Making concurrent systems probabilistic

Samy Abbes (Univ. Paris Cité/IRIF)

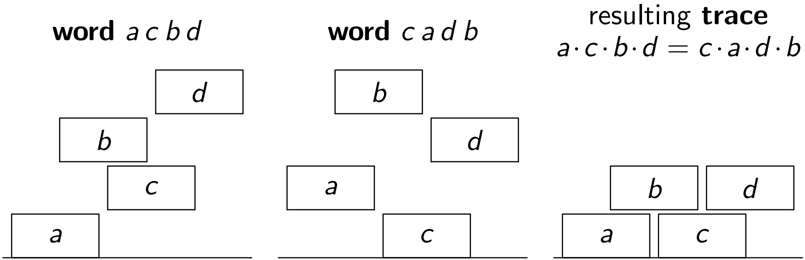
joint work with **Vincent Jugé** (Univ. Gustave Eiffel/LIGM)
Jean Mairesse (CNRS/LIP6)

Workshop *Pomsets and Related Structures (RaPS)*

April 24th, 2024

Outline

Framework (1): an example of trace monoid



- Alphabet $\Sigma = \{a, b, c, d\}$ of **generators**
- **Traces** $\iff$ heaps obtained in a “Tetris”-like way by letting generators fall
- Commutation relations between generators:

$$a \cdot c = c \cdot a \qquad a \cdot d = d \cdot a \qquad b \cdot d = d \cdot b$$

Framework (1): an example of trace monoid

Structure

- Heaps are in bijection with the elements of the trace monoid:

$$\mathcal{M} = \Big\langle \underbrace{a, b, c, d}_{\text{generators}} \;\Big|\; \underbrace{ac = ca,\; ad = da,\; bd = db}_{\text{commutation relations}} \Big\rangle$$

Elements of $\mathcal{M}$ are **words on** $\{a, b, c, d\}$ up to the congruence generated by the 3 **commutation relations**

Framework (1): an example of trace monoid

Structure

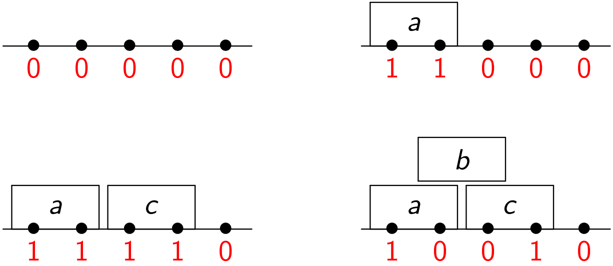
- Heaps are in bijection with the elements of the trace monoid:

$$\mathcal{M} = \big\langle \underbrace{a, b, c, d}_{\text{generators}} \;\big|\; \underbrace{ac = ca,\ ad = da,\ bd = db}_{\text{commutation relations}} \big\rangle$$

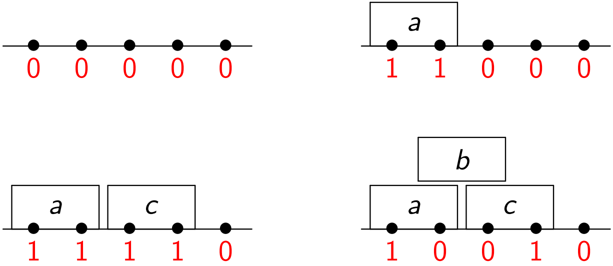
  Elements of $\mathcal{M}$ are **words on** $\{a, b, c, d\}$ up to the congruence generated by the 3 **commutation relations**

- $\mathcal{M}$ comes with:
  - a *monoid structure*, *i.e.*, an associative composition $(x, y) \in \mathcal{M} \times \mathcal{M} \mapsto x \cdot y$ and a neutral element ε (the empty trace)
  - a *length function* $x \in \mathcal{M} \mapsto |x| \in \mathbb{N}$, number of letters in x
  - a partial order relation: $x \leq y \iff \exists z \in \mathcal{M} \quad y = x \cdot z$

