Towards Higher Coalgebras: The Enriched Case

PPOS Online Seminar

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Outline

Introduction and Motivation

Motivation

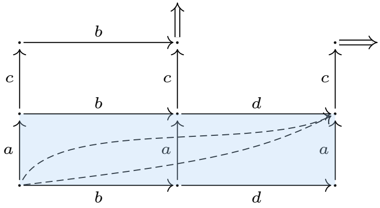
Homotopy theory and algebraic topology for behaviour

- (Weak) homotopy equivalence of systems
- Homotopy-invariant logic
- Homological algebra to find behavioural obstructions

Examples

- Concurrent computing — detecting deadlocks
- Distributed computing — computability results
- Hybrid computing — detecting Zeno behaviour

(i)Pomset Languages of Higher-Dimensional Automata (HDA)



[1] Uli Fahrenberg et al. "Languages of Higher-Dimensional Automata". In: *Math. Struct. Comput. Sci.* 31.5 (2021), pp. 575–613. DOI: 10.1017/S0960129521000293.

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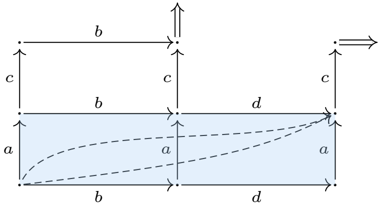

$$\Big\{ \big(a \to b \to c \big), \big(a \to c \to b \big), \big(b \to a \to c \big),$$

$$\big(a \to b \to d \to c \big), \big(b \to a \to d \to c \big), \big(b \to d \to a \to c \big),$$

$$\left(\begin{matrix} a & & \\ & \searrow & \\ & & c \\ & \nearrow & \\ b & & \end{matrix} \right), \left(\begin{matrix} & & a & & \\ & & & \searrow & \\ & & & & c \\ & & & \nearrow & \\ b & \to & d & & \end{matrix} \right), \left(\begin{matrix} a & & & & \\ & \searrow & & & \\ & & d & \to & c \\ & \nearrow & & & \\ b & & & & \end{matrix} \right), \left(\begin{matrix} & & a & & \\ & \nearrow & & \searrow & \\ b & & & & c \\ & \searrow & & \nearrow & \\ & & d & & \end{matrix} \right) \Big\}$$

[1]Uli Fahrenberg et al. "Languages of Higher-Dimensional Automata". In: *Math. Struct. Comput. Sci.* 31.5 (2021), pp. 575–613. DOI: 10.1017/S0960129521000293.

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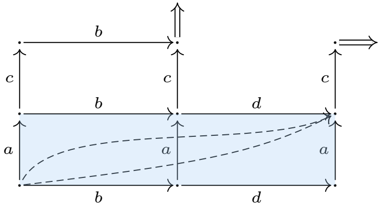

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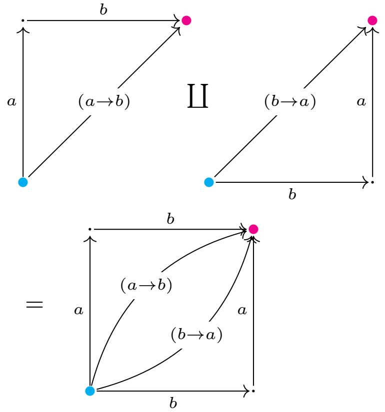
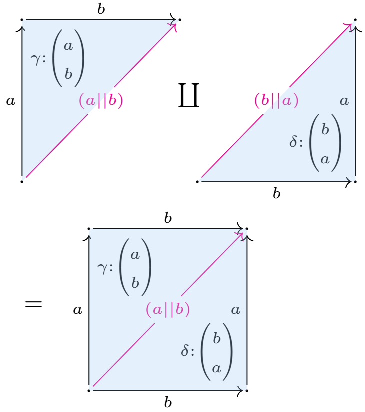
- We want homotopy between pomsets
- Naively imposing them collapses them to free commutative monoid
- Issue: the cubes of HDA are a representation of topological spaces, but HDA conflate behaviour (computation direction and labelling) with the space representation
- Potential solution: simplicial set of ipomsets for labelling, similar to HDA labelling[1]
- Generally: Coalgebras can achieve separation of space and behaviour

[1] Uli Fahrenberg et al. "Languages of Higher-Dimensional Automata". In: *Math. Struct. Comput. Sci.* 31.5 (2021), pp. 575–613. DOI: 10.1017/S0960129521000293.

