The rabbit calculus:
convolution products on double categories
and categorification of rule algebra

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Probability generating functions

Key idea: given a chemical reaction described as a transition

$$\kappa_{i,o} \quad : \quad iX \longrightarrow oX$$

where

- ▷ $i \in \mathbb{N}$ denotes the number of particles X **entering the transition**
- ▷ $o \in \mathbb{N}$ denotes the number of particles X **exiting the transition**
- ▷ $\kappa_{i,o} \in \mathbb{R}_{>0}$ denotes the **base rate** of the transition

the dynamics may be encoded using a **probability generating function**

$$P(t;x) \quad = \quad \sum_{n \geq 0} p_n(t)\, x^n$$

where the scalar

$$p_n(t) \geq 0$$

is the probability at time t that the system is in a state with n particles.

Probability generating functions

Delbruck's formulation of the evolution of the system

$$\frac{\partial}{\partial t} P(t;x) \quad = \quad \mathcal{H} P(t;x)$$

with initial distribution

$$P(0;x) = P_0(x)$$

and evolution operator:

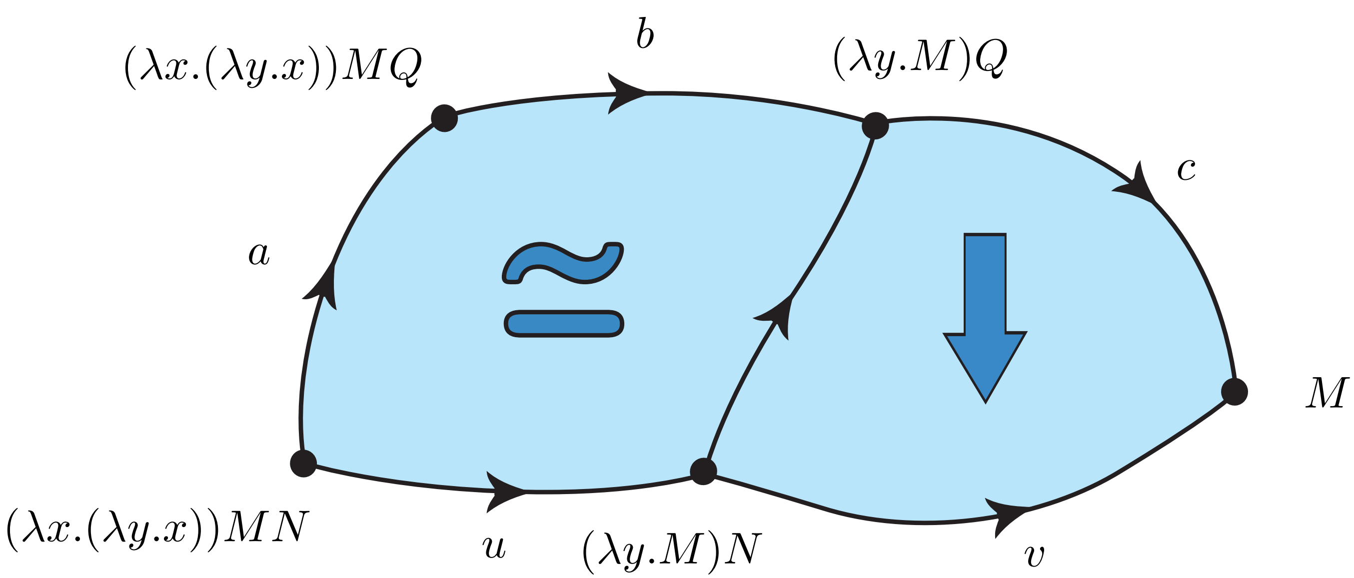
$$\mathcal{H} \quad = \quad \sum_{i,o} \kappa_{i,o} (\hat{x}^o - \hat{x}^i) \Big(\frac{\partial}{\partial x}\Big)^i$$

which can be related to stochastic rewriting theory (cf. Nicolas Behr).

The quest for causality in rewriting theory

An important insight coming from Huet and Lévy:

In order to track the **causality structure** relating different β-redexes, one needs to consider rewriting paths modulo **permutations** of the form



Illustration

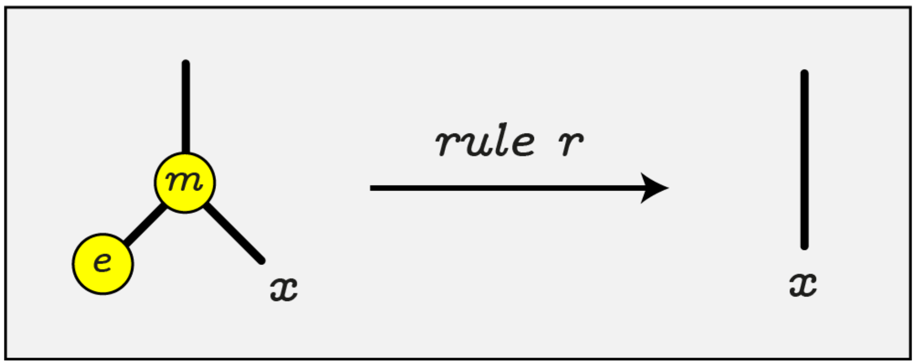
Consider the **term rewriting system** on the signature with two letters:

a binary letter $m : 2$ a constant letter $e : 0$

together with the **unique rewrite rule**

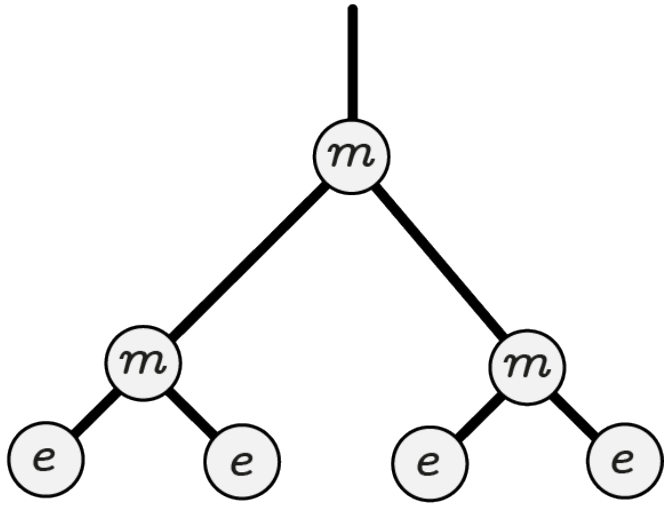
$$r \quad : \quad m(e, x) \longrightarrow x$$

which we depict as follows:



Illustration

The term



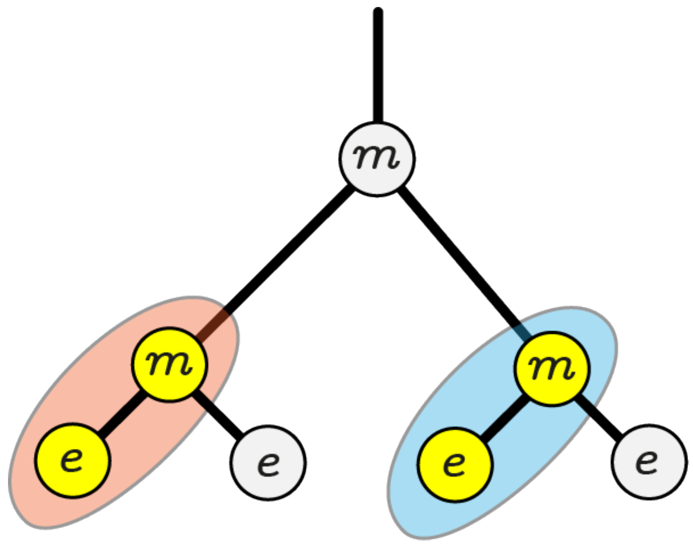
admits **exactly two redexes**

$$m(e, m(e,e)) \xleftarrow{\textit{red}} m(m(e,e), m(e,e)) \xrightarrow{\textit{blue}} m(m(e,e), e)$$

which are **independent** and can be computed in parallel.

Illustration

The term



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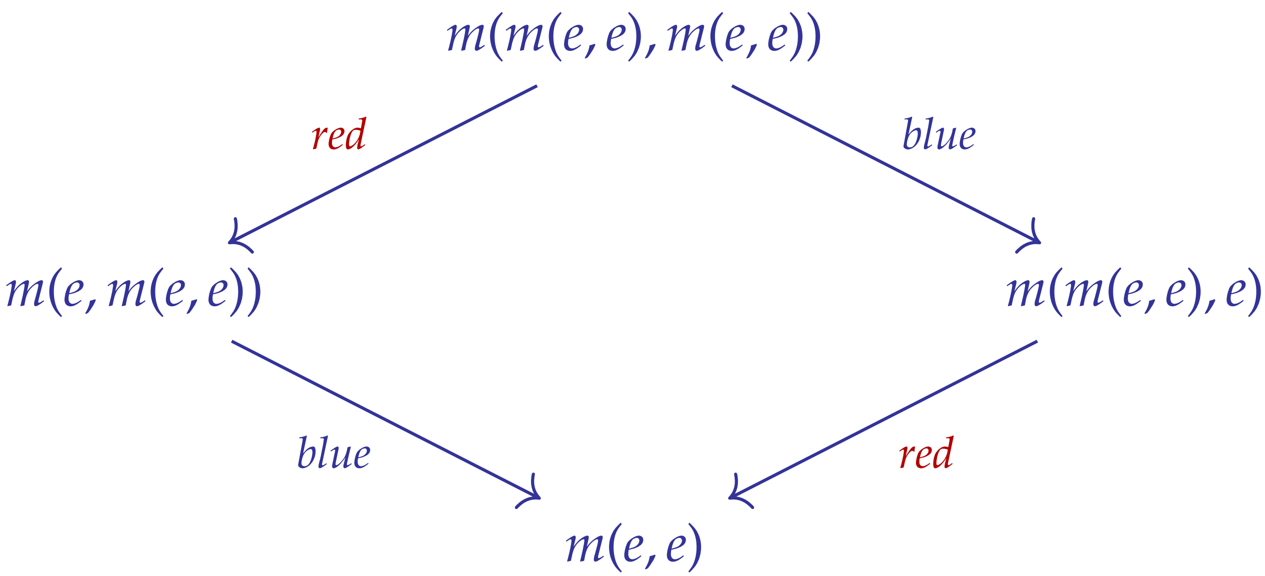
$$m(e, m(e,e)) \xleftarrow{\textit{red}} m(m(e,e), m(e,e)) \xrightarrow{\textit{blue}} m(m(e,e), e)$$

which are **independent** and can be computed in parallel.

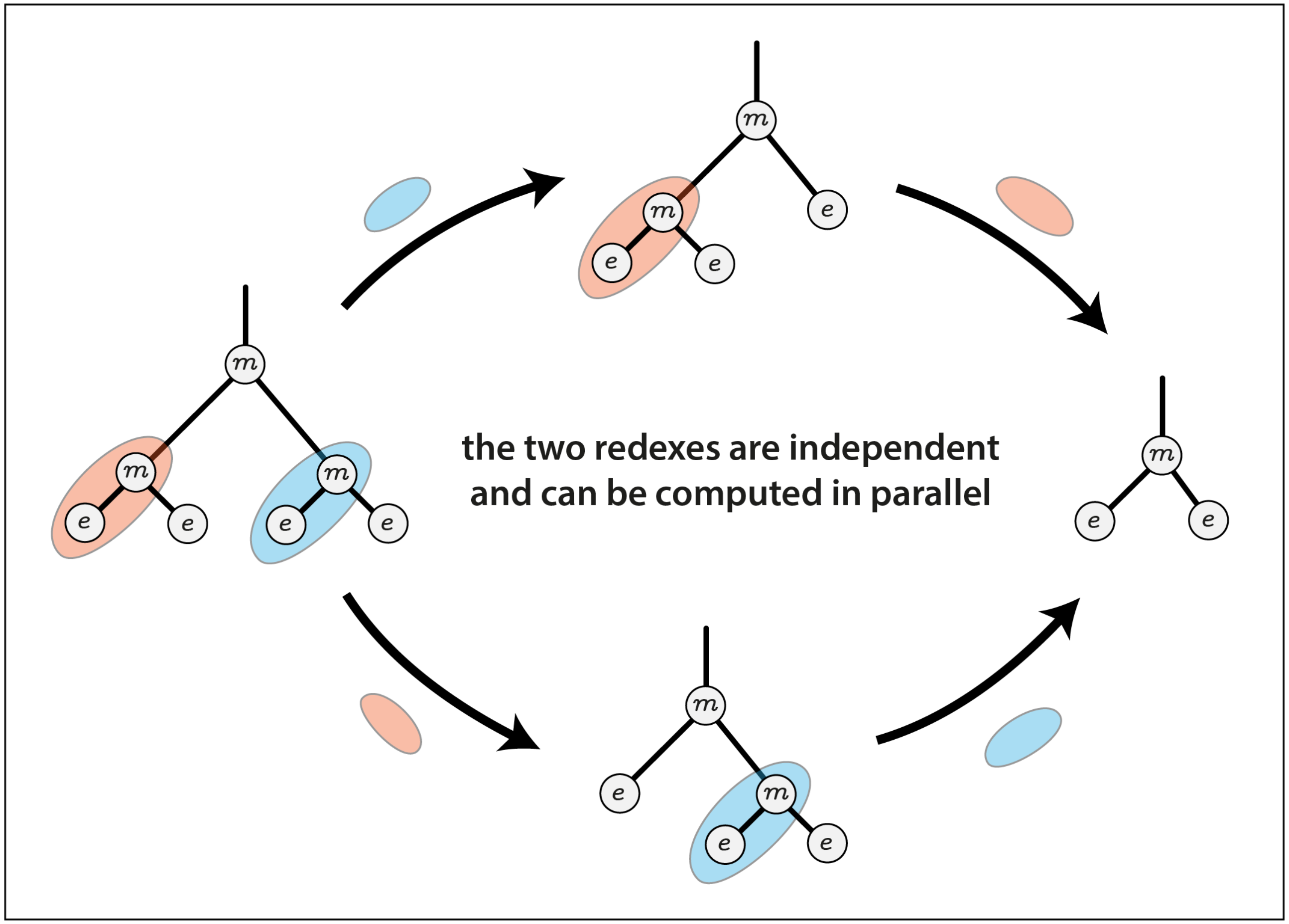
Illustration

One obtains a **local confluence** or **permutation diagram**



expressing that the blue redex and the red redex are **independent**.

Independence and permutation

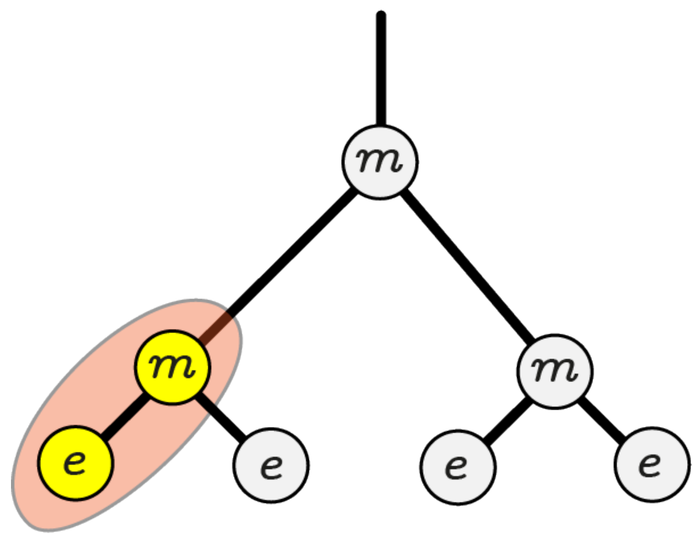


Illustration

At the same time, rewriting the redex

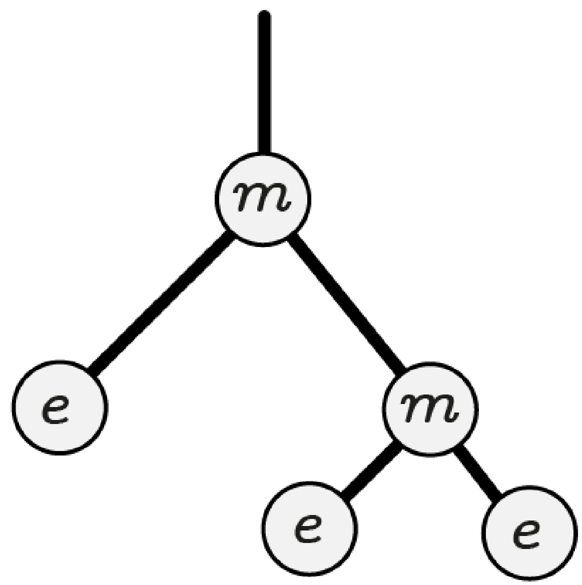
$$m(m(e,e),m(e,e)) \xrightarrow{\textit{red}} m(e,m(e,e))$$

