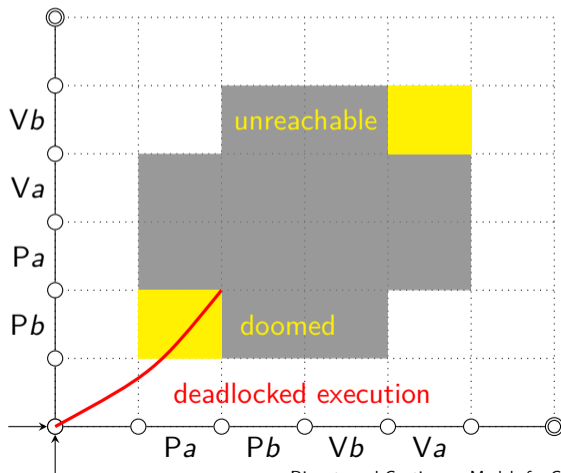
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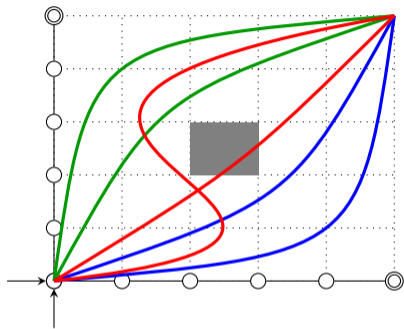


More Holes

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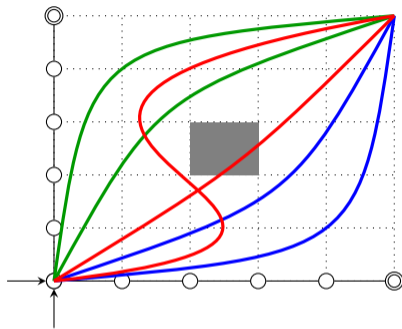


Summing Up



- A program is a topological space
- An execution is a path through said space
- Two executions are equivalent iff their paths are homotopic
- Deadlocks and unreachable states are concave corners

Summing Up



- A program is a **directed** topological space
- An execution is a **directed** path through said space
- Two executions are equivalent iff their **dipaths** are **dihomotopic**
- Deadlocks and unreachable states are concave corners

① Introduction

② Geometric Semantics

③ Directed Algebraic Topology

④ Exercises

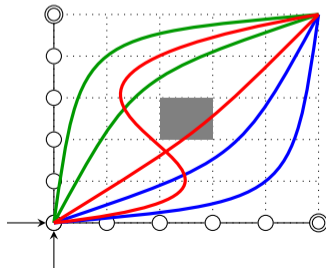
Directed Spaces

Definition (po-space)

A **partially ordered space** is a topological space X together with a partial order \leq on X such that $\leq \subseteq X \times X$ is *closed* in the product topology.

A **morphism** of po-spaces is a \leq -preserving continuous function.

- a **dipath** in a po-space is a continuous & monotone path
 - a morphism $\vec{I} \rightarrow X$
- directed interval $\vec{I} = [0, 1]$ with usual order
- directed square $\vec{I} \times \vec{I}$, cube, etc.
- concatenation \otimes , branching \oplus
- **no loops**



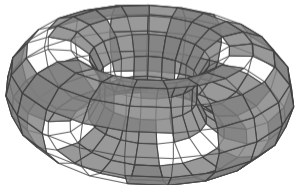
Directed Spaces

Definition (lpo-space)

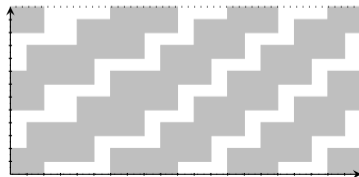
A **locally partially ordered space** is a *Hausdorff* topological space X together with a relation \leq on X in which any $x \in X$ has an open neighborhood $U \ni x$ such that the restriction of \leq to U is a closed partial order.

A **morphism** of po-spaces is a continuous function which is *locally* \leq -preserving.

- \leq may be taken globally reflexive and transitive



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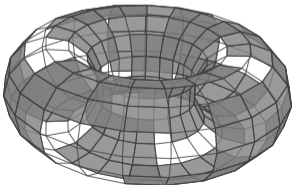
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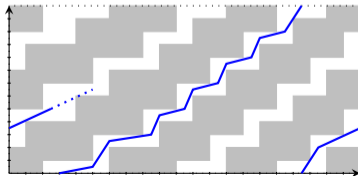
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A **morphism** of po-spaces is a continuous function which is *locally* \leq -preserving.