Homotopy for (i)Pomsets





No relation

$$a \bullet b \not\sim b \bullet a$$

Homotopy via 2-cells

$$a \bullet b \overset{\gamma}{\sim} a || b = b || a \overset{\delta}{\sim} b \bullet a$$

Behaviour via Coalgebras

- Behaviour from repeated observation of a space X via map $c\colon X \to FX$
- Functor $F\colon \mathcal{C} \to \mathcal{C}$ on a category $\mathcal{C}$ determines the type of observations

Example (Hybrid Systems as Coalgebras)

- Hybrid system as space and a coalgebra that specifies the trajectories in the space[2]
- **Top** "convenient" category of topological spaces that is (co)complete, Cartesian closed, and has CW-complexes, like compactly generated Hausdorff spaces or Δ-spaces[3]
- Define a functor $H\colon \mathbf{Top} \to \mathbf{Top}$ by

$$HX = \{(\varrho, d) \in X^{\mathbb{R}_{\geq 0}} \times [0, \infty] \mid \varrho \circ \min(-, d) = \varrho\} \qquad \text{and} \qquad (Hf)(\varrho, d) = (f \circ \varrho, d)$$

- A coalgebra $c\colon X \to HX$ continuously assigns to $x \in X$ a pair (ϱ, d) of trajectory $\varrho\colon \mathbb{R}_{\geq 0} \to X$ that is constant after duration d.
- Can be refined to ensure that the trajectory $c(x)$ has x as starting point etc.

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[2] Renato Neves et al. "Continuity as a Computational Effect". In: *Journal of Logical and Algebraic Methods in Programming. Articles Dedicated to Prof. J. N. Oliveira on the Occasion of His 60th Birthday* 85 (5, Part 2 Aug. 1, 2016), pp. 1057–1085. ISSN: 2352-2208. DOI: 10.1016/j.jlamp.2016.05.005.

[3] J. Peter May. *A Concise Course in Algebraic Topology*. Chicago Lectures in Mathematics. University of Chicago Press, Sept. 1999. 254 pp. ISBN: 978-0-226-51183-2. URL: https://www.math.uchicago.edu/~may/CONCISE/.

Behaviour of Coalgebras

- Behaviour of coalgebra c by recursively expanding observations into a sequence

$$X \xrightarrow{c} FX \xrightarrow{Fc} F(FX) \xrightarrow{F(Fc)} \cdots$$

- Gives in the limit a total view on behaviour of c[4], if that exists
- Traces and logical formulas are partial view on this sequence
- Coalgebra homomorphisms relate the behaviour of systems

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle d} \\ FX & \xrightarrow{Ff} & FY \end{array}$$

- Coalgebra homomorphisms preserve and reflect the behaviour
- Behaviour of the image f in d is equal to that of c
- Often coincide with bisimilarity[5], but we want homotopic behaviour

[4] Michael Barr. "Terminal Coalgebras in Well-Founded Set Theory". In: *Theoretical Computer Science* 114.2 (1993), pp. 299–315. DOI: 10.1016/0304-3975(93)90076-6.

[5] Sam Staton. "Relating Coalgebraic Notions of Bisimulation". In: *Logical Methods in Computer Science* 7.1 (2011), pp. 1–21. DOI: 10.2168/LMCS-7(1:13)2011.