Framework (2): an example of action of trace monoid

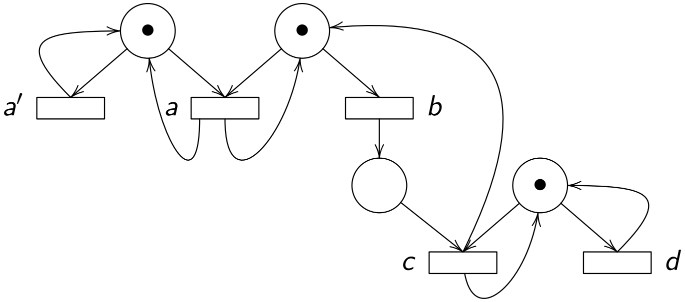


Framework (2): an example of action of trace monoid



$$\mathcal{M} = \langle\, a, b, c, d \mid ac = ca,\ ad = da,\ bd = db \,\rangle$$
$$X \subseteq \{0, 1\}^5$$

Framework (2): an example of 1-safe Petri net



$X = \{0,1\}$ the two markings of the net

$\mathcal{M} = \langle \underbrace{a, a', b, c, d}_{\text{transitions of the net}} \mid \underbrace{a'b = ba', \ldots, bd = db}_{{}^\bullet t^\bullet \cap {}^\bullet t'^\bullet = \emptyset} \rangle$

} True-concurrent semantics: action of $\mathcal{M}$ on X

Franework (2): action of $\mathcal{M}$ on X

A trace monoid $\mathcal{M}$ acts on a set of **states** X with an additional **sink** state $\bot$

$$\begin{cases} \big(X \cup \{\bot\}\big) \times \mathcal{M} & \to & X \cup \{\bot\} \\ (\alpha, x) & \mapsto & \alpha \cdot x \end{cases}$$

$$\alpha \cdot (x \cdot y) = (\alpha \cdot x) \cdot y, \qquad \alpha \cdot \varepsilon = \varepsilon, \qquad \bot \cdot x = \bot$$

Franework (2): action of $\mathcal{M}$ on X

A trace monoid $\mathcal{M}$ acts on a set of **states** X with an additional **sink** state $\perp$

$$\begin{cases} \big(X \cup \{\perp\}\big) \times \mathcal{M} & \to & X \cup \{\perp\} \\ (\alpha, x) & \mapsto & \alpha \cdot x \end{cases}$$

$$\alpha \cdot (x \cdot y) = (\alpha \cdot x) \cdot y, \qquad \alpha \cdot \varepsilon = \varepsilon, \qquad \perp \cdot x = \perp$$

Semantics

- $\alpha \cdot x \neq \perp \iff$
  $\begin{cases} x \text{ is an **execution** of the system from the **initial state** } \alpha \\ \alpha \cdot x \text{ is the **state** reached after playing } x \in \mathcal{M} \text{ from } \alpha \end{cases}$
- $\alpha \cdot x = \perp \iff$ playing x from α is **forbidden**

Outline

Framework

Looking for the probabilistic parameters

Several examples

Semantics and theory

Randomizing the executions

- **Data:** concurrent system $(\mathcal{M}, X, \perp)$
- **Goal:** determine the probabilistic parameters $\lambda_\alpha(a)$ as the "probability of playing the letter $a \in \Sigma$ from the state $\alpha \in X$"

Randomizing the executions

- **Data:** concurrent system $(\mathcal{M}, X, \perp)$
- **Goal:** determine the probabilistic parameters $\lambda_\alpha(a)$ as the "probability of playing the letter $a \in \Sigma$ from the state $\alpha \in X$"