in the same term



Illustration

rewrites to the term

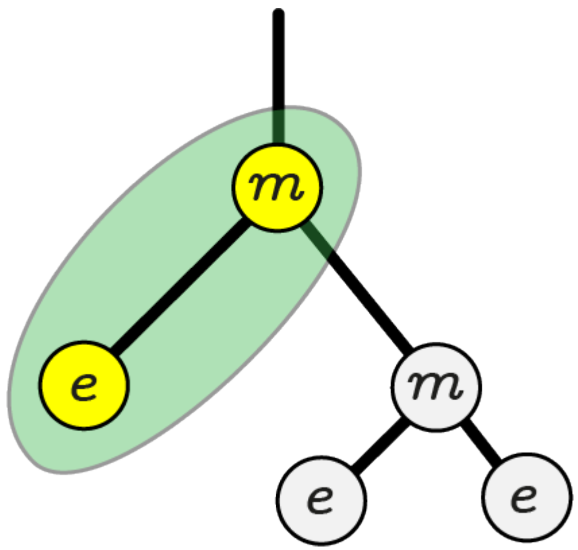


which admits the redex

$$m(e, m(e, e)) \xrightarrow{green} m(e, e)$$

Illustration

rewrites to the term

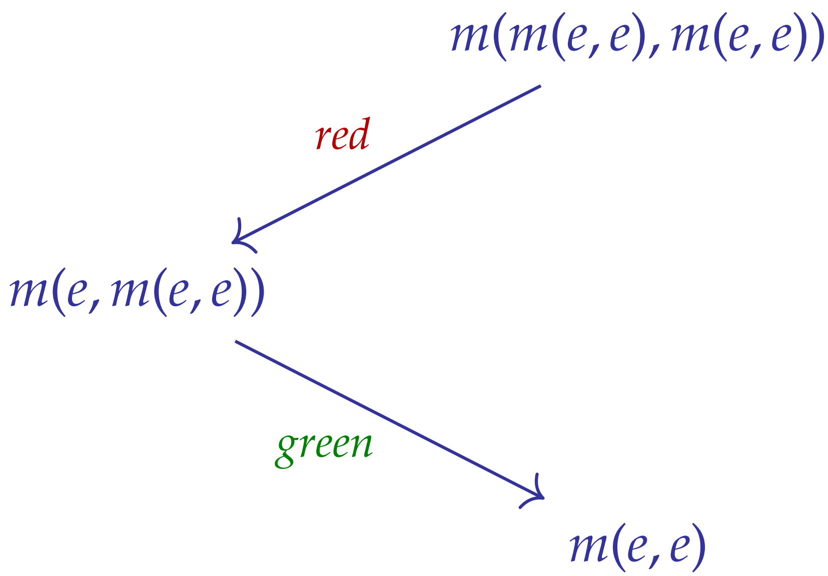


which admits the redex

$$m(e, m(e,e)) \xrightarrow{green} m(e,e)$$

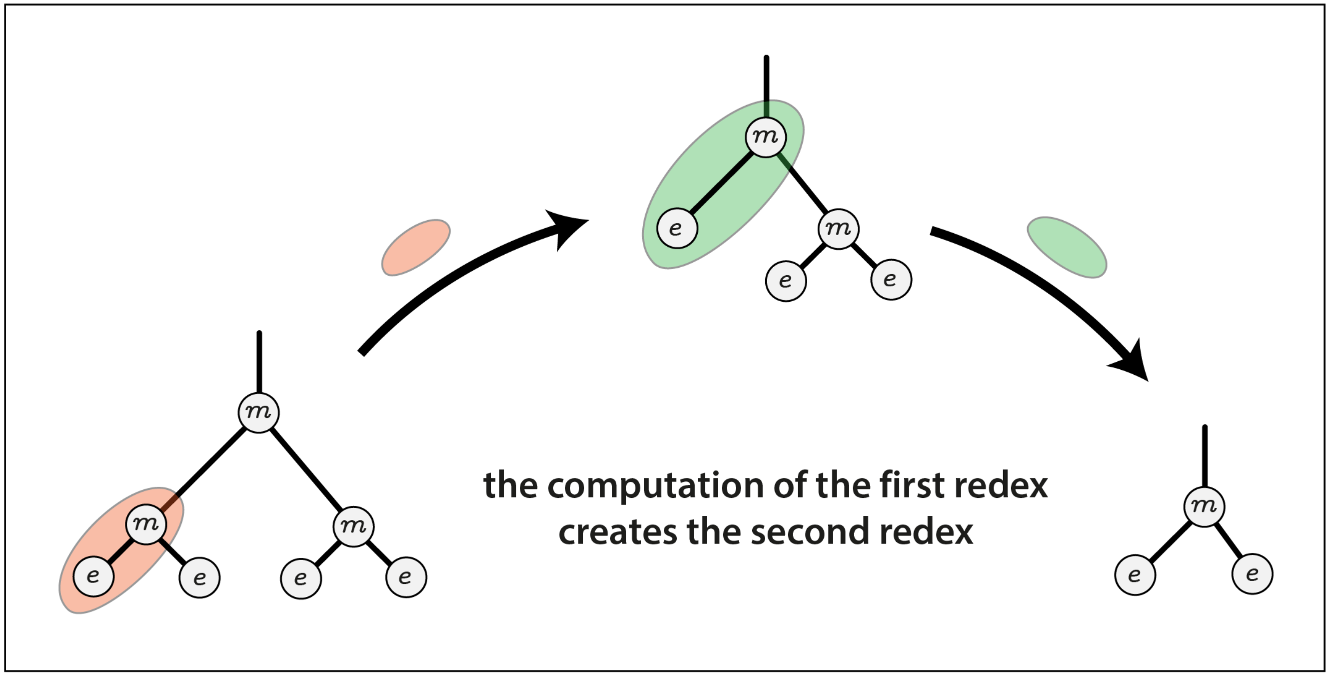
Illustration

Here, the red redex **creates** the green redex

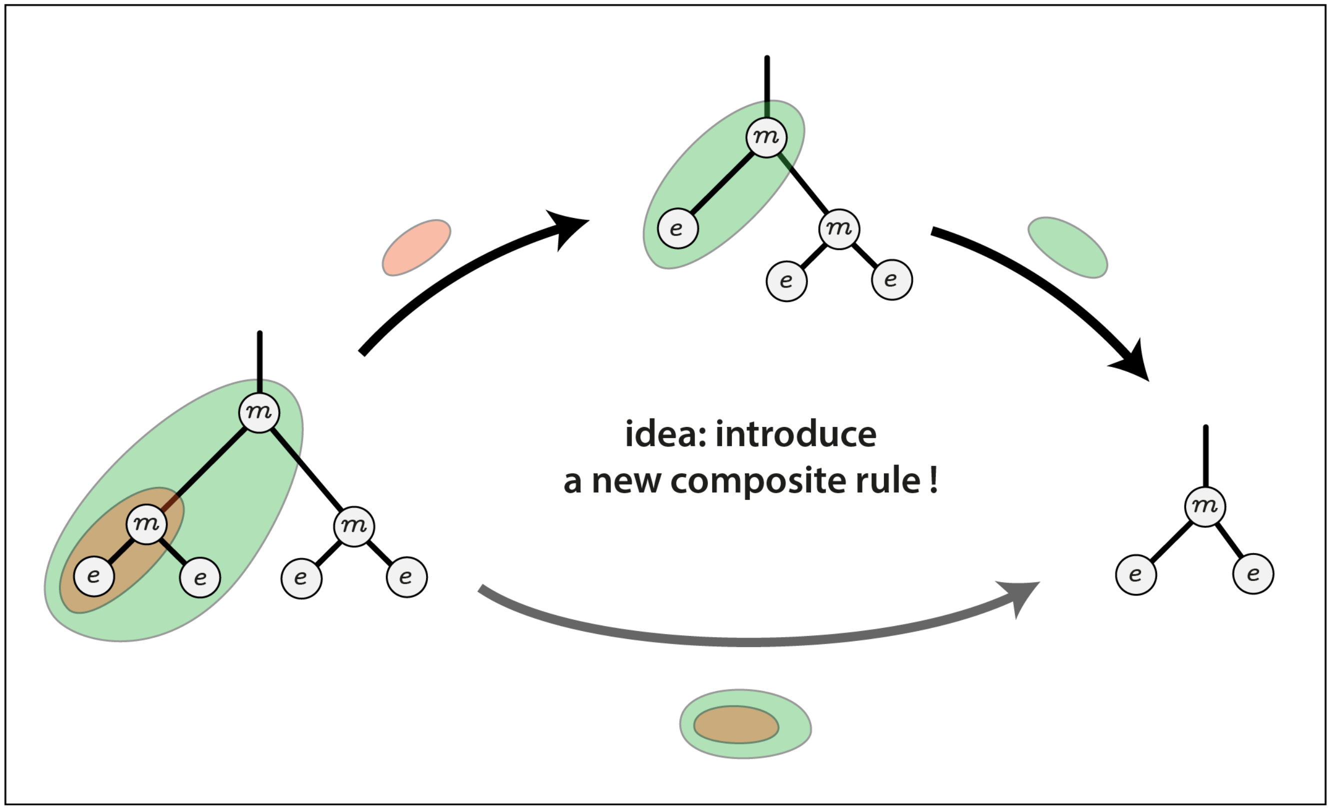


because the green redex **cannot be permuted before** the red redex.

Creation

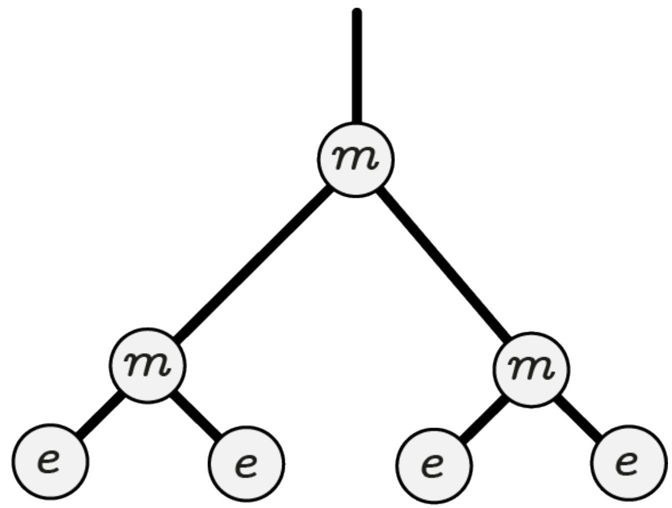

m
m
e
e
m
e
e
m
e
m
e
e
the computation of the first redex
creates the second redex
m
e
e

Composing redexes in term rewriting



Composing redexes in term rewriting

The term

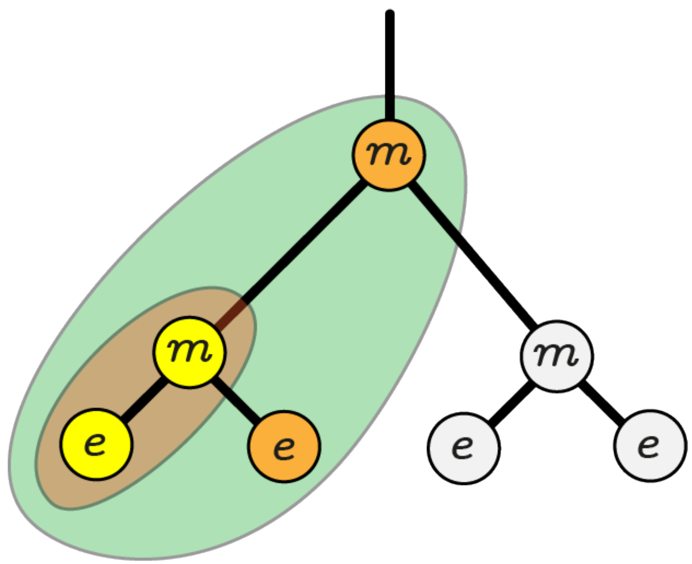


admits one **composite redex** obtained by composing red and green.

$$m(m(e,e),m(e,e)) \xrightarrow{\textit{red}} m(e,m(e,e)) \xrightarrow{\textit{green}} m(e,e)$$

Composing redexes in term rewriting

The term



admits one **composite redex** obtained by composing red and green.

$$m(m(e,e),m(e,e)) \xrightarrow{red} m(e,m(e,e)) \xrightarrow{green} m(e,e)$$

The quest for causality in rewriting theory

In the λ-calculus and term rewriting systems

A well-established tradition based on **optimality** and **residual theory**

- ▷ the notion of **Lévy families** in the λ-calculus (Lévy 1980)
- ▷ their generalisation to any CRS (Asperti, Laneve 1995)
- ▷ a residual theory based on **the notion of trek** (PAM, 2002)

More recently, in categorical graph rewriting

- ▷ the notion of **tracelet** emerging in the work by **Nicolas Behr**.

Our ambition in this work is to initiate a convergence between these lines by revisiting/categorifying the work on tracelets using **double categories.**

Double categories

A convenient framework for term and graph rewriting

Double categories

Definition. A (weak) **double category** $\mathbb{D}$ consists of

- ▷ a category $\mathbb{D}_0$ of objects,
- ▷ a category $\mathbb{D}_1$ of horizontal maps,
- ▷ a pair of **source** and **target** functors

$$\mathbb{D}_0 \xleftarrow{\;T\;} \mathbb{D}_1 \xrightarrow{\;S\;} \mathbb{D}_0$$

- ▷ a **horizontal composition** functor

$$\diamond_h \quad : \quad \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \longrightarrow \mathbb{D}_1$$

- ▷ a **horizontal identity** functor

$$idh \quad : \quad \mathbb{D}_0 \longrightarrow \mathbb{D}_1$$

satisfying a number of **associativity** and **neutrality** properties.

The category $\mathbb{D}_0$ of vertical maps

A morphism in the category $\mathbb{D}_0$ is represented as a **vertical map**

$$\begin{array}{c} A \\ {\scriptstyle a}\big\downarrow \\ A' \end{array}$$

which may be **composed vertically** with other vertical maps.

The category $\mathbb{D}_1$ of horizontal maps

An object in the category $\mathbb{D}_1$ is represented as a **horizontal map**

$$B \xleftarrow{\;r\;} A$$

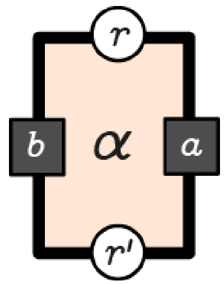
A morphism in the category $\mathbb{D}_1$ is represented as a **double cell**

$$\begin{array}{ccc} B & \xleftarrow{\;r\;} & A \\ {\scriptstyle b}\downarrow & \Downarrow{\scriptstyle \alpha} & \downarrow{\scriptstyle a} \\ B' & \xleftarrow[\;r'\;]{} & A' \end{array}$$

which may be **composed vertically** with other double cells.

The category $\mathbb{D}_1$ of horizontal maps

We often find convenient to use the pictorial notation



for the double cell usually noted

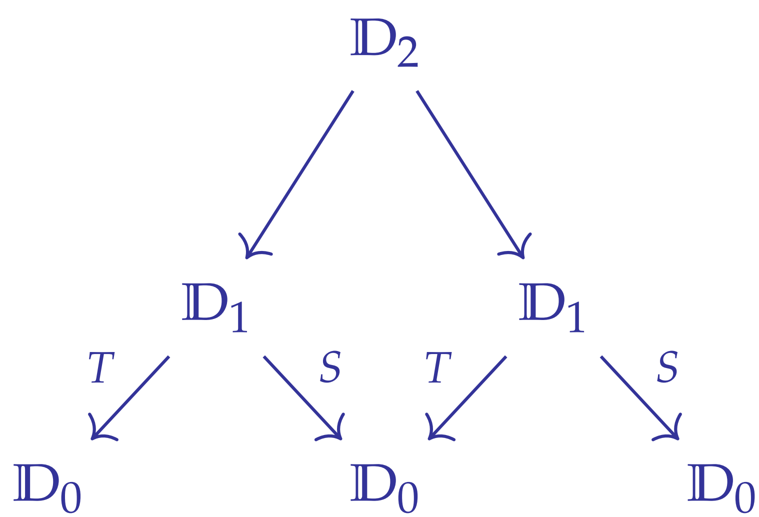
$$\begin{array}{ccc} B & \xleftarrow{\quad r \quad} & A \\ {\scriptstyle b}\downarrow & \Downarrow{\scriptstyle \alpha} & \downarrow{\scriptstyle a} \\ B' & \xleftarrow[\quad r' \quad]{} & A' \end{array}$$

The category $\mathbb{D}_2$ of paths of length 2

Every double category $\mathbb{D}$ comes with

a category $\mathbb{D}_2 = \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1$ of horizontal paths of length 2

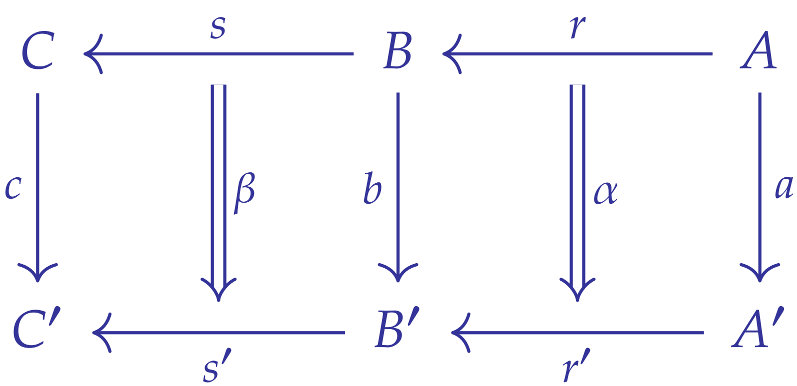
defined as the limit of the diagram of functors



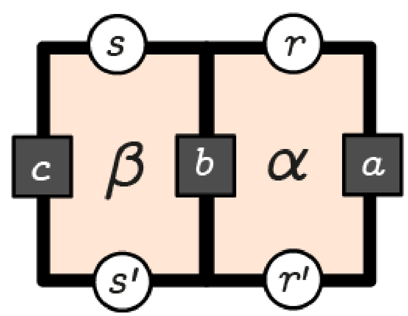
in the category **Cat** of categories and functors.

The category $\mathbb{D}_2$ of paths of length 2

A typical morphism of $\mathbb{D}_2$ has the shape



which we also like to depict as

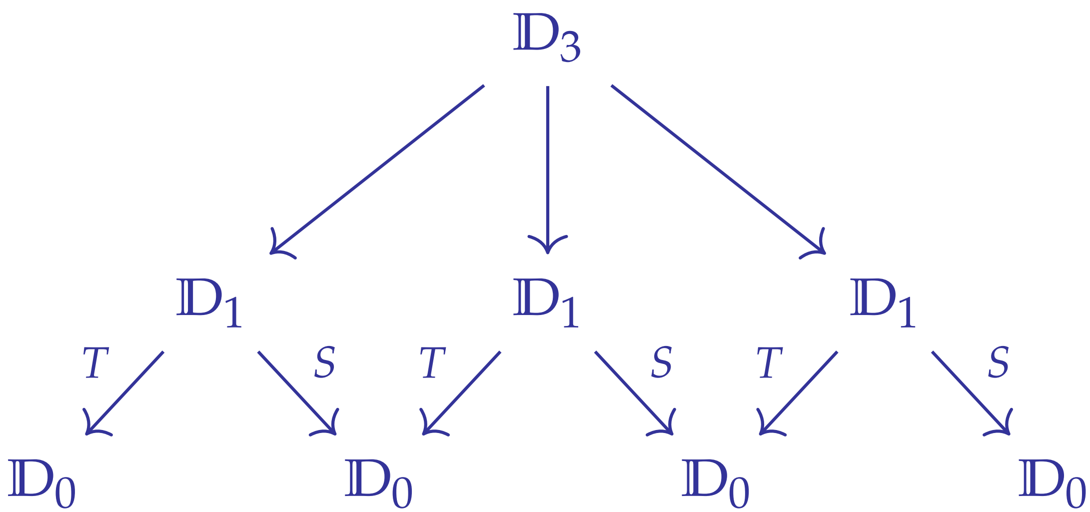


The category $\mathbb{D}_3$ of paths of length 3

Every double category $\mathbb{D}$ comes with

a category $\mathbb{D}_3$ of horizontal paths of length 3

defined as the limit of the diagram of functors



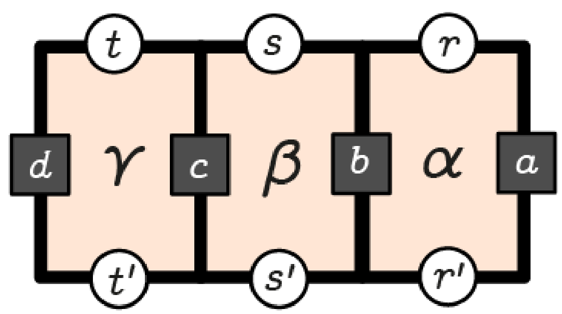
in the category $\mathbf{Cat}$ of categories and functors.

The category $\mathbb{D}_3$ of paths of length 3

A typical morphism of $\mathbb{D}_3$ has the shape

$$\begin{array}{ccccccc}
D & \xleftarrow{\;t\;} & C & \xleftarrow{\;s\;} & B & \xleftarrow{\;r\;} & A \\
{\scriptstyle d}\big\downarrow & \Downarrow{\scriptstyle \gamma} & {\scriptstyle c}\big\downarrow & \Downarrow{\scriptstyle \beta} & \big\downarrow{\scriptstyle b} & \Downarrow{\scriptstyle \alpha} & \big\downarrow{\scriptstyle a} \\
D' & \xleftarrow[\;t'\;]{} & C' & \xleftarrow[\;s'\;]{} & B' & \xleftarrow[\;r'\;]{} & A'
\end{array}$$

which we also like to depict as

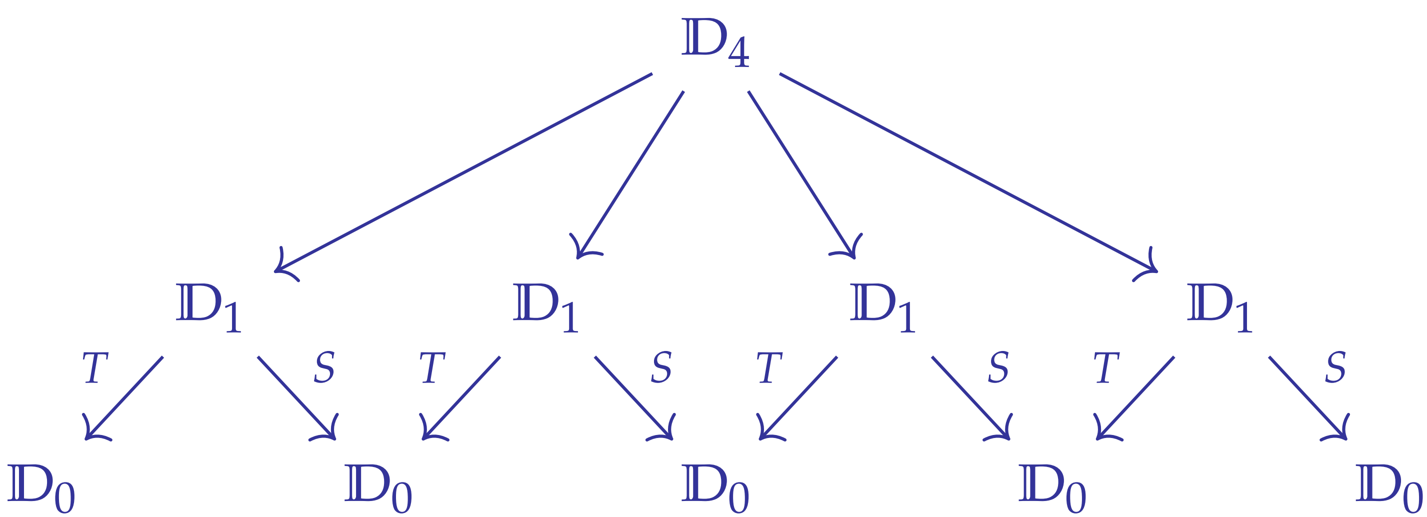


The category $\mathbb{D}_4$ of paths of length 4

Every double category $\mathbb{D}$ comes with

a category $\mathbb{D}_4$ of horizontal paths of length 4

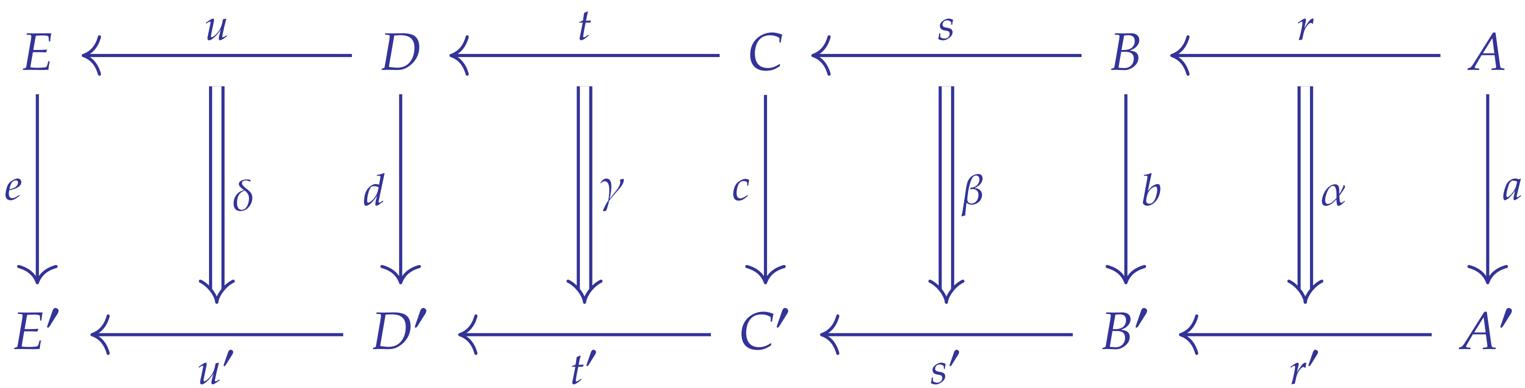
defined as the limit of the diagram of functors



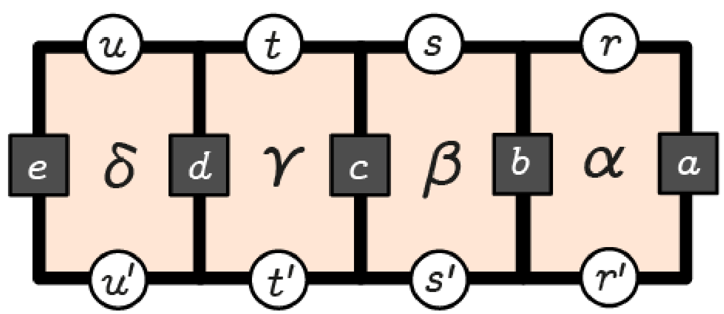
in the category **Cat** of categories and functors.