Higher Coalgebra

Homotopy Coherence via Topological Enrichment

Topological Enrichment

$\underline{\mathcal{C}}$ is a $\mathbf{Top}$-enriched category if

- it has objects
- it has a space $\underline{\mathcal{C}}(X,Y) \in \mathbf{Top}$ for all objects X, Y
- there are continuous composition maps $c_{X,Y,Z} \colon \underline{\mathcal{C}}(Y,Z) \times \underline{\mathcal{C}}(X,Y) \to \underline{\mathcal{C}}(X,Z)$
- there is an identity $\mathrm{id}_X \colon * \to \underline{\mathcal{C}}(X,X)$ for all objects X
- an associativity and two unit diagrams commute

Enrichment (plus other things) enables homotopy theory[6]

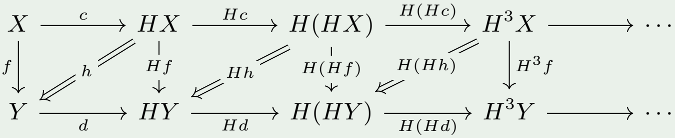
- Define a homotopy $h \colon f \Rightarrow g$ between $f, g \in \underline{\mathcal{C}}(X,Y)$ to be a continuous map $h \colon [0,1] \to \underline{\mathcal{C}}(X,Y)$ with $h(0) = f$ and $h(1) = g$
- Write $f \sim g$ if there is some homotopy $f \Rightarrow g$

[6] Emily Riehl. *Categorical Homotopy Theory*. New Mathematical Monographs 24. Cambridge University Press, 2014. ISBN: 978-1-107-04845-4. URL: https://math.jhu.edu/~eriehl/cathtpy/; Michael Shulman. *Homotopy Limits and Colimits and Enriched Homotopy Theory*. June 30, 2009. DOI: 10.48550/arXiv.math/0610194. arXiv: math/0610194. preprint.

Behaviour up to Homotopy

Example

- Continuous maps form a space $\underline{\mathbf{Top}}(X, Y)$ and composition continuous because $\mathbf{Top}$ is Cartesian closed
- This makes $\underline{\mathbf{Top}}$ a $\mathbf{Top}$-enriched category
- Call $f\colon X \to Y$ a homotopical coalgebra morphism from $c\colon X \to HX$ to $d\colon Y \to HY$ if it comes with a homotopy $h\colon Hf \circ c \Rightarrow d \circ f$
- The functor H is $\mathbf{Top}$-enriched, that is, $H_{X,Y}\colon \mathbf{Top}(X, Y) \to \mathbf{Top}(HX, HY)$ is continuous
- Hence, homotopy $h : f \to g$ can be mapped to a homotopy $Hh\colon Hf \to Hg$ by $Hh = H_{X,Y} \circ h$
- Obtain a sequence of homotopies



Higher Coalgebra

Commutativity up to homotopy

$$\begin{array}{ccc} X & \xrightarrow{c} & FX \\ {\scriptstyle f}\downarrow & \sim & \downarrow{\scriptstyle Ff} \\ Y & \xrightarrow[d]{} & FY \end{array} \qquad \begin{array}{ccccccc} X & \xrightarrow{c} & FX & \xrightarrow{Fc} & F(FX) & \xrightarrow{F(Fc)} & \cdots \\ {\scriptstyle f}\downarrow & \sim & \downarrow{\scriptstyle Ff} & \sim & \downarrow{\scriptstyle F(Ff)} & \sim & \\ Y & \xrightarrow{d} & FY & \xrightarrow{Fd} & F(FY) & \xrightarrow{F(Fd)} & \cdots \end{array}$$

Long-term: homotopy theory of systems as higher coalgebra theory

- inspired by coalgebra[7] and higher algebra[8]
- use $(\infty, 1)$-categories to track homotopies
- the homotopy coherent nerve $N\underline{\mathcal{C}}$ of a $\mathbf{Top}$-enriched category is a model[9]
- homotopy (co)limits, obstruction theory via (co)homology, homotopy-invariant modal logic, ...