With the intuitive properties:

$$\alpha \cdot a = \perp \iff \lambda_\alpha(a) = 0$$

$$a \cdot b = b \cdot a \implies \underbrace{\lambda_\alpha(a)\lambda_{\alpha \cdot a}(b)}_{\substack{\text{probability of} \\ \text{playing } a \cdot b \text{ from } \alpha}} = \underbrace{\lambda_\alpha(b)\lambda_{\alpha \cdot b}(a)}_{\substack{\text{probability of} \\ \text{playing } b \cdot a \text{ from } \alpha}}$$

Randomizing the executions

- **Data:** concurrent system $(\mathcal{M}, X, \perp)$
- **Goal:** determine the probabilistic parameters $\lambda_\alpha(a)$ as the "probability of playing the letter $a \in \Sigma$ from the state $\alpha \in X$"

Without concurrency:

$$\forall \alpha \in X \quad \sum_{a \in \Sigma} \lambda_\alpha(a) = 1$$

What is the correct *normalization condition* with concurrency?

Outline

Basic case: the trace monoid with one state

Trivial action $\{*\} \times \mathcal{M} \to \{*\}$: as if there was no state
$\mathcal{M} = \langle a, b, c, d \mid ac = ca,\ ad = da,\ bd = db \rangle$
Denote by the same letters a, b, c, d the sought probabilistic parameters. Then:

$$1 - a - b - c - d + \underbrace{ac + ad + bd}_{\substack{\text{commutation cliques}\\ \text{of order 2}}} = 0$$

Basic case: the trace monoid with one state

Trivial action $\{*\} \times \mathcal{M} \to \{*\}$: as if there was no state
$\mathcal{M} = \langle a, b, c, d \mid ac = ca,\ ad = da,\ bd = db \rangle$
Denote by the same letters a, b, c, d the sought probabilistic parameters. Then:

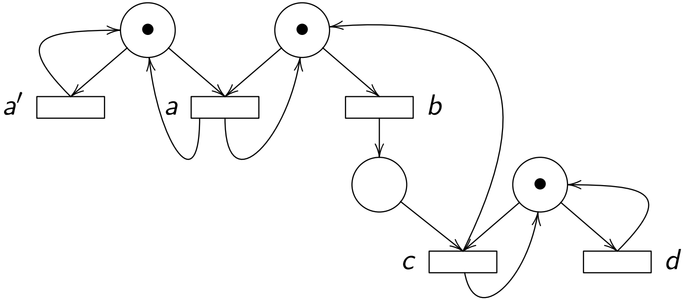
$$1 - a - b - c - d + \underbrace{ac + ad + bd}_{\text{commutation cliques of order 2}} = 0$$

Particular case: all parameters are equal for the *uniform measure*

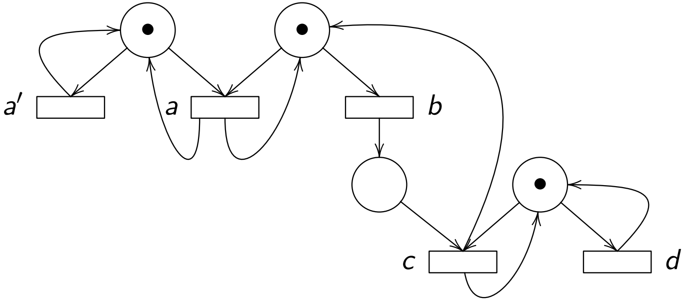
root of the Möbius polynomial of $\mathcal{M}$: $1 - 4t + 3t^2 = 0 \longrightarrow t = \dfrac{1}{3}$ (!!)