The category $\mathbb{D}_4$ of paths of length 4

A typical morphism of $\mathbb{D}_4$ has the shape



which we also like to depict as



Unbiased presentation of a double category

Every double category $\mathbb{D}$ comes equipped with a family of functors

$$h_n \quad : \quad \mathbb{D}_n \longrightarrow \mathbb{D}_1$$

called the **horizontal composition** functors, and satisfying a number of **associativity** and **neutrality** properties.

This leads to an alternative (unbiased) definition of (weak) double category.

Note that the functors h_2 and h_0 coincide with the functors $\diamond_h$ and idh

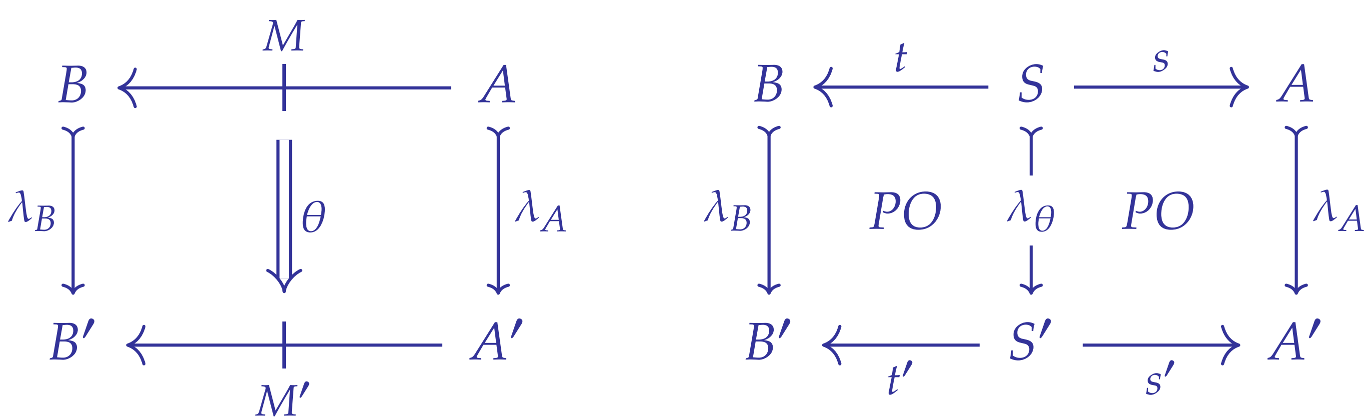
$$h_2 = \diamond_h \quad : \quad \mathbb{D}_2 \longrightarrow \mathbb{D}_1$$

$$h_0 = idh \quad : \quad \mathbb{D}_0 \longrightarrow \mathbb{D}_1$$

The double category $\mathbf{DPO}$ of double pushouts

The double category $\mathbb{D} = \mathbf{DPO}$ on an adhesive category $\mathbf{G}$

- ▷ whose objects are objects A, B, C of the adhesive category $\mathbf{G}$,
- ▷ whose horizontal maps $M = (S, s, t)$ are spans in $\mathbf{G}$,
- ▷ whose vertical maps $\lambda_A : A \to A'$ are monos in $\mathbf{G}$,
- ▷ whose double cells $\theta : M \Rightarrow M'$ are monos $\lambda_\theta : S \to S'$ making the pushout diagram commute:



The double category $\mathbf{LTRS}$ of linear term rewriting

The double category $\mathbb{D} = \mathbf{LTRS}$ on a first-order signature

$$\Sigma = \coprod_{n \in \mathbb{N}} \Sigma_n$$

is defined as follows:

- ▷ its objects are sequences of **closed linear λ-terms**

$$t_1 : A_1 \otimes \ldots \otimes t_n : A_n$$

whose types are generated by the grammar

$$A, B \quad ::= \quad \oplus \mid A \multimap B$$

extended with the rule for each letter $a \in \Sigma_n$ of the signature:

$$\text{Constant} \qquad \frac{}{\vdash \quad a \ : \ \oplus \multimap \cdots \multimap \oplus \multimap \oplus}$$

The double category $\mathbf{LTRS}$ of linear term rewriting

▷ its vertical maps

$$u_1 : A_1 \otimes \ldots \otimes u_p : A_p \xrightarrow{f_1 \otimes \ldots \otimes f_q} v_1 : B_1 \otimes \ldots \otimes v_q : B_q$$

are sequences of **linear λ-terms**

$$\Gamma_1 \vdash f_1 : B_1 \qquad \ldots \qquad \Gamma_q \vdash f_q : B_q$$

separating the context linearly

$$A_1 , \ldots , A_p \quad \cong \quad \Gamma_1 , \ldots , \Gamma_q$$

and satisfying the series of expected equations

$$v_1 = f_1 [u_1, \ldots, u_p] \qquad \ldots \qquad v_q = f_q [u_1, \ldots, u_p]$$

The double category $\mathbf{LTRS}$ of linear term rewriting

- whose double cell are of the form

$$
\begin{array}{ccc}
t_1 \otimes \ldots \otimes t_p & \longrightarrow & u_1 \otimes \ldots \otimes u_p \\
{\scriptstyle f_1 \otimes \ldots \otimes f_q} \downarrow & \Downarrow & \downarrow {\scriptstyle f_1 \otimes \ldots \otimes f_q} \\
t'_1 \otimes \ldots \otimes t'_q & \longrightarrow & u'_1 \otimes \ldots \otimes u'_q
\end{array}
$$

where the horizontal morphism

$$t_1 \otimes \ldots \otimes t_p \longrightarrow u_1 \otimes \ldots \otimes u_p$$

is a **pair of sequences** of closed linear λ-terms with same types.

Rewriting rules as covariant presheaves

Key idea : every **rewriting rule** seen as a **horizontal map** in $\mathbb{D}$

$$r \quad : \quad A \longrightarrow B$$

induces a **representable presheaf**

$$\hat{\Delta}_r \quad : \quad \mathbb{D}_1 \longrightarrow \mathbf{Set}$$

which associates to every horizontal map

$$u \quad : \quad A' \longrightarrow B'$$

the set $\mathbb{D}_1(r, u)$ of double cells

$$\begin{array}{ccc} A & \xrightarrow{\quad r \quad} & B \\ {\scriptstyle f}\downarrow & \Downarrow \theta & \downarrow{\scriptstyle g} \\ A' & \xrightarrow[\quad u \quad]{} & B' \end{array}$$

which **implement** the transformation u as an instance of the rule r.

Illustration

The rewrite rule

$$r \quad : \quad m(e,x) \longrightarrow x$$

implements the red redex using the double cell:

$$\begin{array}{ccc}
\lambda x.m(e,x) : \phi \multimap \phi & \xrightarrow{\mathit{rule}\, r} & \lambda x.x : \phi \multimap \phi \\
\Big\downarrow {\scriptstyle u : \phi \multimap \phi \,\vdash\, m(u(e),m(e,e)) \,:\, \phi} & \Big\Downarrow & \Big\downarrow {\scriptstyle u : \phi \multimap \phi \,\vdash\, m(u(e),m(e,e)) \,:\, \phi} \\
m(m(e,e),m(e,e)) : \phi & \xrightarrow[\mathit{red}]{} & m(e,m(e,e)) : \phi
\end{array}$$

Illustration

The rewrite rule

$$r \quad : \quad m(e,x) \longrightarrow x$$

implements the blue redex using the double cell:

$$\begin{array}{ccc}
\lambda x.m(e,x) : \Phi \multimap \Phi & \xrightarrow{\ rule\ r\ } & \lambda x.x : \Phi \multimap \Phi \\
{\scriptstyle u : \Phi \multimap \Phi \vdash m(m(e,e),u(e)) : \Phi}\ \Big\downarrow & \Downarrow & \Big\downarrow\ {\scriptstyle u : \Phi \multimap \Phi \vdash m(m(e,e),u(e)) : \Phi} \\
m(m(e,e),m(e,e)) : \Phi & \xrightarrow[\ blue\ ]{} & m(m(e,e),e) : \Phi
\end{array}$$

Illustration

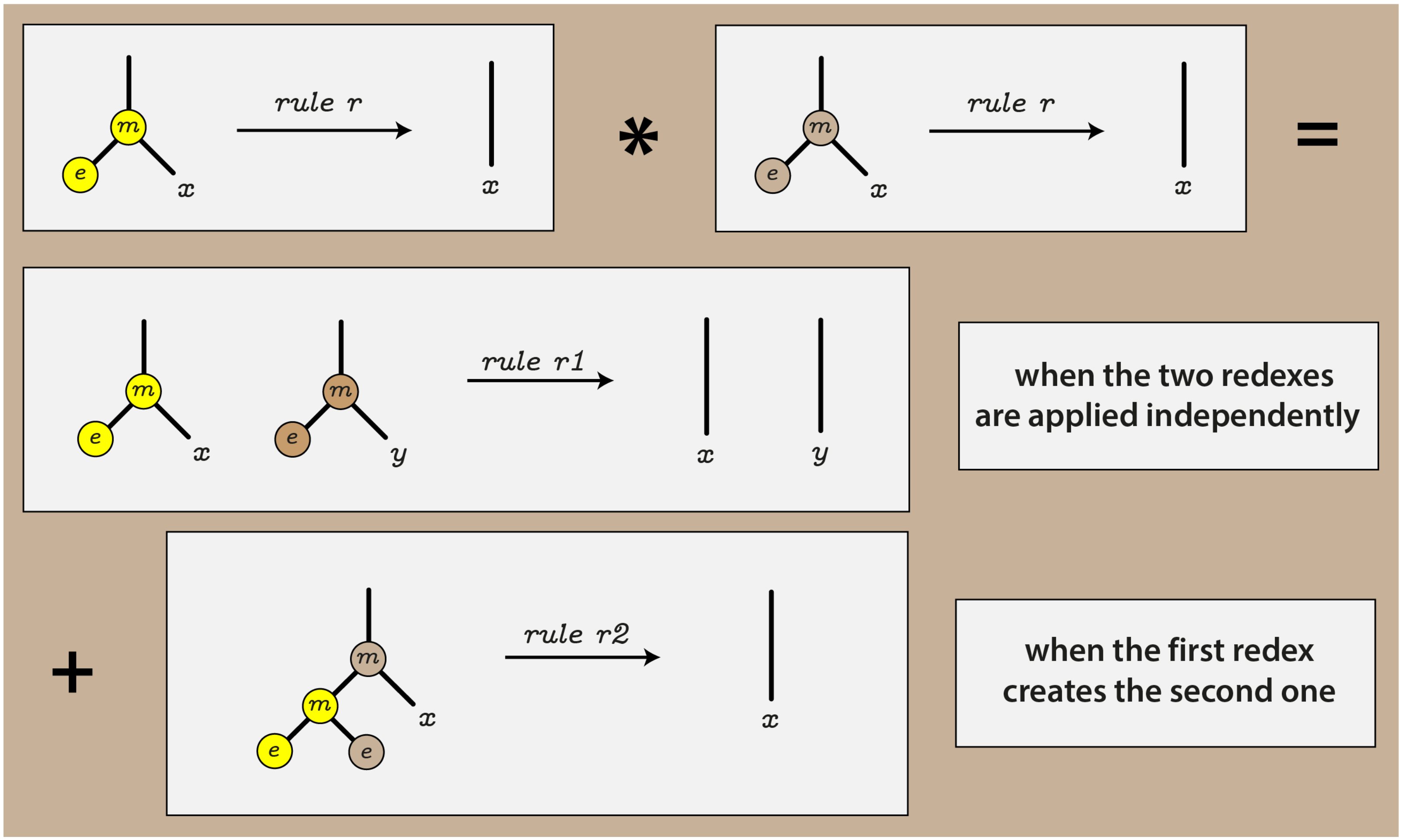
The rewrite rule

$$r \quad : \quad m(e,x) \longrightarrow x$$

implements the green redex using the double cell:

$$\begin{array}{ccc}
\lambda x.m(e,x) : \obar \multimap \obar & \xrightarrow{\textit{rule } r} & \lambda x.x : \obar \multimap \obar \\
{\scriptstyle u : \obar \multimap \obar \ \vdash\ u(m(e,e)) : \obar}\ \Big\downarrow & \Downarrow & \Big\downarrow\ {\scriptstyle u : \obar \multimap \obar \ \vdash\ u(m(e,e) : \obar} \\
m(e,m(e,e)) : \obar & \xrightarrow[\textit{green}]{} & m(e,e) : \obar
\end{array}$$

Goal: composing rules using convolution



Category of elements of a presheaf

The Grothendieck construction

Elements of a covariant presheaf

Recall that an **element**

$$(a, x) \quad \in \quad \mathbf{Elts}\,(F)$$

of a covariant presheaf

$$F \quad : \quad \mathbf{C} \longrightarrow \mathbf{Set}$$

is defined as a pair

$$\Big(\quad a \in \mathbf{C} \quad , \quad x \in F(a) \quad \Big)$$

consisting of

- ▷ an object a of the underlying category $\mathbf{C}$,
- ▷ an element x of the set $F(a)$.

Elements of a covariant presheaf

We find enlightening to draw such a pair

$$\left(\quad a \in \mathbf{C} \quad , \quad x \in F(a) \quad\right) \quad \in \quad \mathbf{Elts}\,(F)$$

in the following way

with the intuition that the element

$$x \in F(a)$$

provides a **witness** of the covariant presheaf F at instance $a \in \mathbf{C}$.

Covariant action of a presheaf

By definition of a covariant presheaf

$$F \quad : \quad \mathbf{C} \longrightarrow \mathbf{Set}$$

every element

$$\Big(\quad a \in \mathbf{C} \quad , \quad x \in F(a) \quad \Big) \quad \in \quad \mathbf{Elts}\,(F)$$

and morphism of the category $\mathbf{C}$

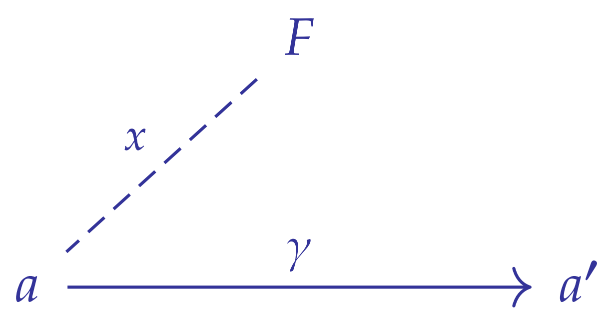
$$\gamma \quad : \quad a \longrightarrow a'$$

induces an element

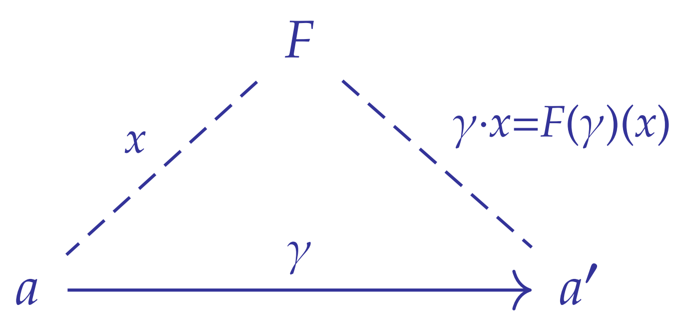
$$\Big(\quad a' \in \mathbf{C} \quad , \quad \gamma \cdot x = F(\gamma)(x) \in F(a') \quad \Big) \quad \in \quad \mathbf{Elts}\,(F)$$

Covariant action of a presheaf

This means that every diagram



can be completed into the diagram



The category of elements

The category $\mathbf{Elts}\,(F)$ of elements of a covariant presheaf

$$F \quad : \quad \mathbf{C} \longrightarrow \mathbf{Set}$$

is defined in the following way:

- ▷ its objects are the elements (a, x) of the covariant presheaf F
- ▷ its morphisms

$$(f, x) \quad : \quad (a, x) \longrightarrow (a', x')$$

are the pairs consisting of a morphism

$$f \quad : \quad a \longrightarrow a'$$

of the category $\mathbf{C}$ and an element $x \in F(a)$ such that

$$f \cdot x \quad = \quad F(f)(x) \quad = \quad x'$$

The category of elements

The category of elements

$$\mathbf{Elts}\,(F)$$

associated to a covariant presheaf

$$F \quad : \quad \mathbf{C} \longrightarrow \mathbf{Set}$$

comes equipped with a **projection functor**

$$\pi_F \quad : \quad \mathbf{Elts}\,(F) \longrightarrow \mathbf{C}$$

which transports every element

$$(a, x) \in \mathbf{Elts}\,(F)$$

to the object $a \in \mathbf{C}$ of the underlying category $\mathbf{C}$.

Fact. The functor π_F defines a **discrete opfibration**.

Grothendieck opfibrations

Definition. A functor

$$p \quad : \quad \mathbf{E} \longrightarrow \mathbf{C}$$

is an **opfibration** when there exists an opcartesian morphism

$$\begin{array}{ccc} R & \xrightarrow{\quad f \quad} & S \\ & \Big\downarrow{\scriptstyle p} & \\ A & \xrightarrow[\quad u \quad]{} & B \end{array}$$

for every object $R \in p^{-1}(A)$ and every morphism $u : A \to B$.

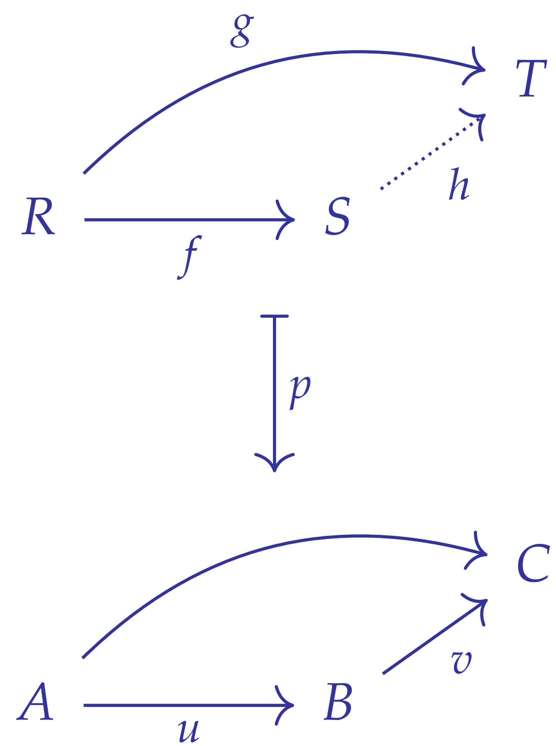
Opcartesian morphisms

A morphism $f : R \to S$ in $\mathbf{E}$ is opcartesian above $u : A \to B$ in $\mathbf{C}$ when the following property holds:

for every map $g : R \to T$

for every map $v : B \to C$
such that $p(g) = v \circ u$

there exists
a unique map $h : S \to T$
such that $h \circ f = g$
and $p(h) = v$.