- \leq may be taken globally reflexive and transitive
- a **dipath** is a morphism $\vec{I} \rightarrow X$



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Discrete and Continuous Models for Concurrent Systems

Directed Spaces

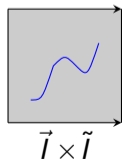
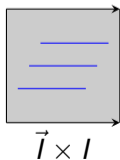
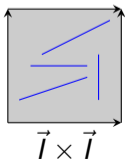
Definition (d-space)

A **directed space** is a topological space X together with a set $\vec{P}X$ of paths $I \rightarrow X$, called **directed paths**, such that

- all constant paths are directed,
- concatenations of directed paths are directed, and
- reparametrizations and restrictions of directed paths are directed.

A **morphism** of d-spaces is a continuous function which preserves directed paths.

- a **dipath** is a morphism $p: \vec{I} \rightarrow X$, equivalently $p \in \vec{P}X$



Directed Spaces

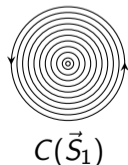
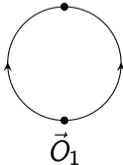
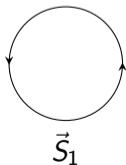
Definition (d-space; unpacked)

A **directed space** is a topological space X together with a set $\vec{P}X \subseteq \text{Top}(I, X)$ such that

- $\forall p \in X : \lambda x.p \in \vec{P}X$ [const]
- $\forall \alpha, \beta \in \vec{P}X : \alpha(1) = \beta(0) \implies \alpha * \beta \in \vec{P}X$ [conc]
- $\forall \alpha \in \vec{P}X, \rho : \vec{I} \rightarrow \vec{I} : \alpha \circ \rho \in \vec{P}X$ [rep-rest]

A **morphism** of d-spaces X, Y is $f \in \text{Top}(X, Y)$ such that $\forall \alpha \in \vec{P}X : f \circ \alpha \in \vec{P}Y$.

Resulting category dTop is complete and cocomplete (and dLCHaus is cartesian closed).



origin is
a **vortex**

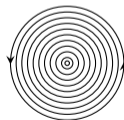
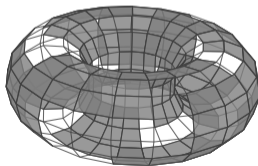
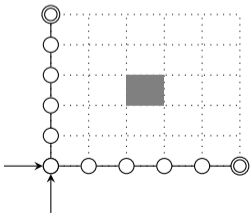
Directed (?) Intervals

Three types of intervals:

- \vec{I} : $I = [0, 1]$ with **usual** order
 - po-space; lpo-space
 - as d-space: \vec{PI} : all **monotone** paths
- I : **trivial** order: $x \leq y$ iff $x = y$
 - po-space; lpo-space
 - as d-space: \vec{PI} : only **constant** paths
- \tilde{I} : **chaotic** d-space
 - d-structure: $\vec{\tilde{P}\tilde{I}}$: **all** paths
 - not an lpo-space (every point is a vortex!), neither a po-space

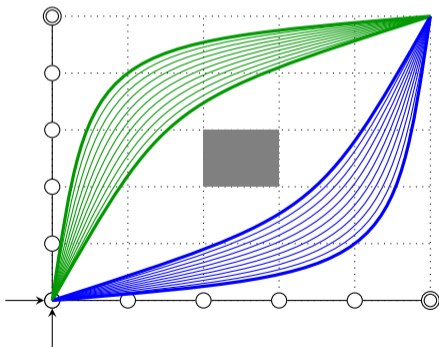
Directed Spaces, Summary

- **po-spaces**: partially ordered topological spaces: no loops, nice but restrictive
- **lpo-spaces**: locally partially ordered top. spaces: loops OK, difficult to work with
- **d-spaces**: top. spaces with distinguished paths: nice category, but include vortices
- (other models exist)
- $\text{po-spaces} \leftrightarrow \text{lpo-spaces} \leftrightarrow \text{d-spaces}$ (not full as categories)



Directed Paths and Homotopies

- the **directed interval** \vec{I} : $([0, 1], \leq)$ (usual order): po-space
- **dipaths** in X : morphisms $\vec{I} \rightarrow X$
 - for d-space $(X, \vec{P}X)$: dipaths $\hat{=} \vec{P}X$
- a **dihomotopy** $H : \vec{I} \times \vec{I} \rightarrow X$:
 - all $H(s, \cdot)$ dipaths
 - $H : I \times I \rightarrow X$ continuous
 - $H(\cdot, 0)$ and $H(\cdot, 1)$ constant
 - (some variants exist)



The Fundamental Category

- central object in algebraic topology: the **fundamental group** of a space X
- for $x \in X$, $\pi_1(X, x) = \{\alpha : I \rightarrow X \mid \alpha(0) = \alpha(1) = x\}$ modulo homotopy
- captures **all** information on homotopy of **paths** / 1-dimensional **holes**

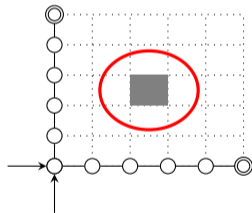
The Fundamental Category

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- captures **all** information on homotopy of **paths** / 1-dimensional **holes**
- in d-spaces, loops carry little information!