Interpretation: All traces of a given length k have the same probability $\left(\frac{1}{3}\right)^k$ to appear at the beginning of a random infinite execution

The Petri net example (1)



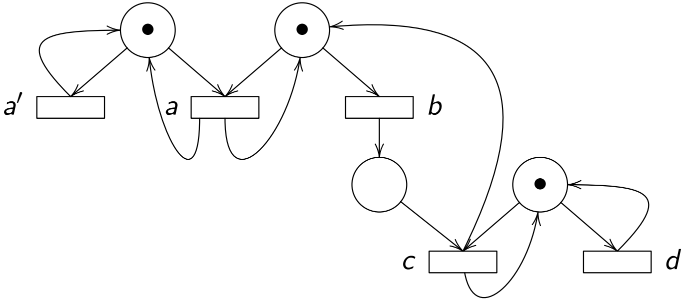
The Petri net example (1)



At marking 0: parameters a'_0, a_0, b_0, c_0, d_0

At marking 1: parameters a'_1, a_1, b_1, c_1, d_1

The Petri net example (1)

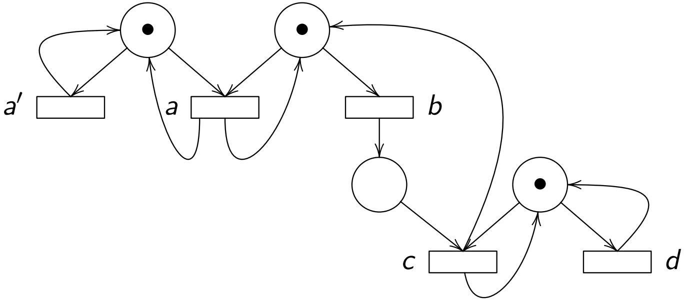


$$0 \cdot c = \bot \implies c_0 = 0$$

$$1 \cdot a = \bot \implies a_1 = 0$$

$$1 \cdot b = \bot \implies b_1 = 0$$

The Petri net example (1)

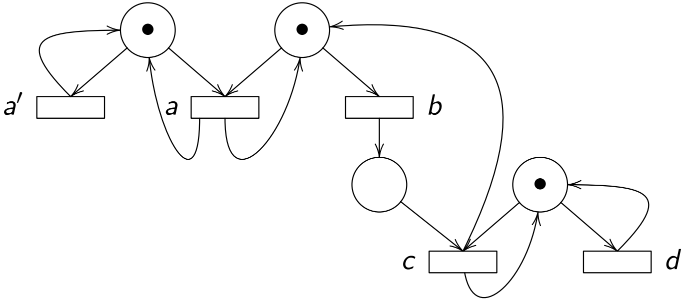


equality in $\mathcal{M}$: $a' \cdot b = b \cdot a' \implies a'_0 b_0 = b_0 a'_1 \implies a'_0 = a'_1$

equality in $\mathcal{M}$: $b \cdot d = d \cdot b \implies b_0 d_1 = d_0 b_0 \implies d_0 = d_1$

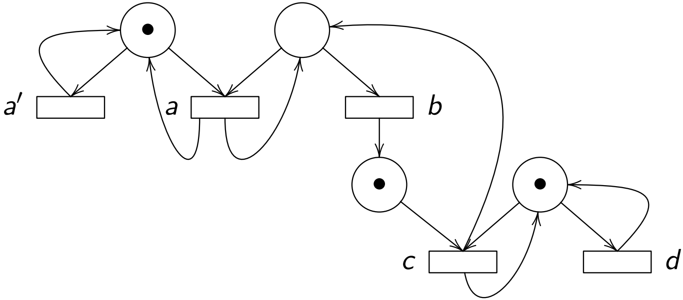
5 parameters only: $a' = a'_0, \quad a = a_0, \quad b = b_0, \quad c = c_1, \quad d = d_1$

The Petri net example (1)