The Grothendieck correspondence

The projection functor

$$\pi_F \quad : \quad \mathbf{Elts}\,(F) \longrightarrow \mathbf{C}$$

is a discrete opfibration. Indeed, every diagram

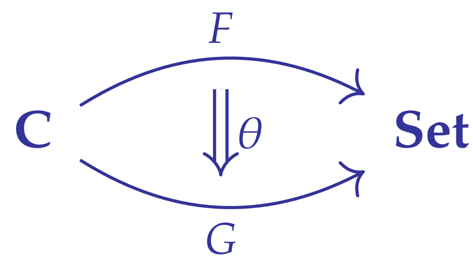
$$\begin{array}{ccc} x & & \\ {\scriptstyle \pi_F}\big\downarrow & & \\ a & \xrightarrow{\quad f \quad} & a' \end{array}$$

can be completed with the opcartesian morphism $(\,f\,,\,x\,)$ as follows:

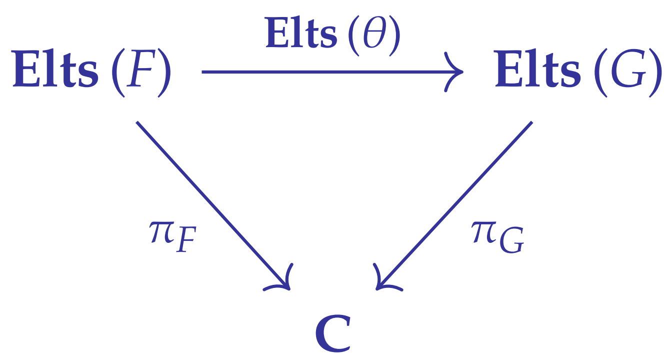
$$\begin{array}{ccc} x & \xrightarrow{\quad (\,f\,,x\,) \in \mathbf{Elts}\,(F) \quad} & f \cdot x \\ {\scriptstyle \pi_F}\big\downarrow & & \big\downarrow{\scriptstyle \pi_F} \\ a & \xrightarrow{\quad f \in \mathbf{C} \quad} & a' \end{array}$$

The Grothendieck correspondence

Moreover, every natural transformation



induces a commutative diagram of discrete opfibrations:



The Grothendieck correspondence

Fact. This induces a categorical equivalence between

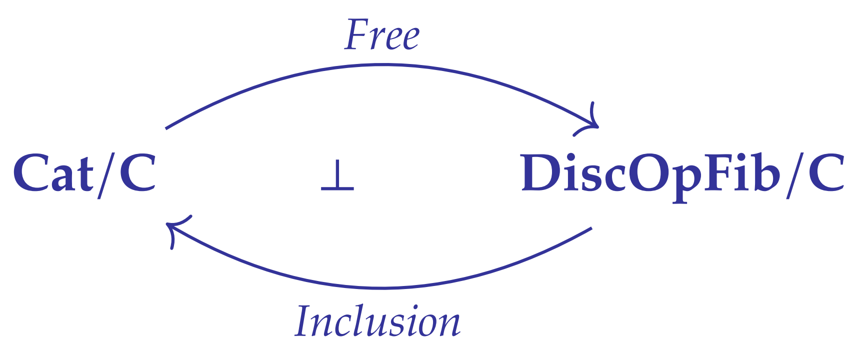
- ▷ The category $[\mathbf{C}, \mathbf{Ens}]$ of **covariant presheaves**

$$F, G \quad : \quad \mathbf{C} \longrightarrow \mathbf{Set}$$

  and natural transformations between them.

- ▷ The slice category $\mathbf{DiscOpFib}/\mathbf{C}$ of **discrete opfibrations** above $\mathbf{C}$.

Moreover, there is an adjunction



The Day convolution product

A construction on monoidal categories

The Day convolution product

Given two covariant presheaves

$$F, G \quad : \quad \mathbf{C} \longrightarrow \mathbf{Set}$$

on a monoidal category $\mathbf{C}$ with tensor product

$$\otimes \quad : \quad \mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C}$$

the **Day convolution product** of F and G is the covariant presheaf

$$G \,\hat{\otimes}\, F \quad : \quad \mathbf{C} \longrightarrow \mathbf{Set}$$

defined by the coend formula

$$G \,\hat{\otimes}\, F \;=\; c \;\mapsto\; \int^{(b,a) \in \mathbf{C} \times \mathbf{C}} \mathbf{C}(b \otimes a, c) \times G(b) \times F(a)$$

The Day convolution product

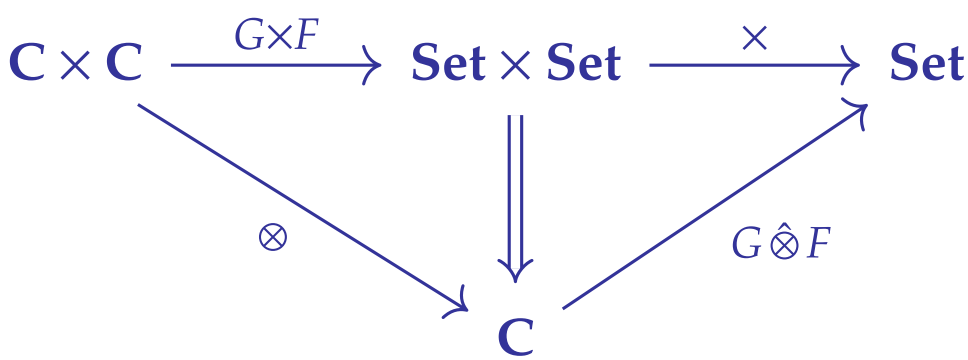
Equivalently, the convolution product

$$G \hat{\otimes} F \quad : \quad \mathbf{C} \longrightarrow \mathbf{Set}$$

may be defined as the **left Kan extension** of the functor

$$\mathbf{C} \times \mathbf{C} \xrightarrow{G \times F} \mathbf{Set} \times \mathbf{Set} \xrightarrow{\quad \times \quad} \mathbf{Set}$$

along the tensor product functor:



What does the coend formula mean?

An element of the coend

$$G \hat{\otimes} F\ (c) \quad = \quad \int^{(b,a) \in \mathbf{C} \times \mathbf{C}} \mathbf{C}(b \otimes a, c) \times G(b) \times F(a)$$

consists of a morphism

$$b \otimes a \xrightarrow{\gamma} c$$

together with a pair of elements

$$y \in G(b) \qquad\qquad x \in F(a)$$

considered modulo an equivalence relation $\sim$.

What does the coend formula mean?

As we did before, we find enlightening to draw the two elements

$$y \in G(b) \qquad\qquad x \in F(a)$$

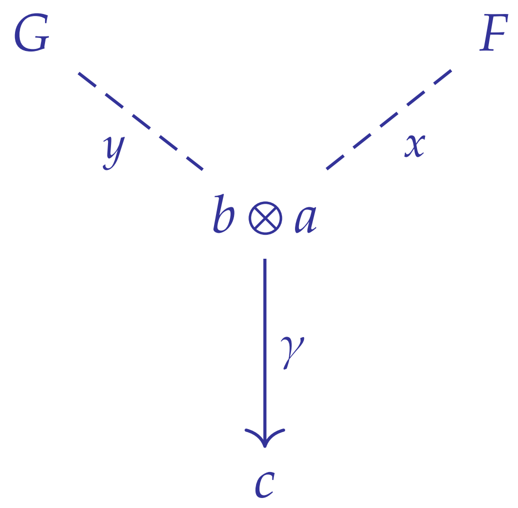
in the following way:

What does the coend formula mean?

Accordingly, we like to draw the triple

$$\left(\quad b \otimes a \xrightarrow{\gamma} c \quad , \quad x \in F(a) \quad , \quad y \in G(b) \quad \right)$$

in the following way:



What does the coend formula mean?

Suppose given a pair of elements

$$x \in F(a) \qquad\qquad y \in G(b)$$

a pair of morphisms

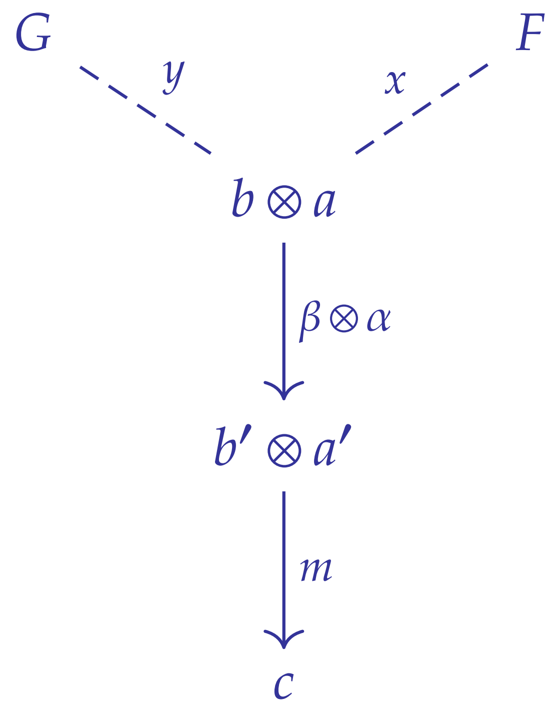
$$\alpha \ : \ a \longrightarrow a' \qquad\qquad \beta \ : \ b \longrightarrow b'$$

and a morphism

$$\gamma \ : \ a' \otimes b' \longrightarrow c$$

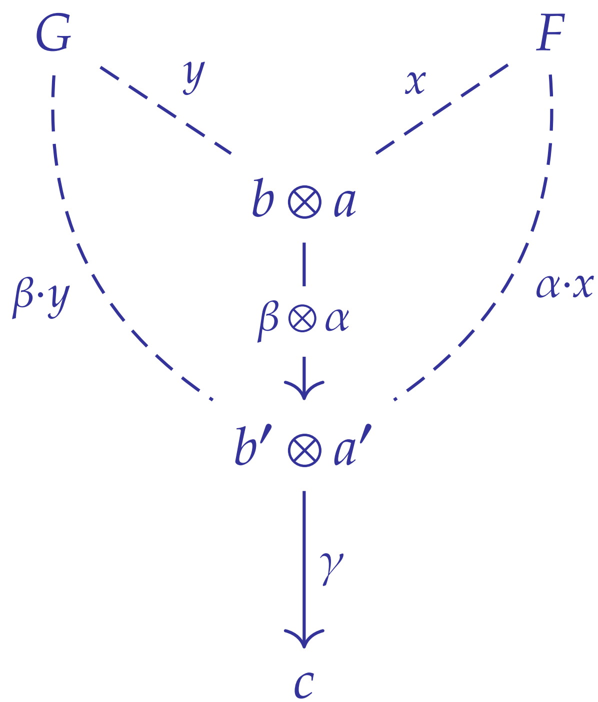
What does the coend formula mean?

The situation may be depicted as follows:



What does the coend formula mean?

The diagram may be completed as follows:



What does the coend formula mean?

This equivalence relation $\sim$ defined by the coend

$$G \hat{\otimes} F\ (c) \quad = \quad \int^{(b,a)\in \mathbf{C}\times\mathbf{C}} \mathbf{C}(b\otimes a, c) \times G(b) \times F(a)$$

identifies every triple of the form

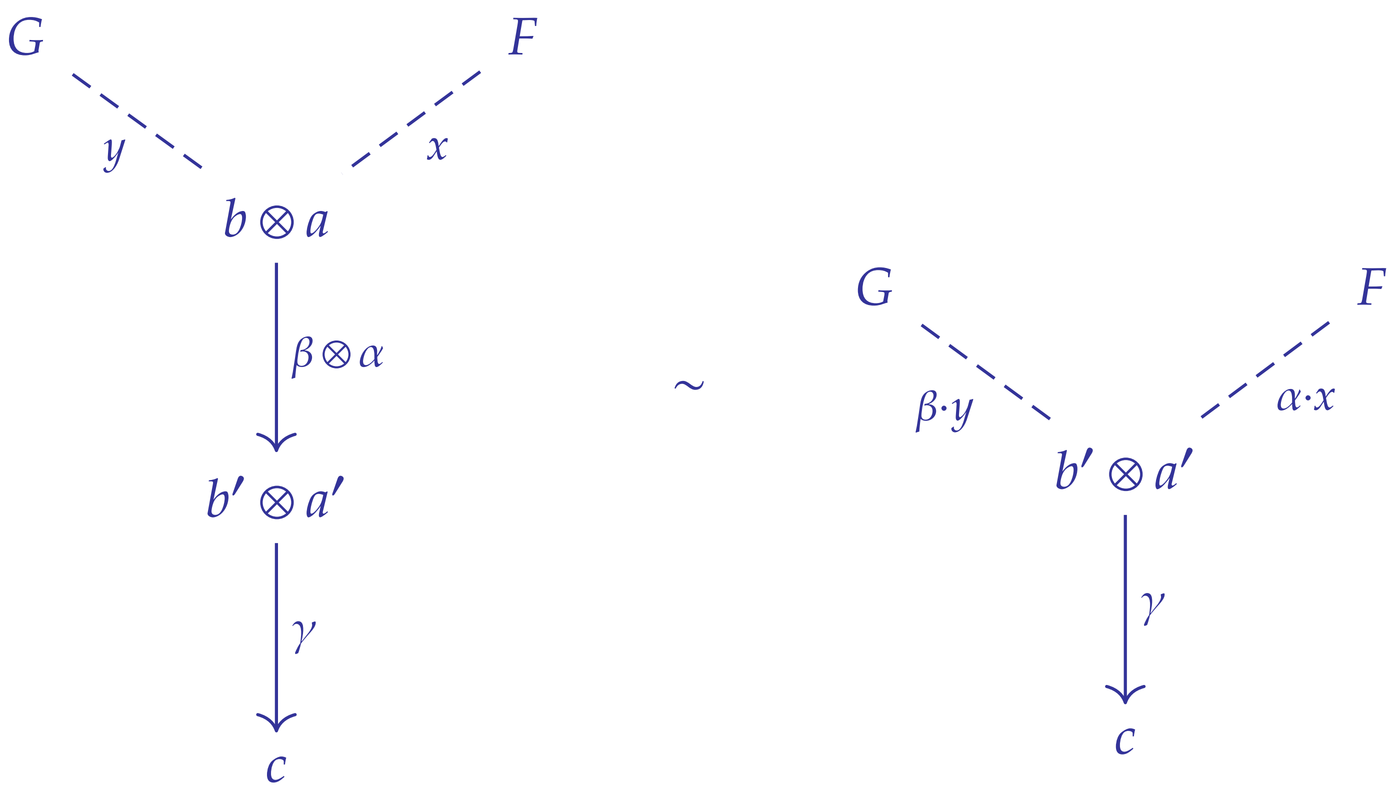
$$\left(\quad b\otimes a \xrightarrow{\beta\otimes\alpha} b'\otimes a' \xrightarrow{\gamma} c \quad , \quad x \in F(a) \quad , \quad y \in G(b) \quad \right)$$

with the corresponding triple

$$\left(\quad b'\otimes a' \xrightarrow{\gamma} c \quad , \quad \alpha\cdot x \in F(a') \quad , \quad \beta\cdot y \in G(b') \quad \right)$$

What does the coend formula mean?

Diagrammatically, the equivalence relation $\sim$ identifies the two triples:



The Day convolution product

Theorem [Day 1970] The convolution product

$$G, F \quad \mapsto \quad G \,\hat{\otimes}\, F$$

on a monoidal category $\mathbf{C}$ with tensor product $\otimes$ defines a functor

$$\hat{\otimes} \quad : \quad [\mathbf{C}, \mathbf{Set}] \times [\mathbf{C}, \mathbf{Set}] \longrightarrow [\mathbf{C}, \mathbf{Set}]$$

which equips the category of covariant presheaves

$$[\mathbf{C}, \mathbf{Set}]$$

with the structure of a monoidal closed category.

In particular, the convolution product is associative:

$$H \,\hat{\otimes}\, (G \,\hat{\otimes}\, F) \quad \cong \quad (H \,\hat{\otimes}\, G) \,\hat{\otimes}\, F$$

Construction of the free discrete opfibration

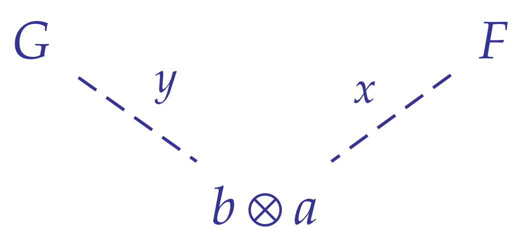
Step 0. We start from the functor

$$\mathbf{Elts}\,(G) \times \mathbf{Elts}\,(F) \xrightarrow{\pi_G \times \pi_F} \mathbf{C} \times \mathbf{C} \xrightarrow{\otimes} \mathbf{C}$$

whose objects in the source category are pairs

$$\Big(\quad x \in F(a) \quad , \quad y \in G(b) \quad \Big)$$

may be depicted in the following way:



Construction of the free discrete opfibration

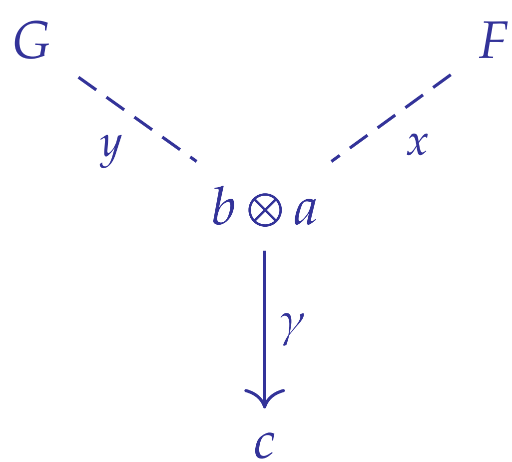
Step 1. We replace the functor by its **free split opfibration**

$$\mathbf{Elts}\,(G, F) \xrightarrow{\quad \pi_{G,F} \quad} \mathbf{C}$$

where the source category $\mathbf{Elts}\,(G, F)$ has objects defined as triples

$$\left(\quad b \otimes a \xrightarrow{\gamma} c \quad , \quad x \in F(a) \quad , \quad y \in G(b) \quad \right)$$

which may be depicted in the following way:

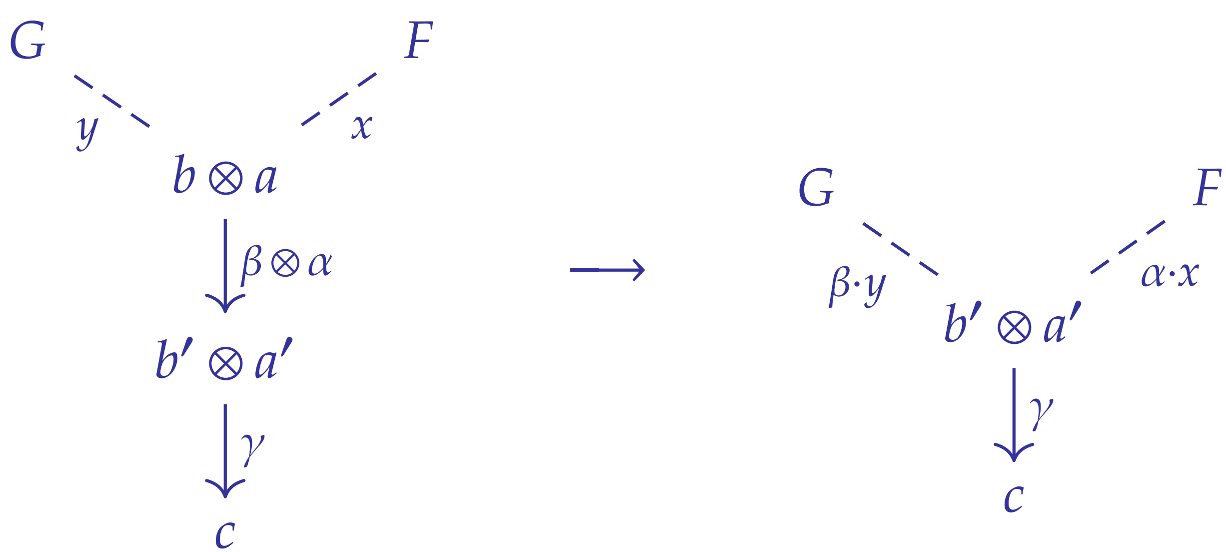


Construction of the free discrete opfibration

Step 1. We replace the functor by its **free split opfibration**

$$\mathbf{Elts}\,(G, F) \xrightarrow{\quad \pi_{G,F} \quad} \mathbf{C}$$

whose morphisms in each fiber above $c \in \mathbf{C}$ are of the form:

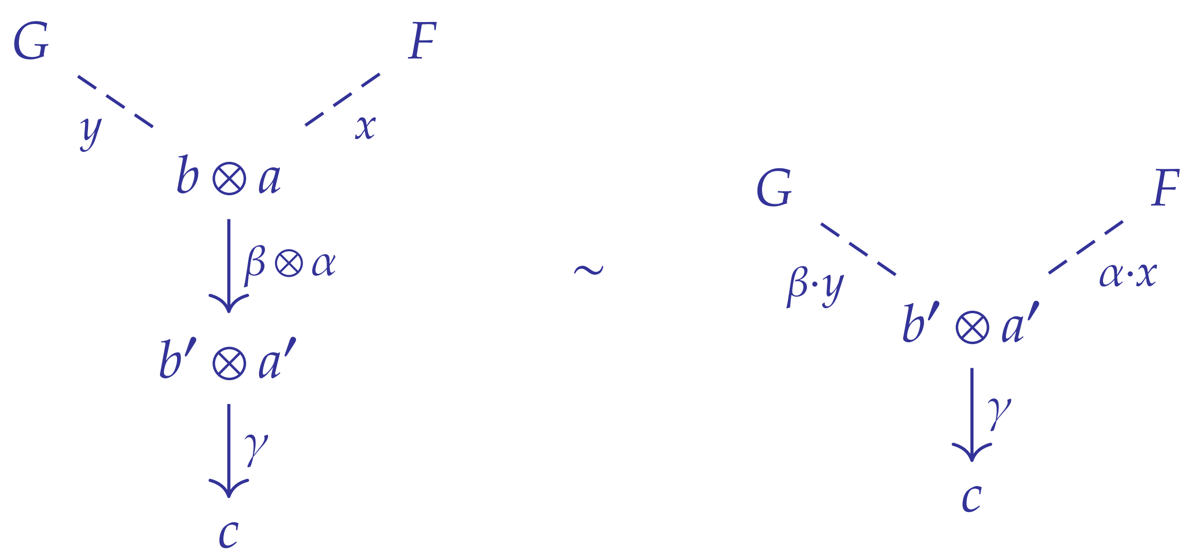


Construction of the free discrete opfibration

Step 2. Replace each fiber category of the opfibration

$$\mathbf{Elts}\,(G,F) \xrightarrow{\quad \pi_{G,F} \quad} \mathbf{C}$$

by its set of **connected components**, using the equivalence relation:

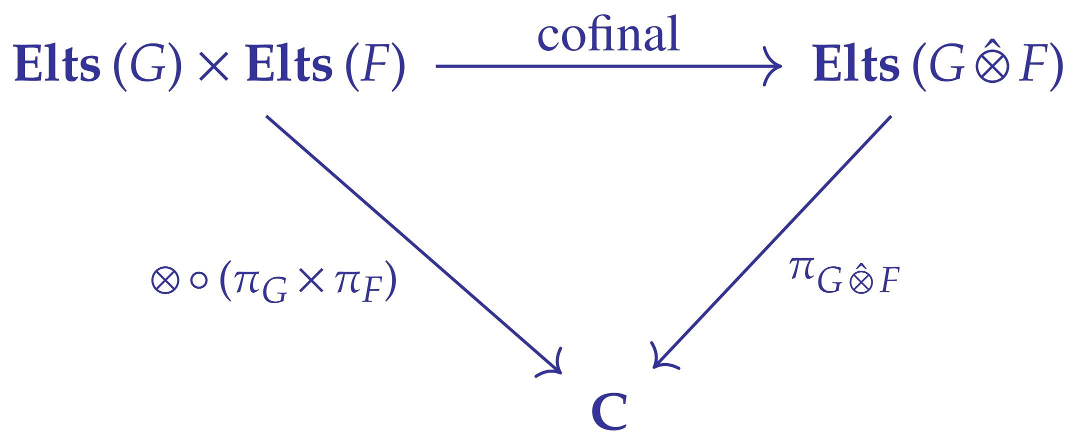


A key observation

From this follows that there exists a cofinal functor

$$\mathbf{Elts}\,(G) \times \mathbf{Elts}\,(F) \longrightarrow \mathbf{Elts}\,(G \,\hat{\otimes}\, F)$$

making the diagram commute:



in the category $\mathbf{Cat}$ of categories and functors.