Definition (fundamental category)

The **fundamental category** $\vec{\pi}_1(X)$ of a d-space X has as

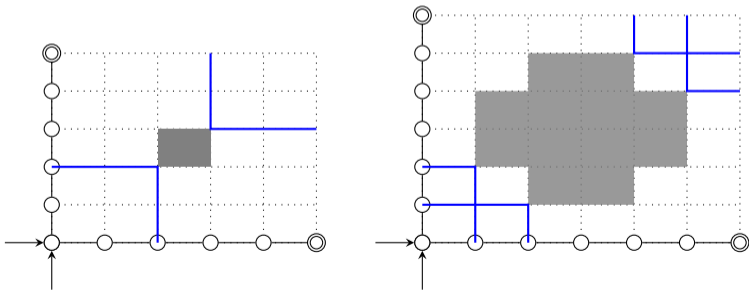
- objects **points** $x \in X$,
- morphisms $\vec{\pi}_1(X)(x, y) = \{\alpha : \vec{I} \rightarrow X \mid \alpha(0) = x, \alpha(1) = y\}$ modulo dihomotopy



The Fundamental Category

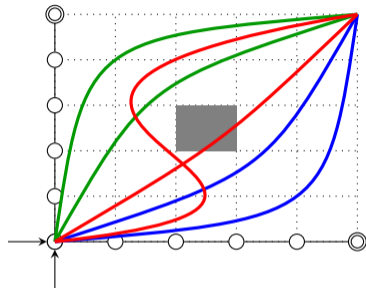
The **fundamental cat** $\vec{\pi}_1(X)$ of a d-space X has as **objects** points $x \in X$ and as **morphisms** $\vec{\pi}_1(X)(x, y) = \{\alpha : \vec{I} \rightarrow X \mid \alpha(0) = x, \alpha(1) = y\}$ modulo dihomotopy.

- related: **fundamental groupoid** of topological spaces (“blow-up” of fundamental group)
- $\vec{\pi}_1(X)$ is **huge** \implies identify **components** “where nothing happens”



Summing Up

- A program is a directed topological space X
 - po-space, lpo-space, d-space, etc.
- An execution is a directed path $\vec{I} \rightarrow X$
- Two executions are equivalent iff they are related by a dihomotopy $\tilde{I} \times \vec{I} \rightarrow X$
- The fundamental category: useful invariant, but too big
- Directed homotopy equivalence; directed coverings; directed homology; directed topological complexity; etc.



Selected Bibliography

- L.Fajstrup, E.Goubault, E.Haucourt, S.Mimram, M.Rausen. *Directed algebraic topology and concurrency*. Springer 2016
- M.Grandis. *Directed algebraic topology*. Cambridge Univ.Press 2009
- R.Brown. *Topology and groupoids*. BookSurge 2006
- L.Fajstrup, E.Goubault, M.Rausen. *Detecting deadlocks in concurrent systems*. CONCUR 1998
- E.Goubault. *Geometry and concurrency: a user's guide*. Math.Struct.Comput.Sci. 2000
- L.Fajstrup, M.Rausen, E.Goubault, E.Haucourt. *Components of the fundamental category*. Appl.Categ.Struct. 2004
- L.Fajstrup, M.Rausen, E.Goubault. *Algebraic topology and concurrency*. Theor.Comput.Sci. 2006
- E.Haucourt. *Non-Hausdorff parallelized manifolds over geometric models of conservative programs*. hal 2025

Exercises

Exercise 1: For each of the following objects: (1) make a drawing to understand it; (2) decide where it is situated in the hierarchy po-space \hookrightarrow lpo-space \hookrightarrow d-space; (3) draw a few dipaths.

- ① The directed square $\vec{I} \times \vec{I}$
- ② The “half-directed” square $\vec{I} \times I$, where I is the po-space with order $x \leq y \iff x = y$
- ③ The “half-twiggly” square $\vec{I} \times \vec{I}$, where \vec{I} is the d-space with $\vec{P}\vec{I} = \text{Top}(I, I)$
- ④ $I \times \vec{I}$
- ⑤ \vec{S}_1 , where $S_1 = \{e^{it} \mid 0 \leq t \leq 2\pi\} \subseteq \mathbb{C}$ is the unit circle and $e^{it_1} \leq e^{it_2} \iff 0 \leq t_2 - t_1 \pmod{2\pi} < \pi$
- ⑥ \vec{O}_1 , where $O_1 = S_1$ and $e^{it_1} \leq e^{it_2} \iff 0 \leq t_1 \leq t_2 \leq \pi$ or $\pi \leq t_2 \leq t_1 \leq 2\pi$
- ⑦ a finite automaton with language given by the regular expression $a(b+c)a$, seen as a “geometric graph” where the vertices are points and the edges, unit intervals
- ⑧ a finite automaton with language given by the regular expression $a(b+c)^*$