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[7] Jan Rutten. "Universal Coalgebra: A Theory of Systems". In: *Theor. Comput. Sci.* 249.1 (2000), pp. 3–80. ISSN: 0304-3975. DOI: `10.1016/S0304-3975(00)00056-6`.

[8] Jacob Lurie. *Higher Algebra*. Sept. 2017. URL: `https://www.math.ias.edu/~lurie/papers/HA.pdf`.

[9] Jacob Lurie. *Higher Topos Theory*. Annals of Mathematics Studies 170. Princeton University Press, 2009. ISBN: 978-0-691-14049-0. arXiv: `math/0608040`.

Homotopy-Invariant Modal Logic

Modal Logic on HDA

Show modalities and homotopy axiom[10]

$$\varphi ::= p \mid \bot \mid \varphi \to \varphi \mid \Diamond^{\uparrow}\varphi \mid \Diamond^{\downarrow}\varphi$$

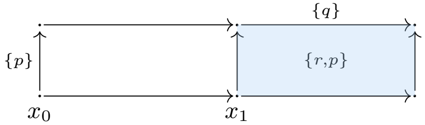
- $\Diamond^{\uparrow}\varphi$ holds if some action can be started and φ holds during execution
- $\Diamond^{\downarrow}\varphi$ holds if some action can be ended and φ holds afterwards

Interpretation over an HDA with cubes X

$$\begin{aligned}
[\![\Diamond^{\uparrow}\varphi]\!]_n &= \{x \in X_n \mid \exists x' \in X_{n+1}.\, x \text{ is a boundary cell of } x' \text{ and } x' \in [\![\varphi]\!]_{n+1}\} \\
[\![\Diamond^{\downarrow}\varphi]\!]_{n+1} &= \{x \in X_{n+1} \mid \exists x' \in X_n.\, x' \text{ is a boundary cell of } x \text{ and } x' \in [\![\varphi]\!]_n\} \\
x \models \varphi &\iff \exists n.\, x \in [\![\varphi]\!]_n
\end{aligned}$$

[10] Cristian Prisacariu. "Modal Logic over Higher Dimensional Automata". In: *Proc. of CONCUR 2010*. 2010, pp. 494–508. DOI: 10.1007/978-3-642-15375-4_34.

Homotopy-Invariance for HDA Logic



Example

$$x_0 \models \Diamond^{\uparrow} p \qquad x_1 \models \Diamond^{\uparrow}\Diamond^{\uparrow}\Diamond^{\downarrow} q$$

$$x_1 \models \Diamond^{\uparrow}\Diamond^{\uparrow} r \wedge p \qquad x_1 \models \Diamond^{\uparrow}\Diamond^{\downarrow}\Diamond^{\uparrow} q$$

Interchange Axioms[11]

$$\Diamond^{\uparrow}\Diamond^{\uparrow}\Diamond^{\downarrow}\varphi \rightarrow \Diamond^{\uparrow}\Diamond^{\downarrow}\Diamond^{\uparrow}\varphi \qquad \text{(A10)}$$

$$\Diamond^{\uparrow}\Diamond^{\downarrow}\Diamond^{\downarrow}\varphi \rightarrow \Diamond^{\downarrow}\Diamond^{\uparrow}\Diamond^{\downarrow}\varphi \qquad \text{(A10')}$$

[11] Cristian Prisacariu. *Higher Dimensional Modal Logic*. 2014. arXiv: 1405.4100. URL: http://arxiv.org/abs/1405.4100. preprint.