A key observation

The category $\mathbf{Cat}/\mathbf{C}$ inherits a tensor product

$$\tilde{\otimes} \quad : \quad \mathbf{Cat}/\mathbf{C} \times \mathbf{Cat}/\mathbf{C} \longrightarrow \mathbf{Cat}/\mathbf{C}$$

from the monoidal structure of the category $\mathbf{C}$.

The Day tensor product

$$\hat{\otimes} \quad : \quad \mathbf{DiscOpFib}/\mathbf{C} \times \mathbf{DiscOpFib}/\mathbf{C} \longrightarrow \mathbf{DiscOpFib}/\mathbf{C}$$

is the monoidal structure obtained by transporting $\tilde{\otimes}$ along the adjunction

$$\mathbf{Cat}/\mathbf{C} \quad \underset{\textit{Inclusion}}{\overset{\textit{Free}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \quad \mathbf{DiscOpFib}/\mathbf{C}$$

Construction of the free discrete opfibration

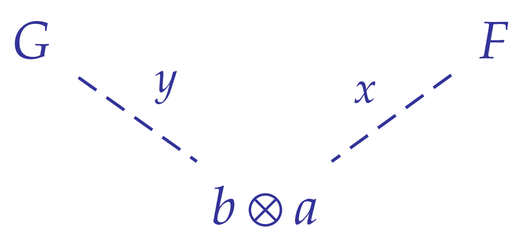
Step 0. We start from the functor

$$\mathbf{Elts}\,(G) \times \mathbf{Elts}\,(F) \xrightarrow{\quad \pi_G \times \pi_F \quad} \mathbf{C} \times \mathbf{C} \xrightarrow{\quad \otimes \quad} \mathbf{C}$$

whose objects in the source category are pairs

$$\Big(\quad x \in F(a) \quad , \quad y \in G(b) \quad \Big)$$

may be depicted in the following way:



Construction of the free discrete opfibration

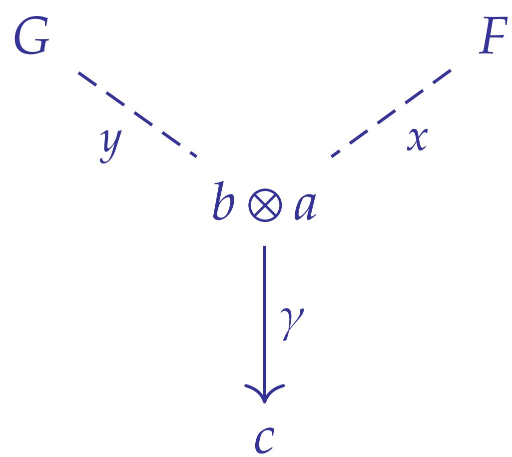
Step 1. We replace the functor by its **free split opfibration**

$$\mathbf{Elts}\,(G, F) \xrightarrow{\quad \pi_{G,F} \quad} \mathbf{C}$$

where the source category $\mathbf{Elts}\,(G, F)$ has objects defined as triples

$$\Big(\quad b \otimes a \xrightarrow{\gamma} c \quad , \quad x \in F(a) \quad , \quad y \in G(b) \quad \Big)$$

which may be depicted in the following way:

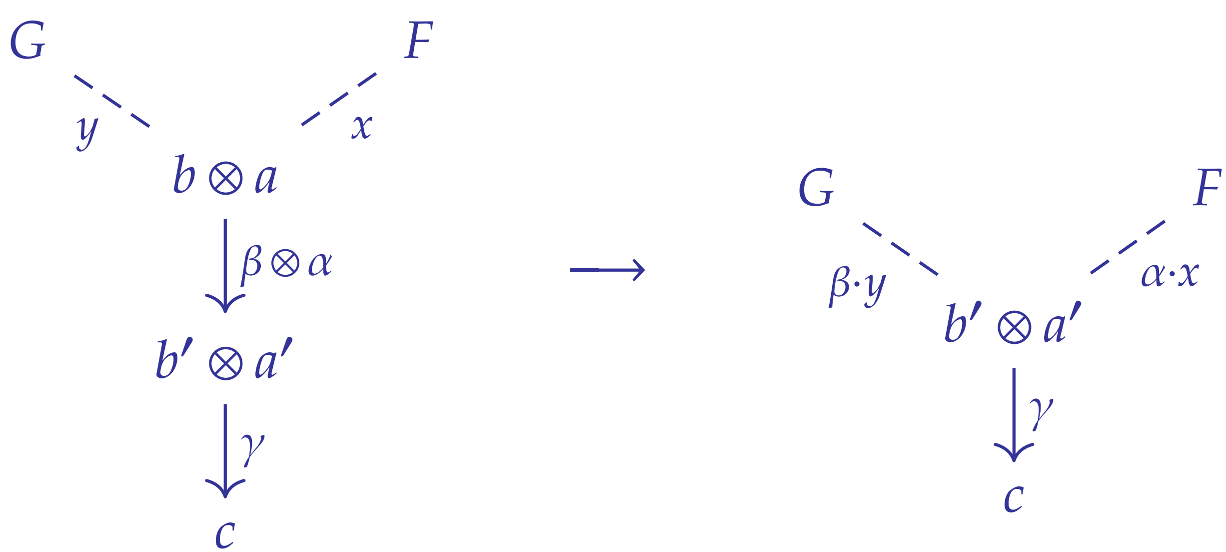


Construction of the free discrete opfibration

Step 1. We replace the functor by its **free split opfibration**

$$\mathbf{Elts}\,(G, F) \xrightarrow{\quad \pi_{G,F} \quad} \mathbf{C}$$

whose morphisms in each fiber above $c \in \mathbf{C}$ are of the form:

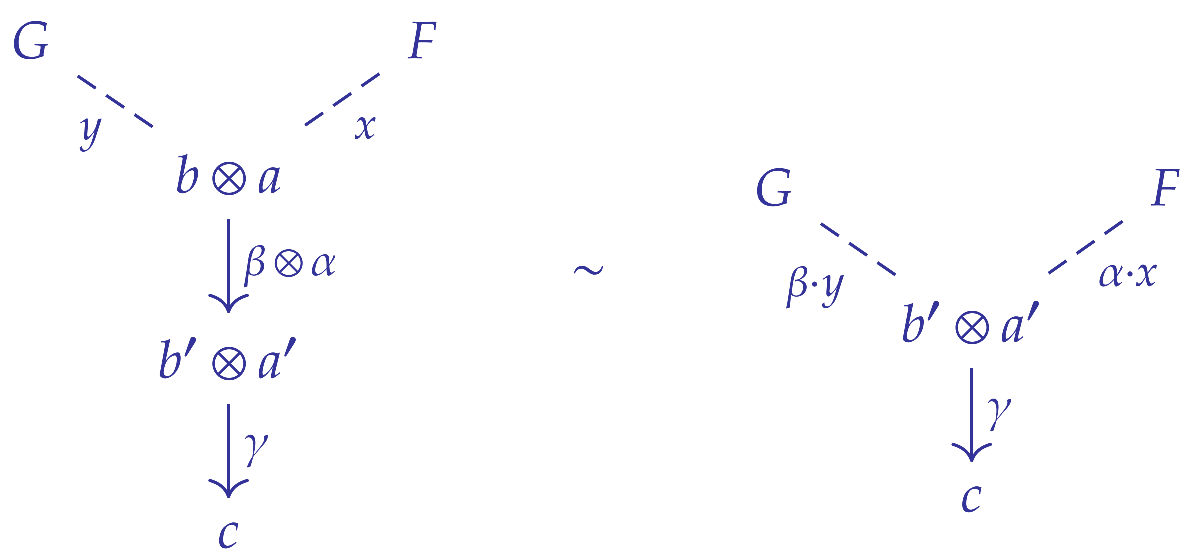


Construction of the free discrete opfibration

Step 2. Replace each fiber category of the opfibration

$$\mathbf{Elts}\,(G, F) \xrightarrow{\pi_{G,F}} \mathbf{C}$$

by its set of **connected components**, using the equivalence relation:



A key observation

From this follows that there exists a cofinal functor

$$\mathbf{Elts}\,(G) \times \mathbf{Elts}\,(F) \longrightarrow \mathbf{Elts}\,(G \mathbin{\hat{\otimes}} F)$$

making the diagram commute:

$$\begin{array}{ccc} \mathbf{Elts}\,(G) \times \mathbf{Elts}\,(F) & \xrightarrow{\text{cofinal}} & \mathbf{Elts}\,(G \mathbin{\hat{\otimes}} F) \\ & {\scriptstyle \otimes \circ (\pi_G \times \pi_F)} \searrow \quad \swarrow {\scriptstyle \pi_{G \hat{\otimes} F}} & \\ & \mathbf{C} & \end{array}$$

in the category **Cat** of categories and functors.

A key observation

The category $\mathbf{Cat}/\mathbf{C}$ inherits a tensor product

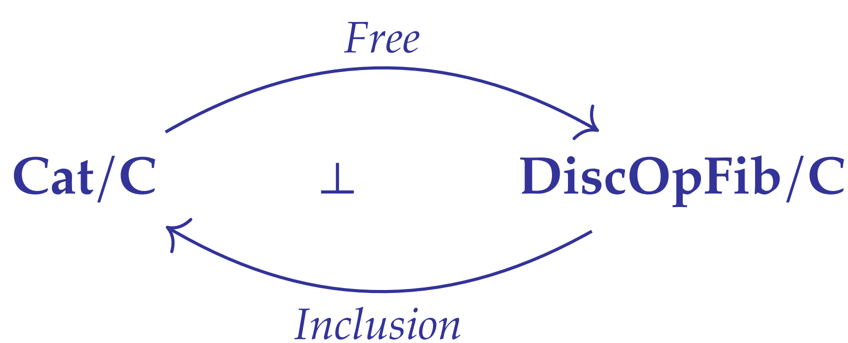
$$\tilde{\otimes} \quad : \quad \mathbf{Cat}/\mathbf{C} \times \mathbf{Cat}/\mathbf{C} \longrightarrow \mathbf{Cat}/\mathbf{C}$$

from the monoidal structure of the category $\mathbf{C}$.

The Day tensor product

$$\hat{\otimes} \quad : \quad \mathbf{DiscOpFib}/\mathbf{C} \times \mathbf{DiscOpFib}/\mathbf{C} \longrightarrow \mathbf{DiscOpFib}/\mathbf{C}$$

is the monoidal structure obtained by transporting $\tilde{\otimes}$ along the adjunction



The convolution product on double categories

Extending the Day construction

The convolution product on double categories

Given two covariant presheaves

$$F, G \quad : \quad \mathbb{D}_1 \longrightarrow \mathbf{Set}$$

on a double category $\mathbb{D}$ with horizontal composition

$$\diamond_h \quad : \quad \mathbb{D}_2 = \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \longrightarrow \mathbb{D}_1$$

the **convolution product** of F and G is the covariant presheaf

$$G * F \quad : \quad \mathbb{D}_1 \longrightarrow \mathbf{Set}$$

defined by the coend formula:

$$G * F \; = \; t \; \mapsto \; \int^{(s,r) \in \mathbb{D}_2} \mathbb{D}_1(s \diamond_h r, t) \times G(s) \times F(r)$$

The convolution product

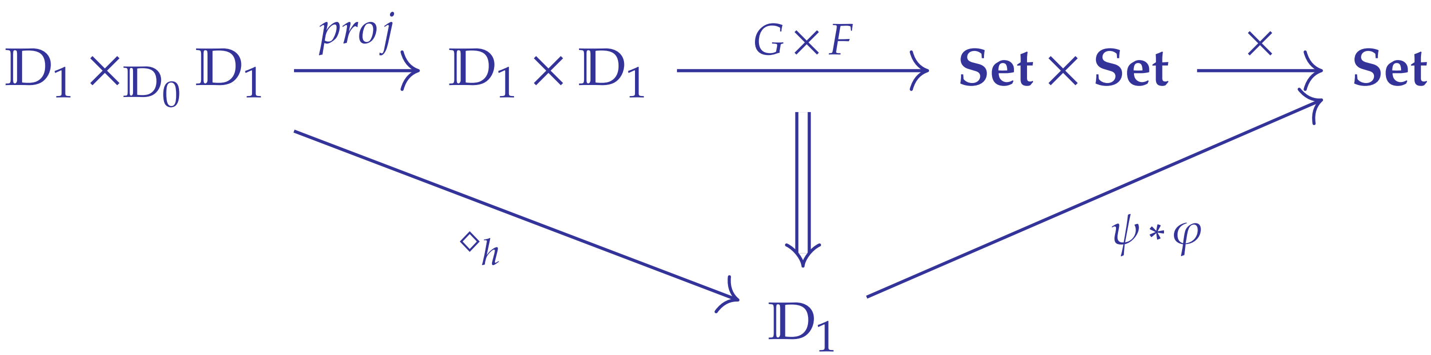
Equivalently, the convolution product

$$G * F \quad : \quad \mathbb{D}_1 \longrightarrow \mathbf{Set}$$

may be defined as the **left Kan extension** of the functor

$$\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \xrightarrow{\ proj\ } \mathbb{D}_1 \times \mathbb{D}_1 \xrightarrow{\ G \times F\ } \mathbf{Set} \times \mathbf{Set} \xrightarrow{\ \times\ } \mathbf{Set}$$

along the tensor product functor:



What does the coend formula mean?

An element of the coend

$$G * F\,(t) \;=\; \int^{(s,r)\,\in\,\mathbb{D}_2} \mathbb{D}_1(s \diamond_h r, t) \times G(s) \times F(r)$$

consists of a double cell of the form

$$\begin{array}{ccccc} B & \xleftarrow{\;s\;} & C & \xleftarrow{\;r\;} & A \\ {\scriptstyle g}\big\downarrow & & \Big\Downarrow{\scriptstyle \gamma} & & \big\downarrow{\scriptstyle f} \\ B' & & \xleftarrow[\;t\;]{\qquad\qquad} & & A' \end{array}$$

together with a pair of elements

$$y \in G(s) \qquad\qquad x \in F(r)$$

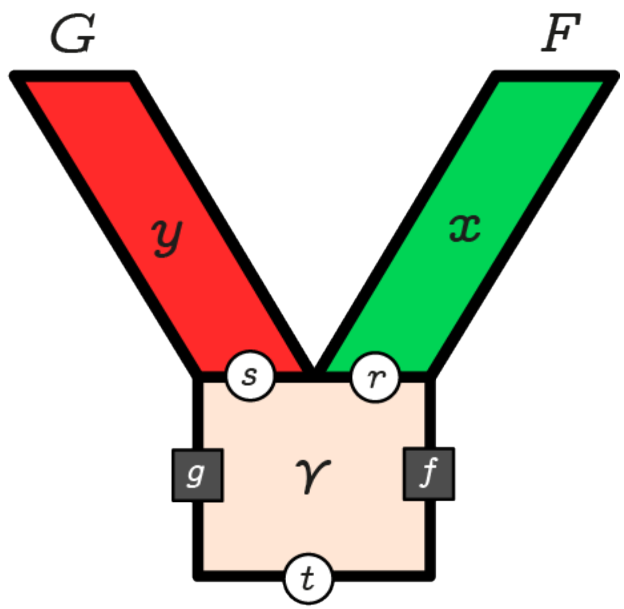
considered modulo an equivalence relation noted $\sim$.

What does the coend formula mean?

We find enlightening to draw the triple

$$\left(\quad s \diamond_h r \xrightarrow{\gamma} t \quad , \quad x \in F(r) \quad , \quad y \in G(s) \quad \right)$$

in the following way:



What does the coend formula mean?

Suppose given a pair of elements

$$x \in F(r) \qquad\qquad y \in G(s)$$

a pair of double cells

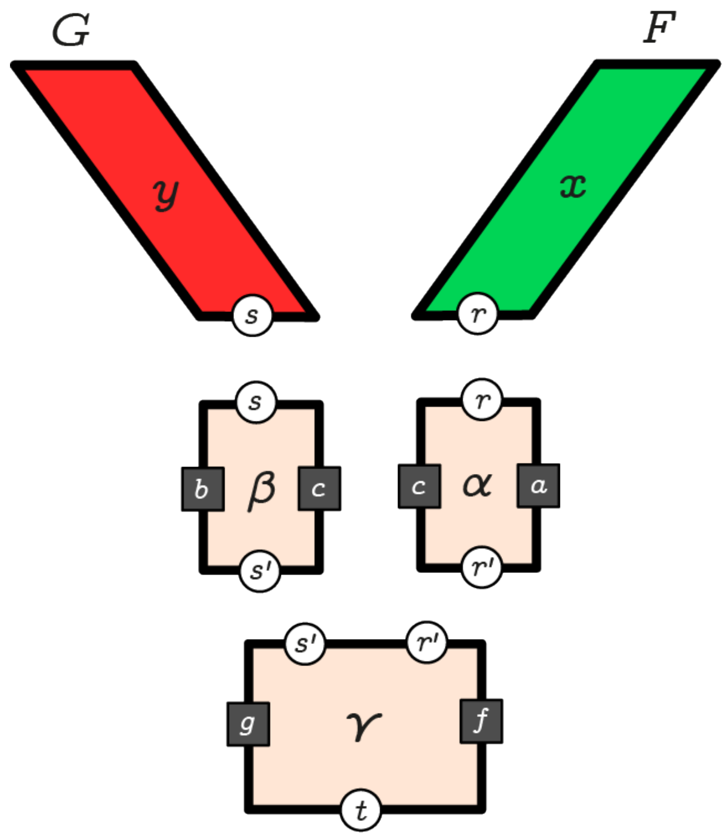
$$\alpha \;:\; r \Longrightarrow r' \qquad\qquad \beta \;:\; s \Longrightarrow s'$$

and a double cell

$$\gamma \;:\; s' \diamond_h r' \Longrightarrow t$$

What does the coend formula mean?