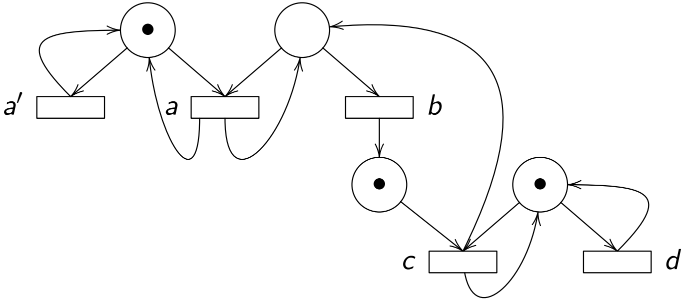
$$\text{At 0:}\quad 1 - \underbrace{a' - a - b - d}_{\substack{\text{commutation cliques}\\ \text{of order 1 that can}\\ \text{play at marking 0}}} + \underbrace{a'b + a'd + ad + bd}_{\textit{idem}\text{ order 2}} - \underbrace{a'bd}_{\textit{idem}\text{ order 3}} = 0$$

The Petri net example (1)



At 1: $1 - a' - c - d + a'c + a'd = 0$

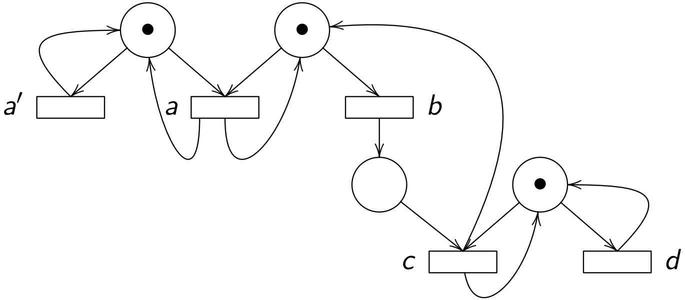
The Petri net example (1)



Question: is there a solution to satisfy both normalization conditions?

$$(1 - a' - a - b + a'b)(1 - d) = 0$$
$$(1 - c - d)(1 - a') = 0$$

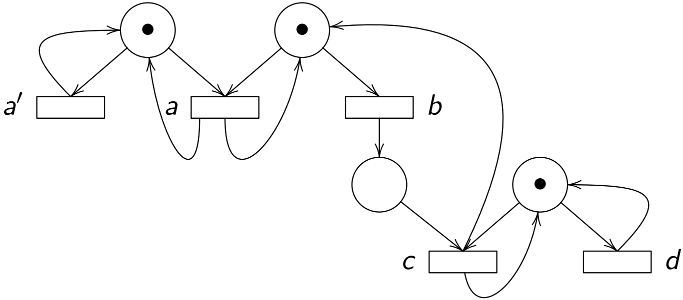
The Petri net example (2): the uniform measure



the Möbius matrix: $M(t) = \begin{pmatrix} 1-3t+2t^2 & -t+2t^2-t^3 \\ -t+t^2 & 1-2t+t^2 \end{pmatrix} \begin{matrix} 0 \\ 1 \end{matrix}$

$$\theta(t) = \det M(t) = (1-t)^3(1-2t-t^2) \longrightarrow \boxed{r = \sqrt{2}-1}$$

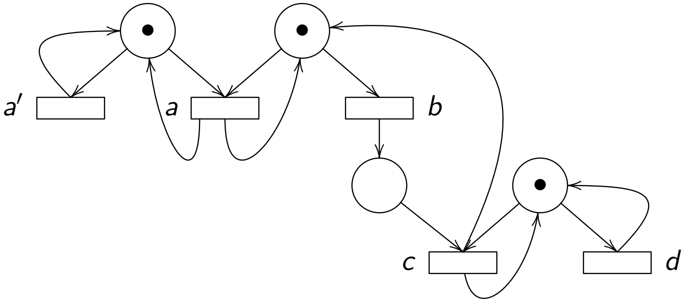
The Petri net example (2): the uniform measure



$$M(r) = \begin{pmatrix} 10 - 7\sqrt{2} & 2(7 - 5\sqrt{2}) \\ 4 - 3\sqrt{2} & 2(3 - 2\sqrt{2}) \end{pmatrix}$$

$$\ker M(r) = \mathrm{Vect}(u) \quad \text{with } u = \begin{pmatrix} 2 \\ \sqrt{2} \end{pmatrix} = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \quad \text{a } \textit{positive} \text{ vector}$$