Coalgebraic Modal Logic

One view based on dual adjunctions, so-called logical connections[12]

$$F \circlearrowright \mathcal{C} \underset{Q}{\overset{P}{\underset{\perp}{\rightleftarrows}}} \mathcal{D}^{\mathrm{op}} \circlearrowleft L^{\mathrm{op}} \quad \text{and} \quad \varrho\colon PF \to L^{\mathrm{op}}P \quad \text{and} \quad \alpha\colon L\Phi \to \Phi$$

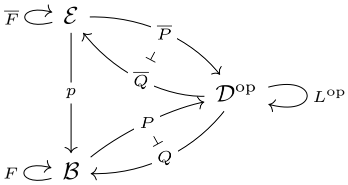
Components

- $\mathcal{C}$ category for “states” in coalgebras
- F behaviour functor to get coalgebras $X \to FX$
- $\mathcal{D}$ typically category of algebras for logical operators
- L specifies modal operators
- initial algebra α for syntax
- distributive law $\varrho\colon LP \to PF$ to give semantics of formulas in a coalgebra
- $P \dashv Q$ is often concrete duality by mapping into dualising object

[12] Dusko Pavlovic, Michael W. Mislove, and James Worrell. “Testing Semantics: Connecting Processes and Process Logics”. In: *Proceedings of Algebraic Methodology and Software Technology, 11th International Conference, AMAST 2006*. Ed. by Michael Johnson and Varmo Vene. Vol. 4019. Lecture Notes in Computer Science. Springer, 2006, pp. 308–322. DOI: 10.1007/11784180_24; Toby Wilkinson. “Enriched Coalgebraic Modal Logic”. PhD thesis. 2013. URL: http://eprints.soton.ac.uk/354112/.

Modal Logic for General Coinductive Predicates

Previous picture is restricted to logic for behavioural equivalence/bisimilarity!



Components[13]

- $p\colon \mathcal{E} \to \mathcal{B}$ fibration
- coalgebras for $\overline{F}$ are proofs of coinductive predicates
- final coalgebras in fibres are typically called coinductive predicates
- soundness (adequacy) and completeness (expressiveness) results provable in this setting

[13] Clemens Kupke and Jurriaan Rot. "Expressive Logics for Coinductive Predicates". In: *Logical Methods in Computer Science* Volume 17, Issue 4 (Dec. 15, 2021). DOI: 10.46298/lmcs-17(4:19)2021.

Homotopy-Invariance for Coinductive Predicates of Hybrid Systems I/III

Focus on the fibration side

Closed predicates

$$\mathbf{cPred} = \begin{cases} \text{objects:} & (X, P) \text{ with } X \in \mathbf{Top}, P \subseteq X \text{ closed} \\ \text{morphisms:} & (X, P) \to (Y, Q) \text{ continuous map } f \colon X \to Y \text{ with } f^{\to}(P) \subseteq Q \end{cases}$$

- Projection $p \colon \mathbf{cPred} \to \mathbf{Top}$ is fibration
- Reindexing by taking preimages: for $f \colon X \to Y$ continuous, define $f^* \colon \mathbf{cPred}_Y \to \mathbf{cPred}_X$ by

$$f^*(Y, Q) = (X, f^{\leftarrow}(Q))$$

- Fibration $p \colon \mathbf{cPred} \to \mathbf{Top}$ is $\mathbf{Top}$-enriched where $\mathbf{cPred}((X, P), (Y, Q))$ has subspace topology

Homotopy-Invariance for Coinductive Predicates of Hybrid Systems I/III

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Enriched Fibrations

Several choices (I know)

1. (large) fibration over the same base $\mathcal{B}$ as $\mathcal{V} \to \mathcal{B}$ and fibred enrichment (Shulman[14])
2. base and total category enriched over a fixed $\mathcal{V}$ (Wong and Beardsley[15])
3. internal fibration in $\mathcal{V}$-Cat (equivalent to previous, Wong[16])
4. enriched in $\mathcal{V}$ and fibration on underlying category (B.)
5. total and base separately enriched, such that hom-objects and composition agree (Vasilakopoulou[17]); can be reformulated as generalisation of Shulman's version

[14] Michael Shulman. "Enriched Indexed Categories". In: *Theory and Applications of Categories* 28.21 (2013), pp. 616–695. URL: http://www.tac.mta.ca/tac/volumes/28/21/28-21abs.html.