The five components may be depicted as follows:

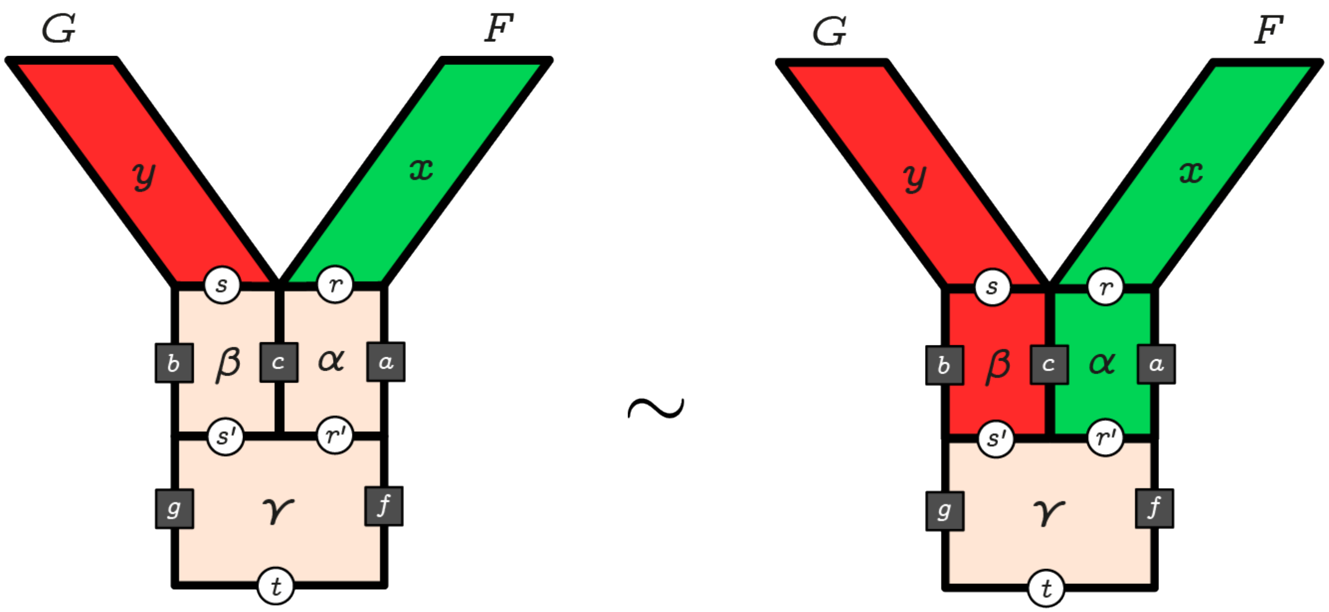


What does the coend formula mean?

The equivalence relation $\sim$ defined by the coend

$$G * F\ (t)\ =\ \int^{(s,r)\in\mathbb{D}_2} \mathbb{D}_1(s \diamond_h r, t) \times G(s) \times F(r)$$

identifies every triple of the form



Key observation

Theorem [Behr, PAM, Zeilberger]

The convolution product

$$G, F \quad \mapsto \quad G * F$$

on a double category $\mathbb{D}$ defines a functor

$$* \quad : \quad \widehat{\mathbb{D}} \times \widehat{\mathbb{D}} \longrightarrow \widehat{\mathbb{D}}$$

which equips the category of covariant presheaves

$$\widehat{\mathbb{D}} \quad := \quad [\mathbb{D}_1, \mathbf{Set}]$$

with the structure of an **oplax monoidal closed category**.

What oplax monoidal means...

The category of covariant presheaves

$$\widehat{\mathbb{D}} \quad := \quad [\mathbb{D}_1, \mathbf{Set}]$$

comes equipped with a **family of convolution products**

$$*_n \quad : \quad \widehat{\mathbb{D}} \times \cdots \times \widehat{\mathbb{D}} \longrightarrow \widehat{\mathbb{D}}$$

where we use the notation

$$(F_n * \cdots * F_1) \quad := \quad *_n \, (F_n, \ldots, F_1)$$

for the n-ary product of n covariant presheaves

$$F_n, \ldots, F_1 \quad : \quad \mathbb{D}_1 \longrightarrow \mathbf{Set}.$$

The ternary convolution product

Typically, the ternary convolution product

$$H * G * F \quad : \quad \mathbf{C} \longrightarrow \mathbf{Set}$$

of three covariant presheaves H, G, F is defined by the coend formula

$$H * G * F \;=\; u \;\mapsto\; \int^{(t,s,r)\in\mathbb{D}_3} \mathbb{D}_1(t \diamond_h s \diamond_h r, u) \times H(t) \times G(s) \times F(r)$$

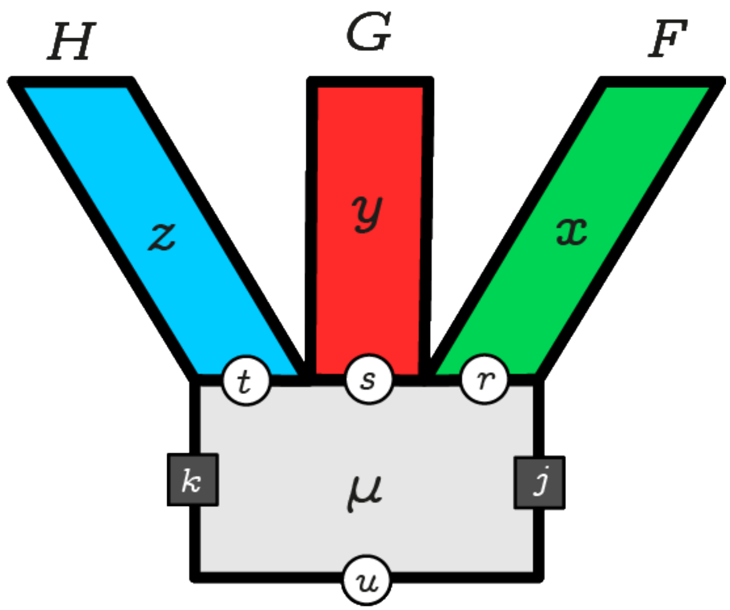
where $\mathbb{D}_3$ is the category of horizontal paths of length 3.

The ternary convolution product

The elements of the ternary convolution product are quadruples

$$\left(\quad t \diamond_h s \diamond_h r \xRightarrow{\delta} u \quad , \quad x \in F(r) \quad , \quad y \in G(s) \quad , \quad z \in G(t) \quad \right)$$

which may be depicted in the following way:

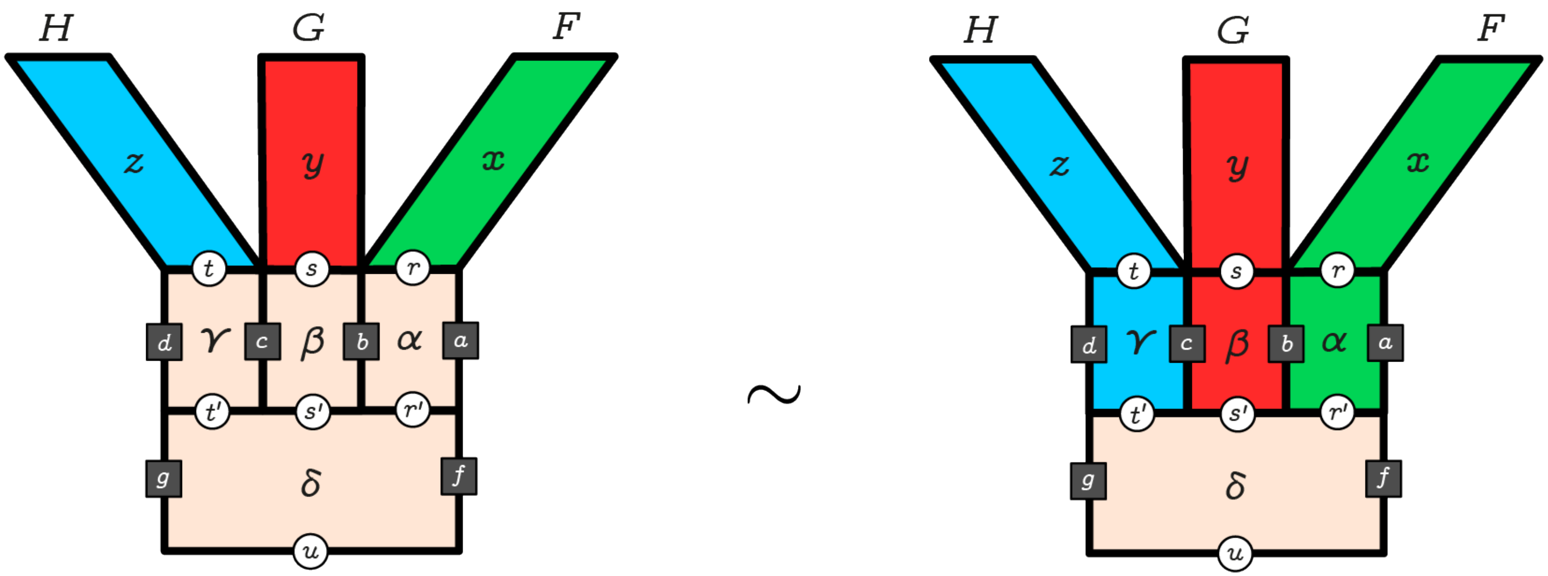


The ternary convolution product

The elements of the convolution product

$$\left(\quad t \diamond_h s \diamond_h r \overset{\delta}{\Longrightarrow} u \quad , \quad x \in F(r) \quad , \quad y \in G(s) \quad , \quad z \in G(t) \quad \right)$$

are identified modulo the equivalence relation:

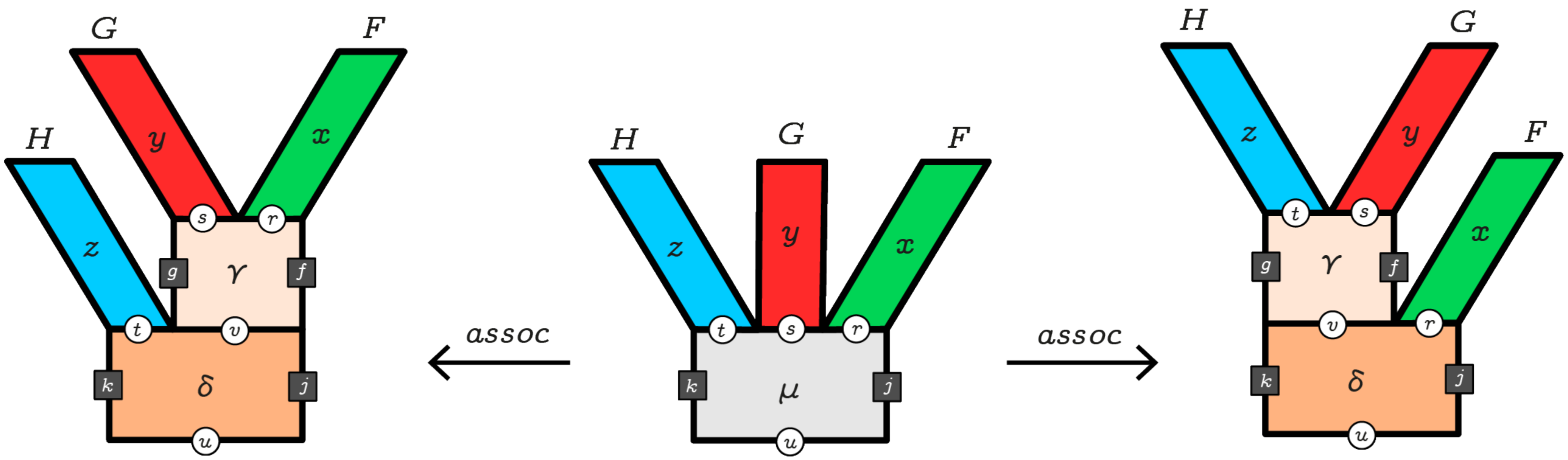


What oplax monoidal means...

The convolution products are related by associativity maps such as

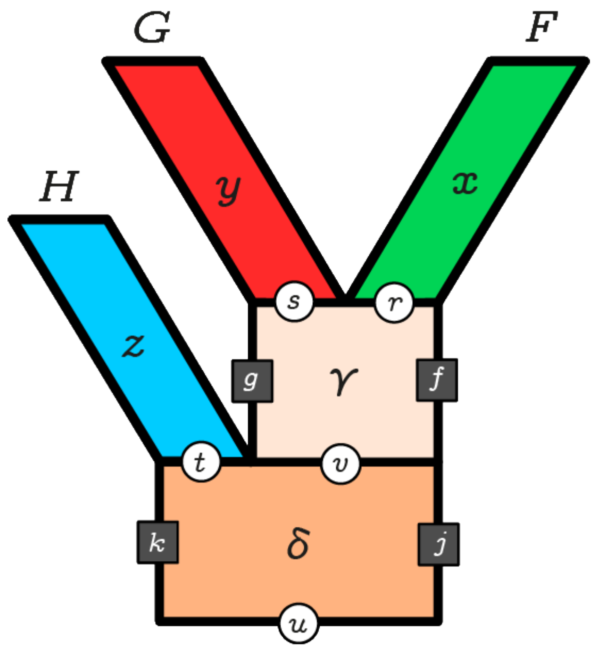
$$H * (G * F) \xleftarrow{assoc} (H * G * F) \xrightarrow{assoc} (H * G) * F$$

which are **not reversible** in general, for the following reason:



What oplax monoidal means...

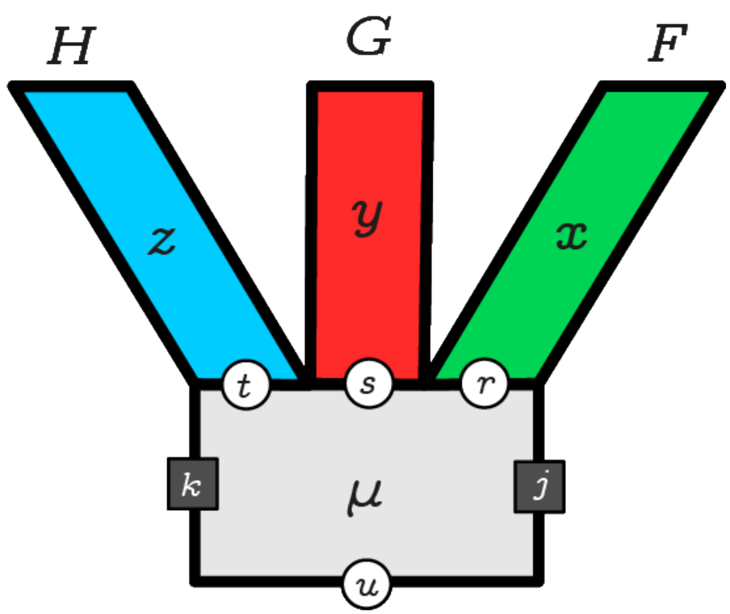
In a general double category $\mathbb{D}$, not every composite shape of the form



defining an element of the presheaf $H * (G * F)$ at instance $u : A \longrightarrow A'$

What oplax monoidal means...

is equivalent modulo $\sim$ in $\mathbb{D}$ to a ternary shape of the form



defining an element of $H * G * F$ at the same instance $u : A \longrightarrow A'$.

Cylindrical decomposition property

A sufficient condition to ensure strong associativity

Towards strong associativity

We want to find a **sufficient condition** on a double category

$$(\,\mathbb{D}\,,\, h_n : \mathbb{D}_n \longrightarrow \mathbb{D}_1\,)$$

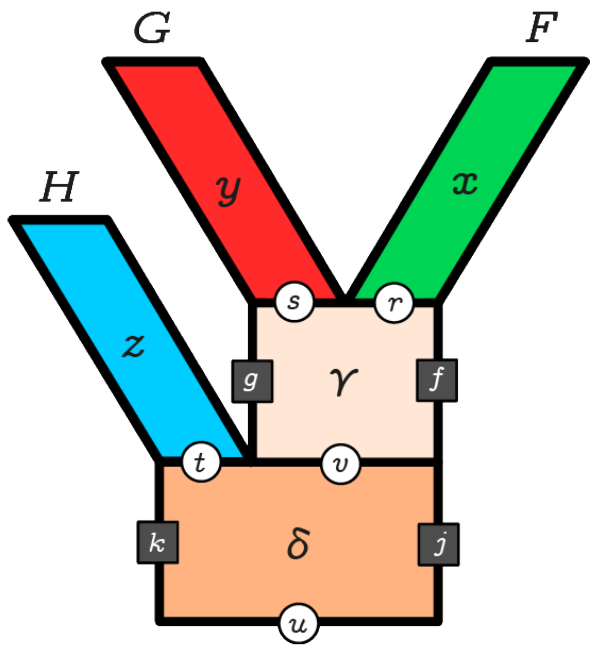
ensuring that the **associativity maps** of the convolution product

$$H * (G * F) \xleftarrow{\;assoc\;} (H * G * F) \xrightarrow{\;assoc\;} (H * G) * F$$

are **reversible**.

Towards strong associativity

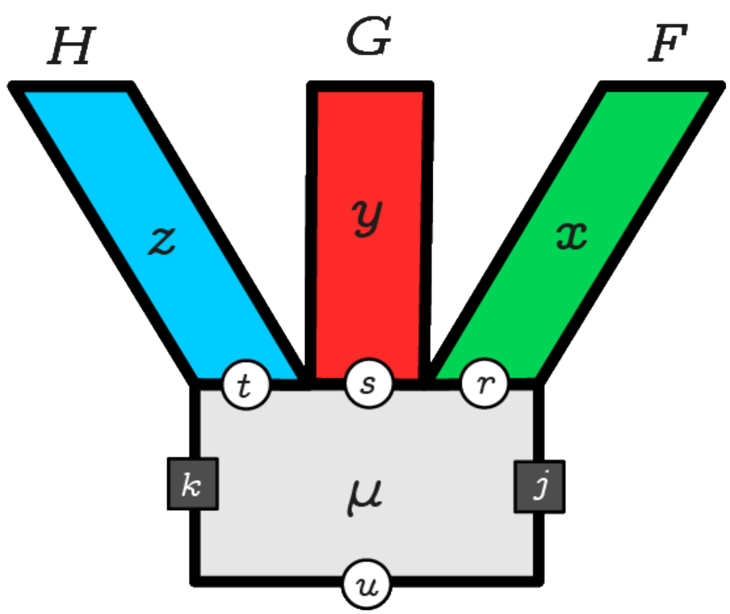
In particular, this requires to show that every **composite shape**



defining an element of the presheaf $H * (G * F)$ at instance $u : A \longrightarrow A'$

Towards strong associativity

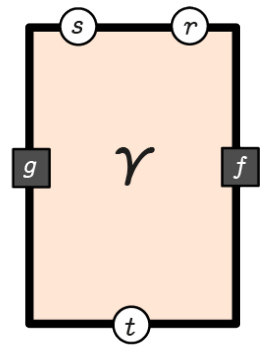
is equivalent modulo $\sim$ in $\mathbb{D}$ to a ternary shape of the form



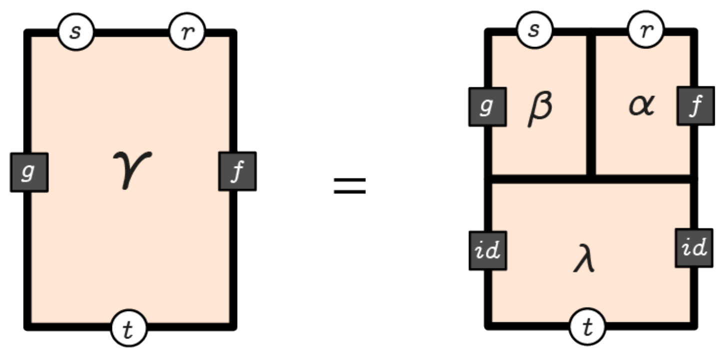
defining an element of $H * G * F$ at the same instance $u : A \longrightarrow A'$.