The Petri net example (2): the uniform measure

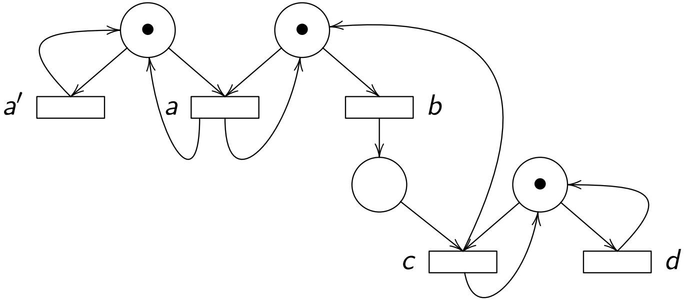


Use the following normalized parameters:

$$\text{for } \alpha = 0, 1 \text{ and } \sigma = a', a, b, c, d \quad \lambda_\alpha(\sigma) = r\frac{u_{\alpha\cdot\sigma}}{u_\alpha}$$

$$\text{and then for any } x = \sigma_1 \cdot \ldots \cdot \sigma_k \in \mathcal{M}: \quad f_\alpha(x) = r^{|x|}\frac{u_{\alpha\cdot x}}{u_\alpha}$$

The Petri net example (2): the uniform measure



$\sigma \in \Sigma$	a'	a	b	c	d
$\lambda(\sigma)$	$-1+\sqrt{2}$	$-1+\sqrt{2}$	$1-\frac{1}{2}\sqrt{2}$	$2-\sqrt{2}$	$-1+\sqrt{2}$
$\approx \lambda(\sigma)$	.4142	.4142	.2929	.5858	.4142

Outline

Framework

Looking for the probabilistic parameters

Several examples

Semantics and theory

Semantics

For a trace monoid $\mathcal{M}$:

- $\partial\mathcal{M}$ is the space of *infinite traces*
- for $x \in \mathcal{M}$ the *visual cylinder* of base x is $\uparrow x = \{\omega \in \partial\mathcal{M} \ : \ x \leq \omega\}$

Semantics

For a trace monoid $\mathcal{M}$:

- $\partial\mathcal{M}$ is the space of *infinite traces*
- for $x \in \mathcal{M}$ the *visual cylinder* of base x is $\uparrow x = \{\omega \in \partial\mathcal{M} \;:\; x \leq \omega\}$
- a *probability measure* ν on $\partial\mathcal{M}$ is entirely determined by the countable collection of values $\nu\big(\uparrow x\big)$, for $x \in \mathcal{M}$

Semantics

For a trace monoid $\mathcal{M}$:

- $\partial\mathcal{M}$ is the space of *infinite traces*
- for $x \in \mathcal{M}$ the *visual cylinder* of base x is $\uparrow x = \{\omega \in \partial\mathcal{M} \ : \ x \leq \omega\}$
- a *probability measure* ν on $\partial\mathcal{M}$ is entirely determined by the countable collection of values $\nu\big(\uparrow x\big)$, for $x \in \mathcal{M}$
- $\nu(\uparrow x)$ is the probability of seing x at the "beginning" of a random infinite execution

Semantics

For a concurrent system $(\mathcal{M}, X, \bot)$:

- a *Markov measure* is a collection $(\nu_\alpha)_{\alpha \in X}$ of probability measures on $\partial\mathcal{M}$ such that:

$$\forall \alpha \in X \quad \forall x \in \mathcal{M} \quad \nu_\alpha(\uparrow x) > 0 \iff \alpha \cdot x \neq \bot$$

and satisfying the chain rule: $\forall \alpha \in X \quad \forall x, y \in \mathcal{M}$

$$\nu_\alpha(\uparrow(x \cdot y)) = \nu_\alpha(\uparrow x)\nu_{\alpha \cdot x}(\uparrow y)$$

Semantics

For a concurrent system $(\mathcal{M}, X, \bot)$:

- a *Markov measure* is a collection $(\nu_\alpha)_{\alpha \in X}$ of probability measures on $\partial\mathcal{M}$ such that:

$$\forall \alpha \in X \quad \forall x \in \mathcal{M} \quad \nu_\alpha(\uparrow x) > 0 \iff \alpha \cdot x \neq \bot$$

and satisfying the chain rule: $\forall \alpha \in X \quad \forall x, y \in \mathcal{M}$

$$\nu_\alpha(\uparrow(x \cdot y)) = \nu_\alpha(\uparrow x)\nu_{\alpha \cdot x}(\uparrow y)$$

- Question 1: do Markov measures always exist?
- Question 2: determine all of them
- Question 3: how to simulate a given Markov measure?

Semantics

Reformulation

The *valuation* associated to the Markov measure ν is the family $f = (f_\alpha)_{\alpha \in X}$ of functions $f_\alpha : \mathcal{M} \to \mathbb{R}_{\geq 0}$ defined by

$$f_\alpha(x) = \nu_\alpha(\uparrow x)$$

Semantics

Reformulation

The *valuation* associated to the Markov measure ν is the family $f = (f_\alpha)_{\alpha \in X}$ of functions $f_\alpha : \mathcal{M} \to \mathbb{R}_{\geq 0}$ defined by

$$f_\alpha(x) = \nu_\alpha(\uparrow x)$$

The functions f_α satisfy:

$$1)\ f_\alpha(x) > 0 \iff \alpha \cdot x \neq \bot$$
$$2)\ f_\alpha(x \cdot y) = f_\alpha(x) f_{\alpha \cdot x}(y)$$

$$\implies f_\alpha(a \cdot b \cdot c) = f_\alpha(b) f_{\alpha \cdot a}(b) f_{\alpha \cdot a \cdot b}(c)$$

f is determined by the *finite* collection of parameters

$$\lambda_\alpha(a) = f_\alpha(a), \qquad (\alpha, a) \in X \times \Sigma$$

Semantics

Reformulation

The *valuation* associated to the Markov measure ν is the family $f = (f_\alpha)_{\alpha \in X}$ of functions $f_\alpha : \mathcal{M} \to \mathbb{R}_{\geq 0}$ defined by

$$f_\alpha(x) = \nu_\alpha(\uparrow x)$$

- **Conversely:** What are the normalization conditions on $\big(\lambda_\alpha(a)\big)_{(\alpha,a)\in X\times\Sigma}$ for the associated valuation f to be *probabilistic* (*i.e.*, induced by some probability measure)?

Theory: probabilistic trace monoids

In case of a trace monoid with the trivial action
$\{*\} \times \mathcal{M} \to \{*\}$ **(no state, no constraint)**

The valuation f is a single function $f : \mathcal{M} \to \mathbb{R}_{>0}$ satisfying:

$$\forall x, y \in \mathcal{M} \quad f(x \cdot y) = f(x)f(y)$$

and determined by the collection $(f(a))_{a\in\Sigma}$

$$f(a_1 \cdot \ldots \cdot a_k) = f(a_1)\ldots f(a_k)$$

The associated probability measure is said to be *memoryless* (rather than *Markov*)

Theory: probabilistic trace monoids

Given a function $f : \mathcal{M} \to \mathbb{R}_{>0}$, its *Möbius* transform is the function $h : \mathscr{C} \to \mathbb{R}$

$$\forall \gamma \in \mathscr{C} \quad h(\gamma) = \sum_{\gamma' \in \mathscr{C} \,:\, \gamma \leq \gamma'} (-1)^{|\gamma'| - |\gamma|} f(\gamma')$$

where $\mathscr{C}$ is the (finite) set of *commutation cliques* of $\mathcal{M}$

and then the *Möbius inversion formula* holds:

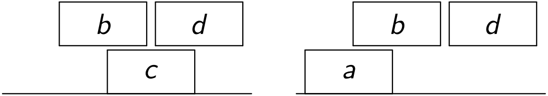
$$\forall \gamma \in \mathscr{C} \quad f(\gamma) = \sum_{\gamma' \in \mathscr{C} \,:\, \gamma \leq \gamma'} h(\gamma')$$

Theory: probabilistic trace monoids

Normal form of traces

- A pair (γ, γ') of cliques is *normal*, denoted $\gamma \to \gamma'$, if:

$$\forall z \in \gamma' \quad \exists y \in \gamma \quad \neg(y \parallel z)$$



$OK \quad c \to (b \cdot d)$ $\qquad$ $not\ OK \quad a \not\to (b \cdot d)$

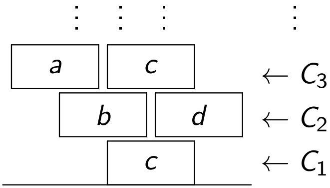
Theory: probabilistic trace monoids

Normal form of traces

- For every trace x there exists a unique integer $k \geq 0$ and a unique sequence $(c_1, \ldots, c_k)$ of cliques such that

$$1)\ c_i \to c_{i+1} \qquad\qquad 2)\ x = c_1 \cdot \ldots \cdot c_k$$

- The same works for *infinite traces*
- If $\partial\mathcal{M}$ is equipped with a probability measure ν, then the sequence $(C_i)_{i\geq 1}$ is a *random process* (every C_i is a *finite random variable*)



Theory: probabilistic trace monoids

Proposition

If ν is a probability measure on $\partial\mathcal{M}$ and $f(x) = \nu(\uparrow x)$, then the law of C_1 is h, the Möbius transform of f:

$$\nu(C_1 = \gamma) = \sum_{\gamma' \in \mathscr{C} \,:\, \gamma \leq \gamma'} (-1)^{|\gamma'|-|\gamma|} f(\gamma')$$

Theory: probabilistic trace monoids

Consequence

If $f(a) = r$ for all $a \in \Sigma$, then $f(x) = r^{|x|}$

total probability law for C_1: $$\sum_{\gamma \in \mathscr{C}} \nu(C_1 = \gamma) = 1$$

yielding $\boxed{\mu(r) = 0}$ (Möbius polynomial of $\mathcal{M}$)

Actually: r must be the root of **smallest modulus** of $\mu(t)$, which is unique and in $(0, 1)$

Example

$$\mathcal{M} = \langle a_0, \ldots, a_4 \mid a_i a_j = a_j a_i \quad |i - j| \mod 5 \geq 2 \rangle$$

$$\mu(t) = 1 - 5t + 5t^2 \quad \to r = \underbrace{\frac{5 \pm \sqrt{5}}{10}}_{\text{both in } (0,1)} \quad \to r = \frac{5 - \sqrt{5}}{10}$$

Theory: probabilistic trace monoids

Law of the first clique for the uniform measure on $\mathcal{M} = \langle a, b, c, d \mid ac = ca,\ ad = da,\ bd = db \rangle$

$$\mu(t) = 1 - 4t + 3t^2 \quad \to r = \frac{1}{3}$$

$$h(a) = f(a) - f(a \cdot c) - f(a \cdot d) = r - 2r^2 = \frac{1}{9}$$

$$h(b) = f(b) - f(b \cdot d) = r - r^2 = \frac{2}{9}$$

$$h(a \cdot c) = f(a \cdot c) = r^2 = \frac{1}{9}$$

$\gamma \in \mathscr{C}$	a	b	c	d	$a \cdot c$	$a \cdot d$	$b \cdot d$
$h(\gamma)$ (proba of $C_1 = \gamma$)	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$
$f(\gamma)$ (proba of $C_1 \geq \gamma$)	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$

Theory: probabilistic concurrent systems

For concurrent systems, an extension of the previous techniques works in connection with

- linear algebra
- analytic combinatorics