[15] Jonathan Beardsley and Liang Ze Wong. "The Enriched Grothendieck Construction". In: *Advances in Mathematics* 344 (Feb. 2019), pp. 234–261. ISSN: 00018708. DOI: 10.1016/j.aim.2018.12.009. arXiv: 1804.03829 [math].

[16] Liang Ze Wong. "The Grothendieck Construction in Enriched, Internal and ∞-Category Theory". Thesis. 2019. URL: https://digital.lib.washington.edu:443/researchworks/handle/1773/44365.

[17] Christina Vasilakopoulou. *On Enriched Fibrations*. July 6, 2018. DOI: 10.48550/arXiv.1801.01386. arXiv: 1801.01386. preprint.

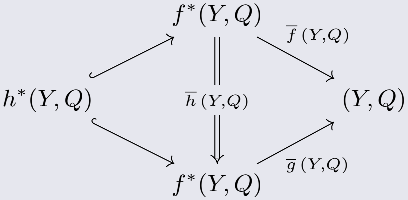
Homotopy-Invariance for Coinductive Predicates of Hybrid Systems II/III

Reindexing along a homotopy

- For homotopy $h\colon f \Rightarrow g\colon X \to Y$, define

$$h^*(Y,Q) = \{x \in X \mid \forall s.\, h(t)(x) \in Q\}$$

- We obtain a homotopy between Cartesian liftings



Homotopy-Invariance for Coinductive Predicates of Hybrid Systems III/III

Example (A simple logic for hybrid systems)

- Family of modalities $\{\Box_t\}_{t\in\mathbb{R}_{\geq 0}}$
- Idea: $\Box_t\varphi$ holds at x if φ holds along the trajectory that leaves x up to (and including) time t
- Define liftings $\{U_t\colon \mathbf{cPred} \to \mathbf{cPred}\}_{t\in\mathbb{R}_{\geq 0}}$ of H with

$$U_t(X,P) = (HX, \{(\gamma,d) \in HX \mid \gamma^{\rightarrow}[0,t] \subseteq P\}) \qquad \text{and} \qquad U_t(f) = Hf$$

- Semantics of modalities in coalgebra $c\colon X \to HX$ as $\Psi_t^c = c^* \circ U_t\colon \mathbf{cPred}_X \to \mathbf{cPred}_X$
- A homotopy $h\colon Hf \circ c \Rightarrow d \circ f$ (f a homotopical coalgebra morphism) induces homotopy $\sigma\colon \Psi_t^c\left(\overline{f}\,P\right) \Rightarrow \overline{f}\left(\Psi_t^d P\right)$ for $P \subseteq X$ closed
- To make the logic homotopy-invariant, any such homotopy must be axiomatised

Wrapping Up

Outlook

1. Model HDA and directed spaces as coalgebras, which requires Vietories-like functor on $\mathbf{Top}$ or simplicial sets
2. Integration with homotopical/model categories (so-called enriched homotopy theory)
3. Higher coalgebra in quasicategories
4. General theory of coalgebraic modal logic, type theory and obstruction theory via (co)homology
5. Reconciliation with directed approaches[18]
6. Various questions on enriched fibrations (Grothendieck constructions, relations, adjunction lifting)
7. Integration with type theory (synthetic $(\infty, 1)$-categories, possibly directed)

[18] Jérémy Dubut, Eric Goubault, and Jean Goubault-Larrecq. "The Directed Homotopy Hypothesis". In: *25th EACSL Annual Conference on Computer Science Logic (CSL 2016)*. Ed. by Jean-Marc Talbot and Laurent Regnier. Vol. 62. LIPIcs. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2016, 9:1–9:16. ISBN: 978-3-95977-022-4. DOI: 10.4230/LIPIcs.CSL.2016.9.

Thank you for your attention!