Towards strong associativity

Suppose that every double cell of the form

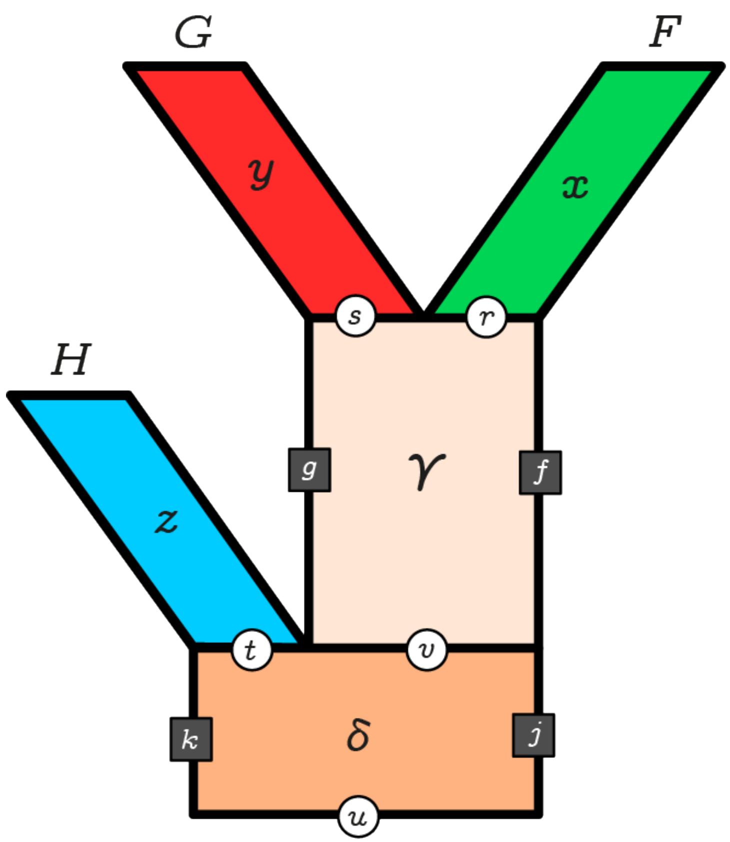


factors in the following way:



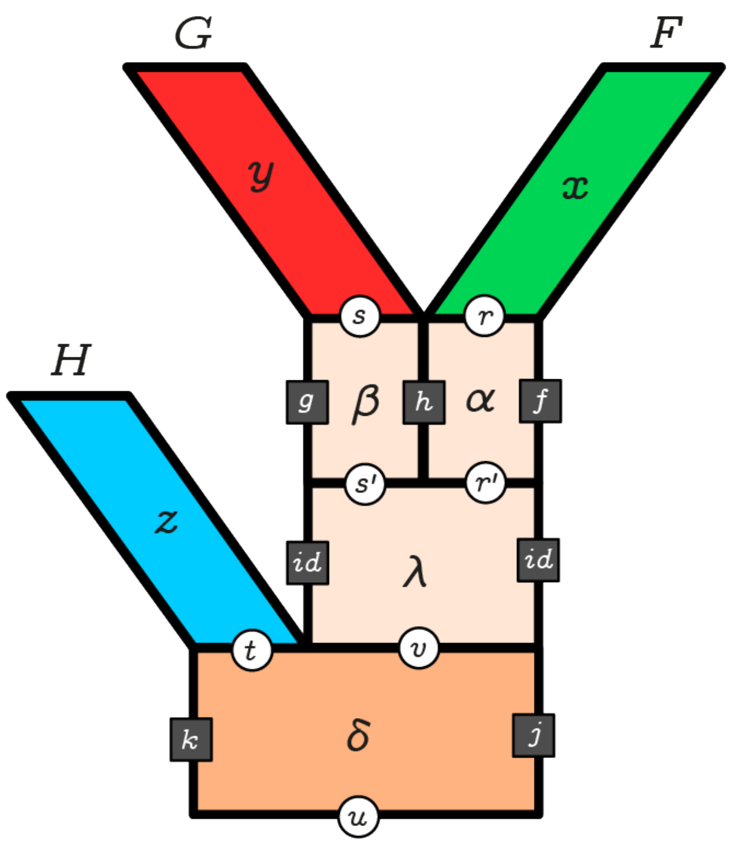
Towards strong associativity

In that case, one can rewrite the original composite shape



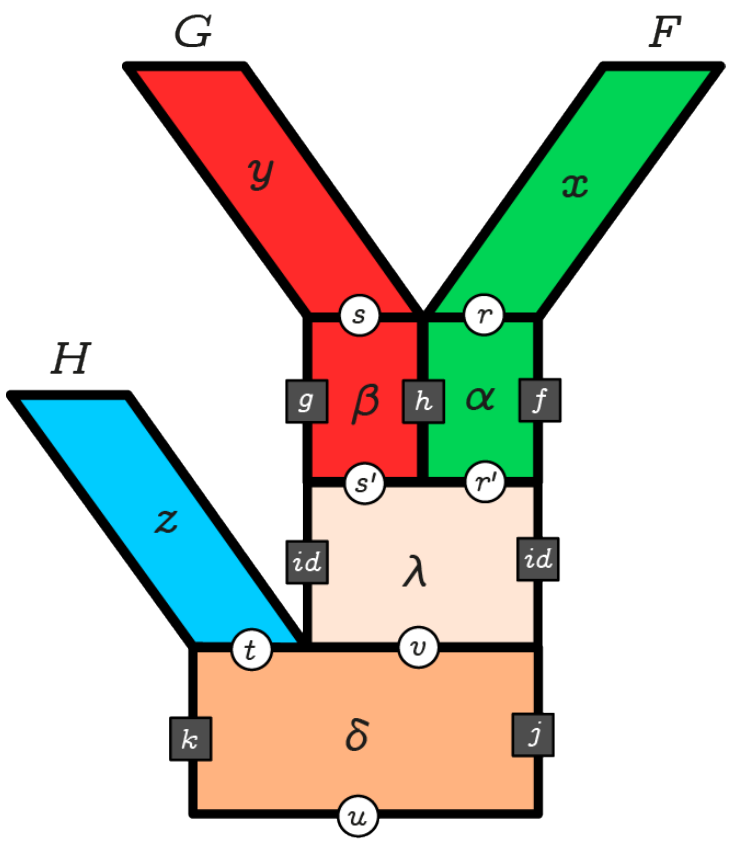
Towards strong associativity

We then into the shape where the cell γ has been factored:



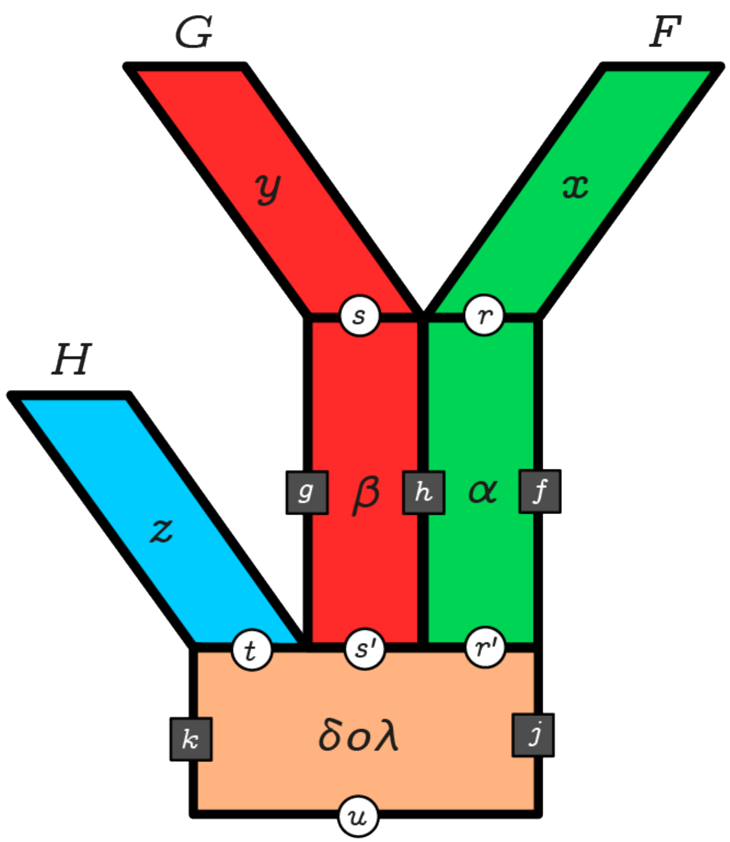
Towards strong associativity

then into the equivalent shape using the equivalence relation $\sim$



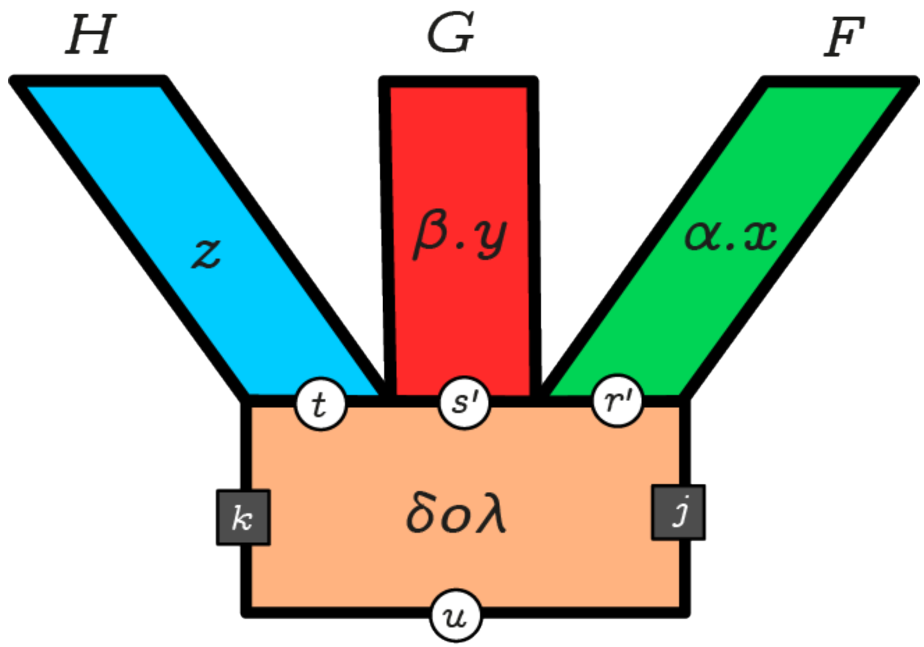
Towards strong associativity

then into the equal shape by vertical composition:



Towards strong associativity

and finally in the ternary shape we were looking for:



The cylinder categories

Every double category $\mathbb{D}$ comes equipped with a family of categories

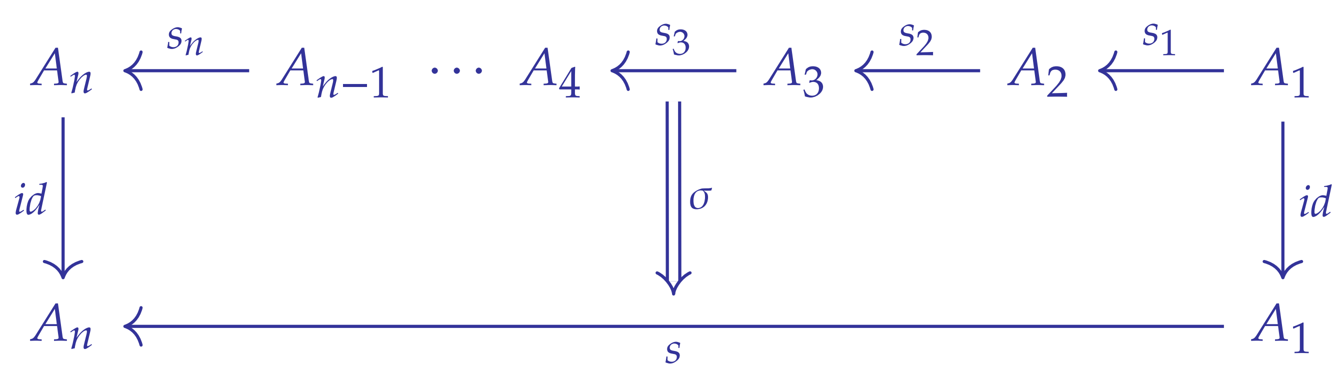
$$\mathbf{Cyl}_{\mathbb{D}}\,[n]$$

called **cylinder categories** and defined in the following way:

▷ the objects of $\mathbf{Cyl}_{\mathbb{D}}\,[n]$ are the tuples

$$\sigma \quad = \quad (\, s_n\,,\, \ldots\,,\, s_1\,,\, s\,,\, \sigma : s_n \diamond_h \cdots \diamond_h s_1 \Rightarrow s\,)$$

defining a **globular cell** of the form



The cylinder categories

▷ given globular cells

$$\sigma \quad = \quad (\, s_n \,,\, \ldots \,,\, s_1 \,,\, s \,,\, \sigma : s_n \diamond_h \cdots \diamond_h s_1 \;\Rightarrow\; s \,)$$

$$\tau \quad = \quad (\, t_n \,,\, \ldots \,,\, t_1 \,,\, t \,,\, \tau : t_n \diamond_h \cdots \diamond_h t_1 \;\Rightarrow\; t \,)$$

the morphisms of $\mathbf{Cyl}_{\mathbb{D}}\,[n]$ of the form

$$(\varphi_n, \cdots, \varphi_1, \varphi) \quad : \quad \sigma \longrightarrow \tau$$

are tuples consisting of a map in $\mathbb{D}_n$

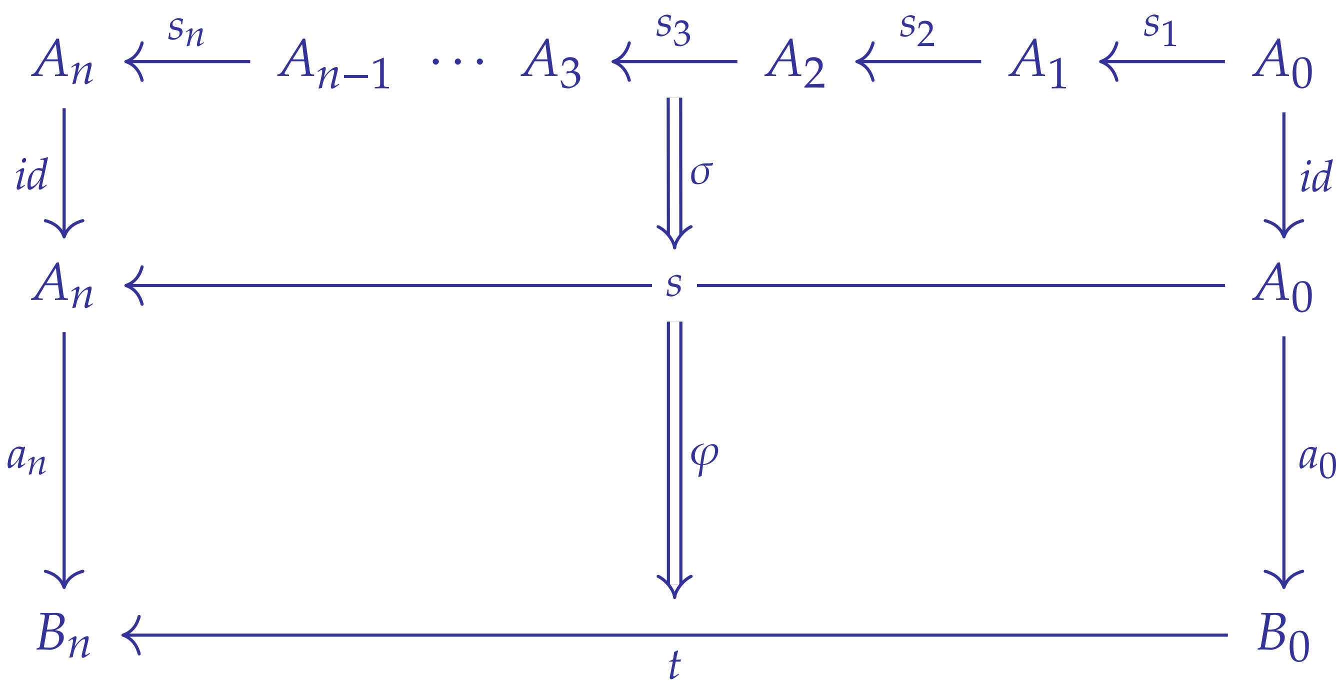
$$(\varphi_n, \ldots, \varphi_1) \quad : \quad (\, s_n \,,\, \ldots \,,\, s_1 \,) \Rightarrow (\, t_n \,,\, \ldots \,,\, t_1 \,)$$

and of a double cell

$$\varphi \quad : \quad s \Rightarrow t$$

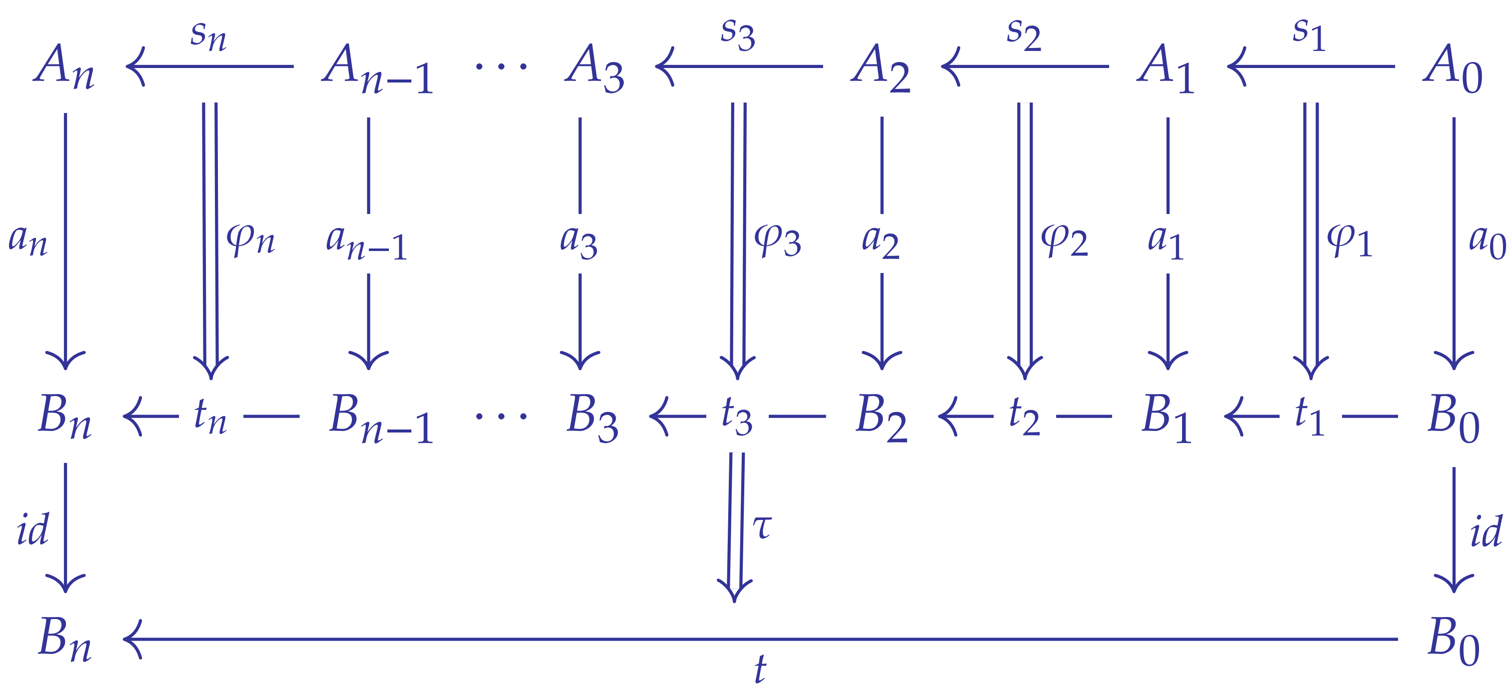
The cylinder categories

such that the double cell $\varphi \circ \sigma$ depicted below



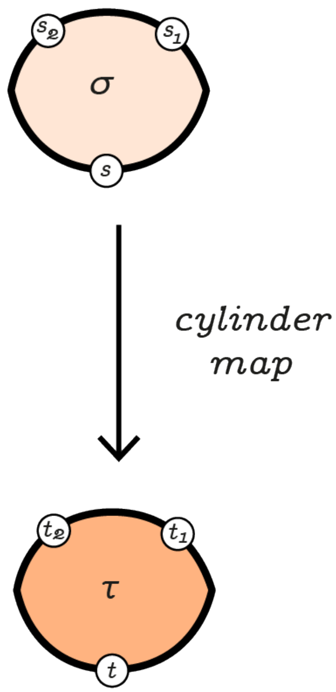
The cylinder categories

is equal to the double cell $\tau \circ (\varphi_n \diamond_h \ldots \diamond_h \varphi_1)$ depicted below



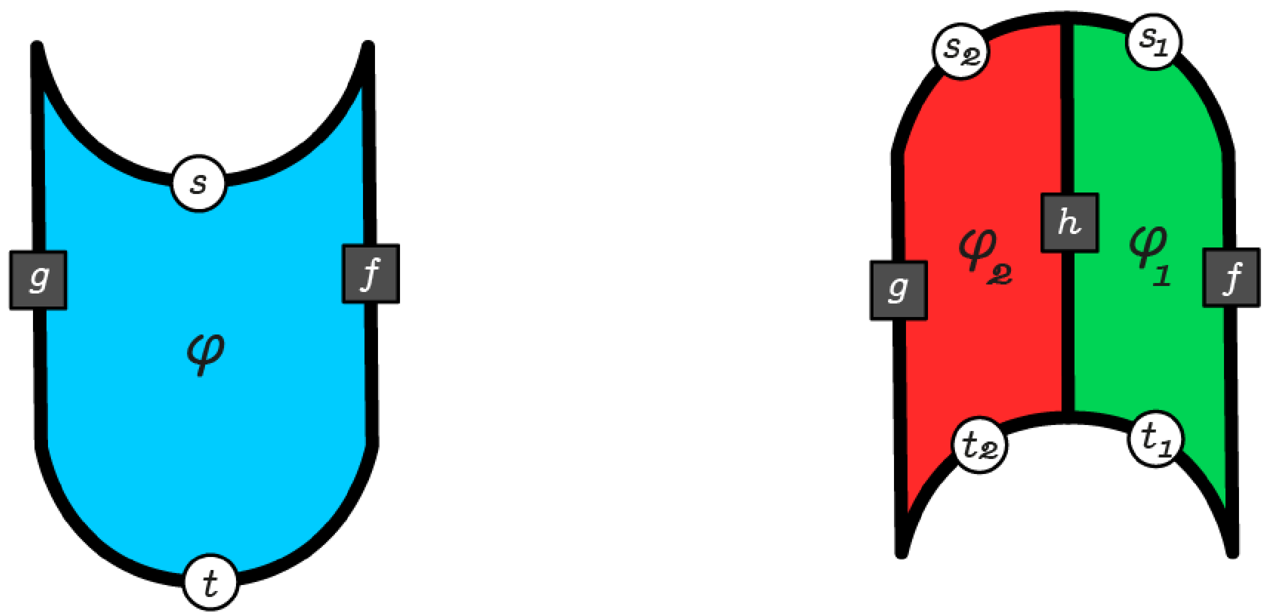
The cylinder categories

Typically, a map of the cylinder category $\mathbf{Cyl}_{\mathbb{D}}\,[2]$ of the form



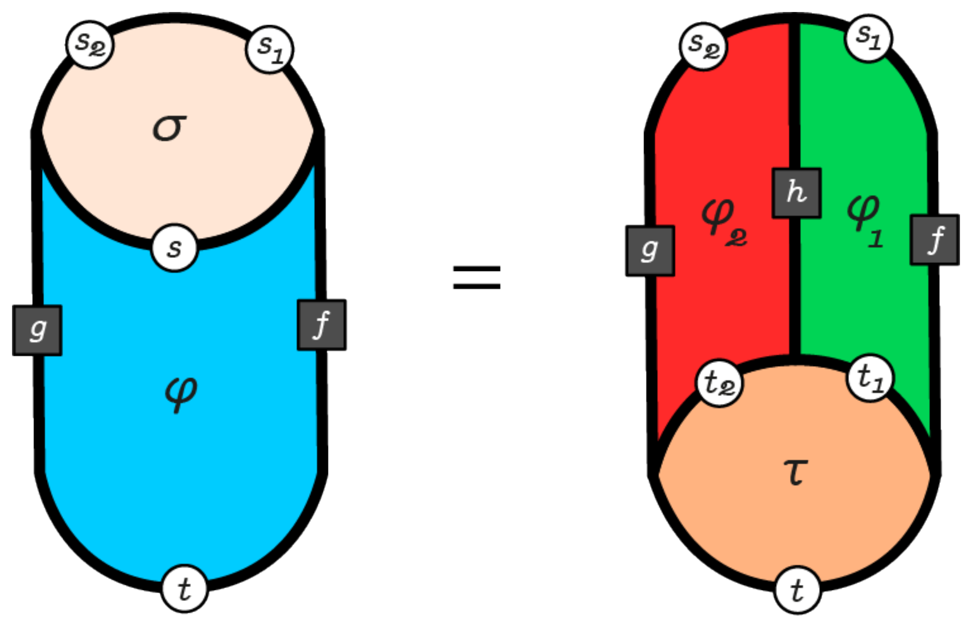
The cylinder categories

is defined as a tuple of double cells $\varphi, \varphi_1, \varphi_2$ of double cells of the form



The cylinder categories

satisfying the equation:



This justifies to see every $\mathbf{Cyl}_{\mathbb{D}}[n]$ as a **cylinder category** of $\mathbb{D}$.

The cylindrical decomposition property

Key observation: each composition functor

$$h_n \quad : \quad \mathbb{D}_n \longrightarrow \mathbb{D}_1$$

of the double category $\mathbb{D}$ factors as

$$\mathbb{D}_n \longrightarrow \mathbf{Cyl}_{\mathbb{D}}[n] \xrightarrow{\pi_n} \mathbb{D}_1$$

Definition. A double category $\mathbb{D}$ satisfies

the n-cylindrical decomposition property (n-CDP)

when the functor

$$\mathbf{Cyl}_{\mathbb{D}}[n] \xrightarrow{\pi_n} \mathbb{D}_1$$

is an opfibration (not necessarily discrete).

Main theorem

Theorem [Behr, PAM, Zeilberger in this FSCD]

Suppose that a double category $\mathbb{D}$ satisfies

the n-cylindrical decomposition property (n-CDP)

for all $n \in \mathbb{N}$.

In that case, the convolution product defines a functor

$$* \quad : \quad \widehat{\mathbb{D}} \times \widehat{\mathbb{D}} \longrightarrow \widehat{\mathbb{D}}$$

which equips the category of covariant presheaves

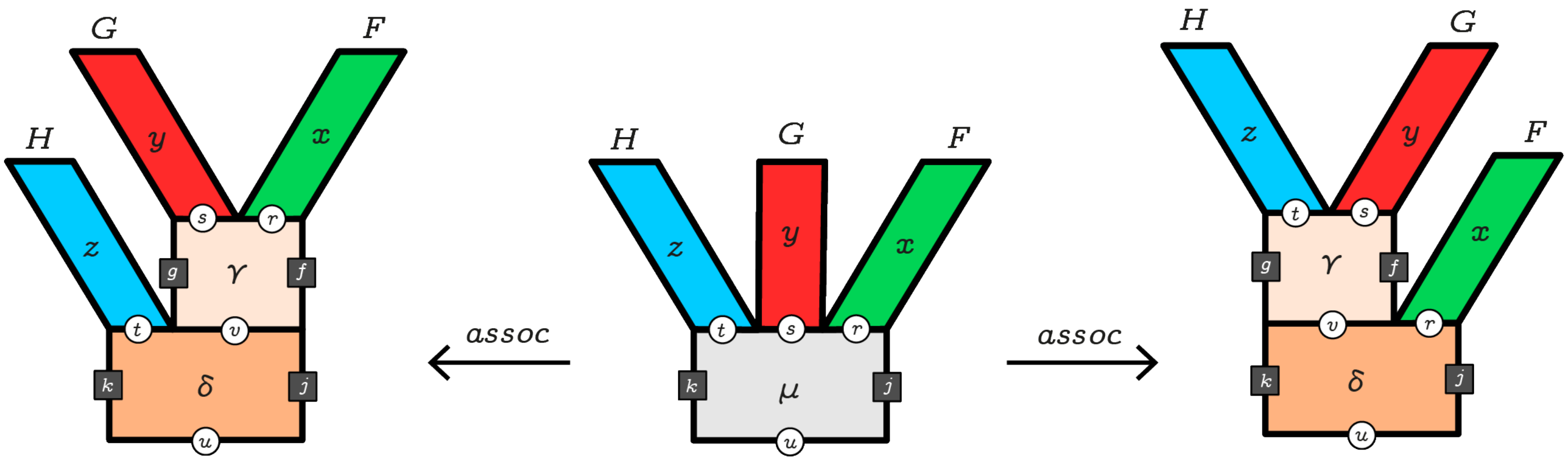
$$\widehat{\mathbb{D}} \quad := \quad [\mathbb{D}_1, \mathbf{Set}]$$

with the structure of an **strong monoidal closed category**.

Main theorem

In particular, the associativity maps are **reversible** in that case:

$$H * (G * F) \xleftarrow{assoc} (H * G * F) \xrightarrow{assoc} (H * G) * F$$



Reversibility comes from the cylindrical decomposition property of $\mathbb{D}$.

Illlustrations

The theorem applies to the following situations:

- ▷ every **bicategory** $\mathbb{D} = \mathcal{W}$ satisfies n-CDP,
- ▷ every **framed bicategory** $\mathbb{D} = \mathcal{W}$ satisfies n-CDP for $n \geq 1$,
- ▷ the double category $\mathbb{D} = \mathbf{DPO}$ satisfies n-CDP for $n \geq 1$.
- ▷ the double category $\mathbb{D} = \mathbf{LTRS}$ of term rewriting satisfies n-CDP.

More generally, the theorem enables us to use the convolution product for a number of categorical graph and term rewriting frameworks.

Categorifying rule algebras

Composing representable presheaves by convolution

Categorification of rule algebras

One main ingredient of **rule algebras** is the following equation

$$\delta(r) \star \delta(s) \quad = \quad \sum_{\mu \in \mathcal{M}_r(s)} \delta(r_\mu s)$$

where

- $\mathcal{M}_r(s)$ is the set of **admissible matches** of rule r into rule s
- $r_\mu s$ denotes one possible way to get a **composite rule** from r and s.

Similarly, we want to find sufficient conditions on $\mathbb{D}$ such that

$$\hat{\Delta}_r * \hat{\Delta}_s \quad = \quad \sum_{\mu \in \mathcal{M}_r(s)} \hat{\Delta}_{r_\mu s}$$

where the sum is now **set-theoretic union**.

Multi-sums

Suppose that A and B are objects in a category $\mathbf{C}$.

Definition. A **multi-sum** of A and B is a family of cospans

$$(A \xrightarrow{a_i} U_i \xleftarrow{b_i} B)_{i \in I}$$

such that for any cospan

$$A \xrightarrow{f} X \xleftarrow{g} B$$

there exists a unique $i \in I$ and a unique morphism

$$[f, g] \quad : \quad U_i \xrightarrow{f} X$$

such that

$$f = [f, g] \circ a_i \quad \text{and} \quad g = [f, g] \circ b_i.$$

Categorification of rule algebras

Theorem. Assume $\mathbb{D}$ is a small double category satisfying

- ▷ the vertical category $\mathbb{D}_0$ has multi-sums,
- ▷ the source and target functors $S, T : \mathbb{D}_1 \to \mathbb{D}_0$ are opfibrations.

In that case, the convolution product of two representable presheaves is isomorphic to the sum of representables

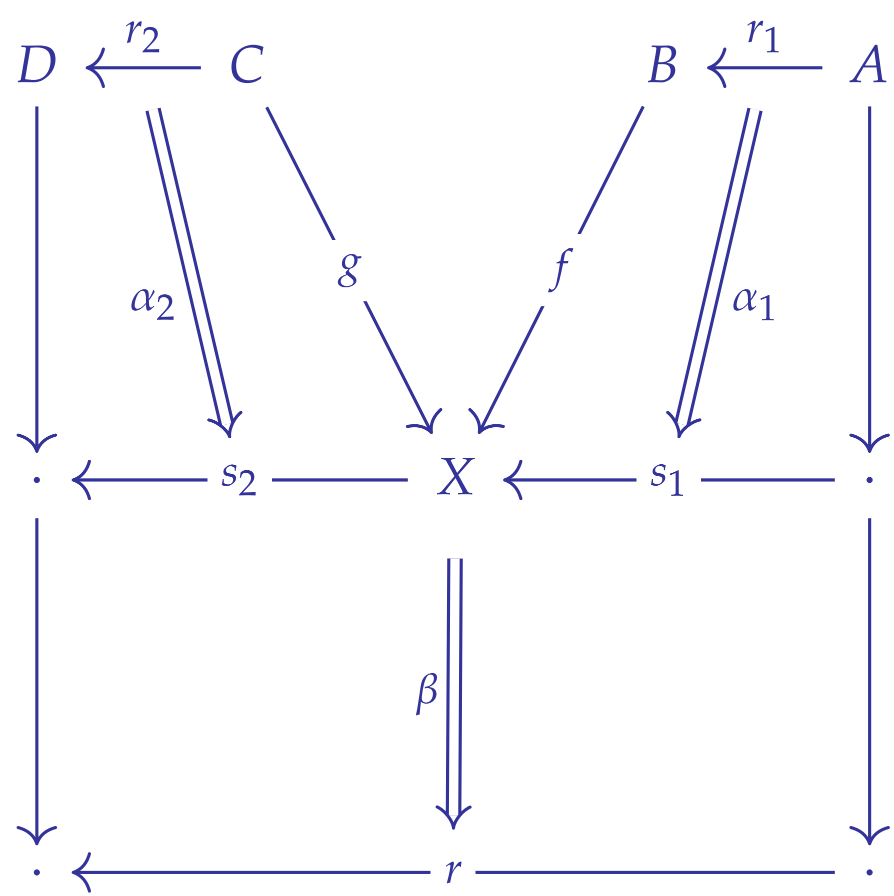
$$\hat{\Delta}_{r_2} * \hat{\Delta}_{r_1} \quad \cong \quad \sum_{i \in I} \quad \hat{\Delta}_{r_2\langle c_i \rangle \diamond_h \langle b_i \rangle r_1}$$

where the multi-sum of B and C is given by a family of cospans

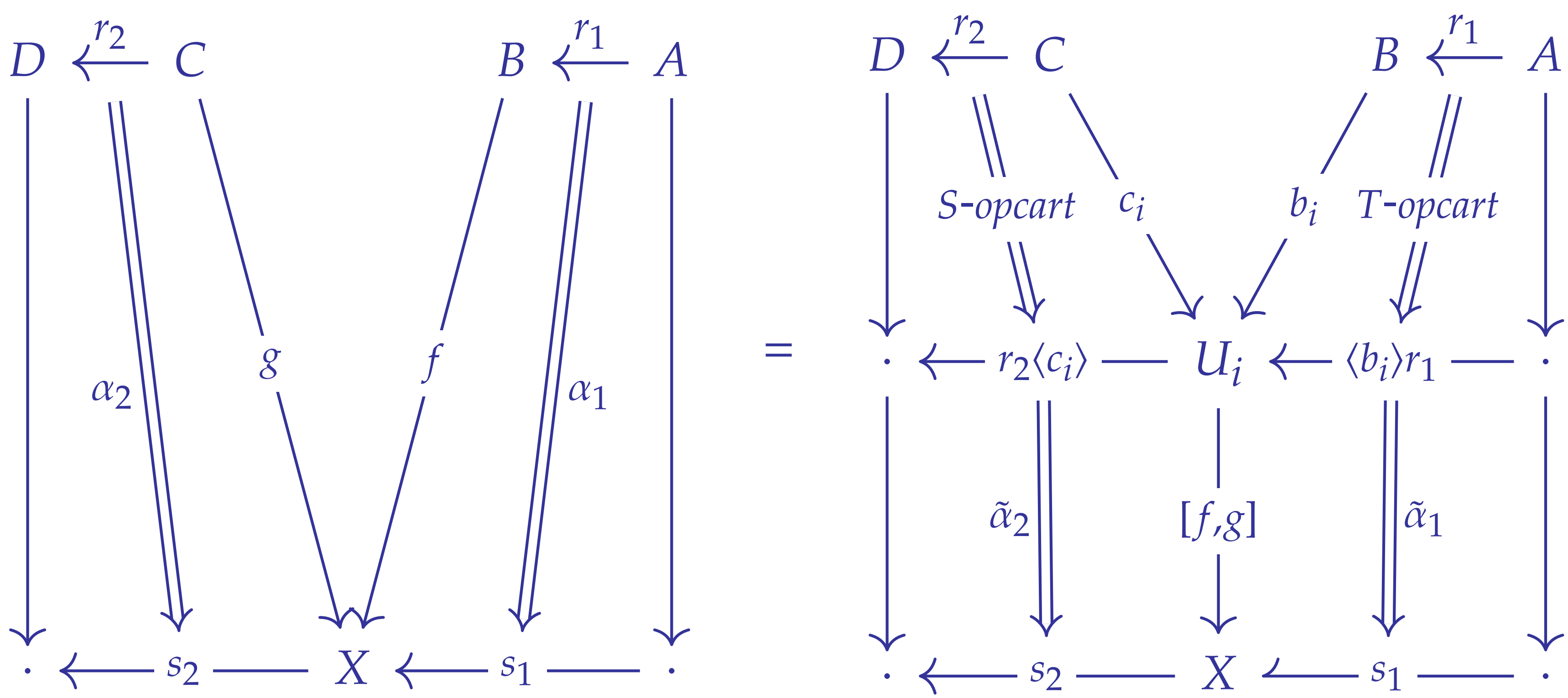
$$(\ B \xrightarrow{\ b_i\ } U_i \xleftarrow{\ c_i\ } C \ )_{i \in I}$$

and where $r_2\langle c_i \rangle$ denotes the S-pushforward of r_2 along c_i
and $\langle b_i \rangle r_1$ denotes the T-pushforward of r_1 along b_i.

Sketch of the proof



Sketch of the proof



Illustration

From this, we obtain that the convolution product with itself

$$\hat{\Delta}_r * \hat{\Delta}_r \quad : \quad \mathbb{D}_1 \longrightarrow \mathbf{Set}$$

of the representable presheaf

$$\hat{\Delta}_r \quad : \quad \mathbb{D}_1 \longrightarrow \mathbf{Set}$$

is isomorphic to **the sum of two representable presheaves**

$$\hat{\Delta}_r * \hat{\Delta}_r \quad \cong \quad \hat{\Delta}_{r_1} + \hat{\Delta}_{r_2}$$

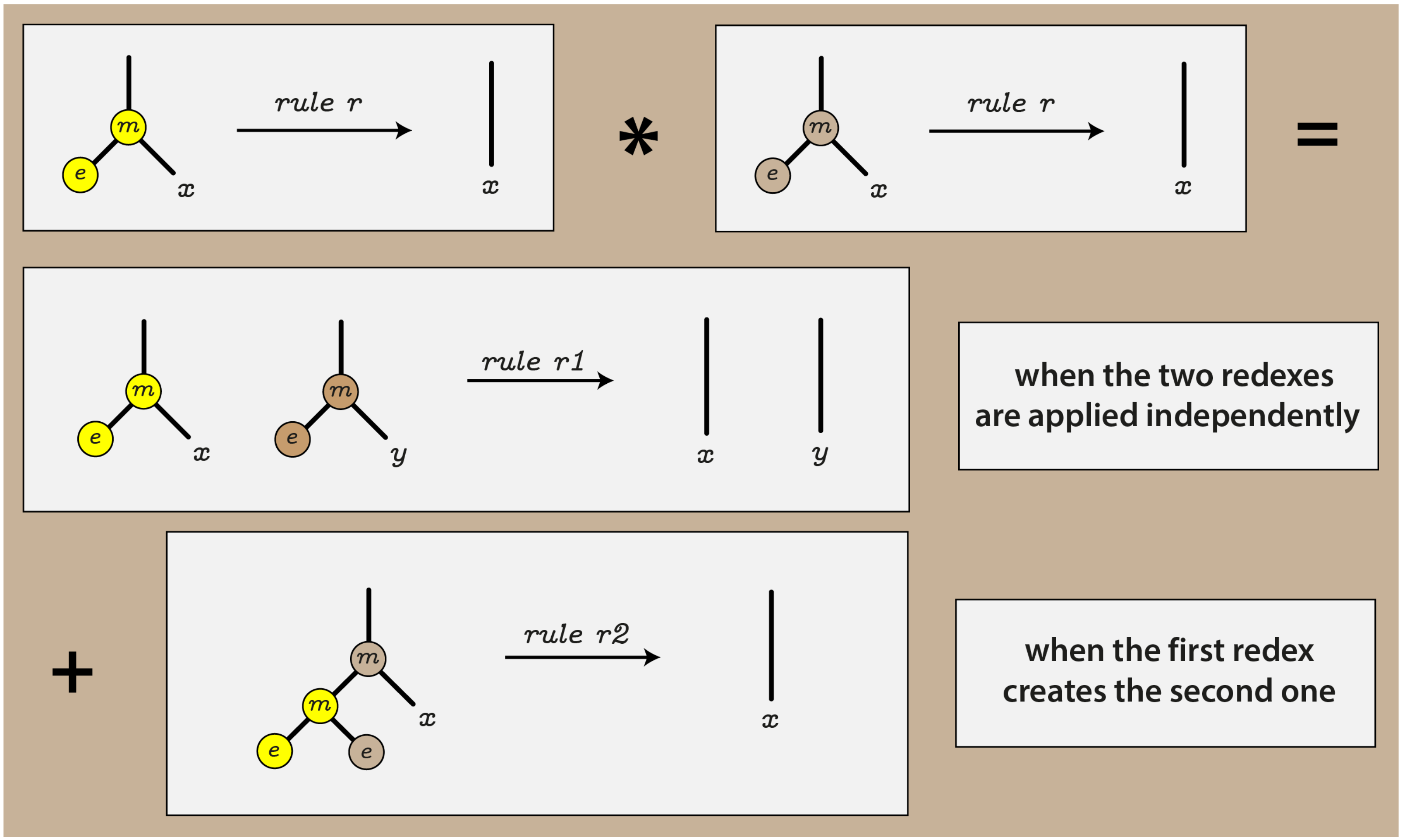
associated to the rewrite rules r_1 and r_2

$$\lambda x.m(e,x) : \mathbb{O} \multimap \mathbb{O} \ , \ \lambda y.m(e,y) : \mathbb{O} \multimap \mathbb{O} \xrightarrow{\quad rule\, r_1 \quad} \lambda x.x : \mathbb{O} \multimap \mathbb{O} \ , \ \lambda y.y : \mathbb{O} \multimap \mathbb{O}$$

$$\lambda x.m(m(e,e),x) : \mathbb{O} \multimap \mathbb{O} \xrightarrow{\quad rule\, r_2 \quad} \lambda x.x : \mathbb{O} \multimap \mathbb{O}$$

in the double category $\mathbb{D} = \mathbf{LTRS}$.

Illustration



Conclusion and future works

What we have done in the FSCD paper:

- a **categorification of tracelets** and rule algebras
- an **axiomatic and unified framework** for term and graph rewriting
- a **convolution product** $G, F \mapsto G * F$ for double categories
- a **cylindrical decomposition property** for strong associativity

A few topics we like to think about:

- make sure the framework works for **higher-order rewrite systems**
- categorify the more **quantitative and stochastic** aspects of tracelets
- improve our understanding of **causality** in rewriting systems

Thank you!

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