Simplicial and Higher Coalgebra

Workshop on Pomsets and Related Structures

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Outline

Introduction and Motivation

Motivation

Homotopy theory and algebraic topology for behaviour

- (Weak) homotopy equivalence of systems
- Homotopy-invariant logic
- Homological algebra to find behavioural obstructions

Examples

- Concurrent computing — detecting deadlocks[1]
- Distributed computing — computability results[2]
- Hybrid computing — detecting and handling Zeno behaviour[3]
- Modal logic for higher dimensional automata[4]

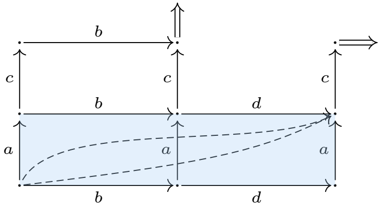
[1] Lisbeth Fajstrup et al. *Directed Algebraic Topology and Concurrency*. Springer, 2016, p. 167. 1 p. ISBN: ISBN 978-3-319-15397-1. DOI: 10.1007/978-3-319-15398-8.

[2] Maurice Herlihy, Dmitry Kozlov, and Sergio Rajsbaum. *Distributed Computing Through Combinatorial Topology*. 1st ed. San Francisco, CA, USA: Morgan Kaufmann Publishers Inc., Nov. 2013. 336 pp. ISBN: 978-0-12-404578-1.

[3] Aaron D. Ames and Shankar Sastry. “Characterization of Zeno Behavior in Hybrid Systems Using Homological Methods”. In: *Proceedings of the 2005, American Control Conference, 2005.* ACC 2005. June 2005, 1160–1165 vol. 2. DOI: 10.1109/ACC.2005.1470118.

[4] Cristian Prisacariu. *Higher Dimensional Modal Logic*. 2014. arXiv: 1405.4100. URL: http://arxiv.org/abs/1405.4100. preprint.

(i)Pomset Languages of Higher-Dimensional Automata (HDA)



$$\Big\{ (a \to b \to c), (a \to c \to b), (b \to a \to c),$$
$$(a \to b \to d \to c), (b \to a \to d \to c), (b \to d \to a \to c),$$
$$\begin{pmatrix} a & \searrow & \\ & & c \\ b & \nearrow & \end{pmatrix}, \begin{pmatrix} & a & \searrow & \\ & & & c \\ b \to d & & \nearrow & \end{pmatrix}, \begin{pmatrix} a & \searrow & & \\ & & d \to c \\ b & \nearrow & \end{pmatrix}, \begin{pmatrix} & \nearrow & a & \searrow & \\ b & & & & c \\ & \searrow & d & \nearrow & \end{pmatrix} \Big\}$$

- We want homotopy between pomsets
- Naively imposing them collapses them to free commutative monoid
- Issue: the cubes of HDA are a representation of topological spaces, but HDA conflate behaviour (computation direction and labelling) with the space representation
- Potential solution: simplicial set of (i)pomsets for labelling, similar to HDA labelling[5]
- Generally: Coalgebras can achieve separation of space and behaviour

[5] Uli Fahrenberg et al. "Languages of Higher-Dimensional Automata". In: *Math. Struct. Comput. Sci.* 31.5 (2021), pp. 575–613. DOI: 10.1017/S0960129521000293.

Behaviour via Coalgebras

- Behaviour from repeated observation of a space X via map $c\colon X \to FX$
- Functor $F\colon \mathcal{C} \to \mathcal{C}$ on a category $\mathcal{C}$ determines the type of observations

Example (Finitely-branching labelled transition systems (LTS))

- $\mathbf{Set}$ – Category of sets and maps
- $F\colon \mathbf{Set} \to \mathbf{Set}$ given by

$$FX = \mathcal{P}_\omega(A \times X) \quad \text{and} \quad Ff = \mathcal{P}_\omega(\mathrm{id}_A \times f)$$

- Coalgebra $c\colon X \to FX$ is A-labelled, finitely-branching transition system
- Equivalent to relation $R \subseteq X \times A \times X$ with $R(x) \subseteq A \times X$ finite for all x

Concurrency in coalgebras

Can we give an analogue of finitely-branching LTS for truly concurrent systems?

Simplicial Coalgebras

Concurrent Systems as Simplicial Coalgebras

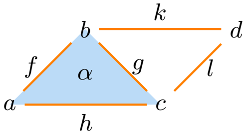
Sketch

- Combine idea of higher-dimensional automata to model n parallel actions as n-dimensional objects
- with that idea of computation as paths in directed spaces
- all of that as coalgebras
- on simplicial sets

Simplicial Sets – A Combinatorial Model of Spaces

Intuition

- X_n set of n-cells (dimension n)
- Boundary maps $d_n^k \colon X_{n+1} \to X_n$
- Degeneracy maps $s_n^k \colon X_n \to X_{n+1}$



— 0-cells
— 1-cells
— 2-cell

$d_2^0(\alpha) = g$

$d_1^0(g) = b$

$d_1^1(d_2^1(\alpha)) = d_1^1(f) = b$

Simplicial sets formally

- Δ – category of non-empty finite linearly ordered sets and monotone maps
- Generated by $[n] = \{0 < 1 < \cdots < n\}$
- Simplicial set is a functor $X \colon \Delta^{\mathrm{op}} \to \mathbf{Set}$ and simplicial maps are natural transformations
- Write $\mathbf{sSet}$ for category of simplicial sets and maps

Simplicial Set of Parallel Actions

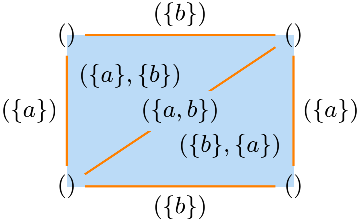
Monoid of parallel actions

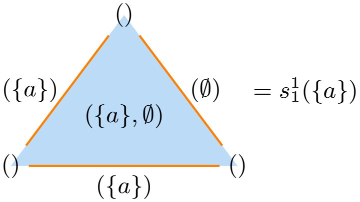
- A set of actions
- $\mathcal{L} = (\mathcal{P}_\omega(A), \emptyset, \cup)$ monoid of finite subsets of A
- Union $U \cup V$ puts all actions in U and V in parallel

Nerve of the monoid $\mathcal{L}$

Define a simplicial set $N\mathcal{L}$ by

- $(N\mathcal{L})_n = \mathcal{L}^n$ (tuples of length n)
- $d_n^0(U_1, \ldots, U_n) = (U_2, \ldots, U_n)$
- $d_n^k(U_1, \ldots, U_k, U_{k+}, \ldots, U_n) = (U_1, \ldots, U_k \cup U_{k+1}, \ldots, U_n)$
- $s_n^k(U_1, \ldots, U_n) = (U_1, \ldots, U_k, \emptyset, U_{k+1}, \ldots, U_n)$
- Compare this to the event order in pomsets
- A map $X \to N\mathcal{L}$ labels n-cell in X with n-tuple ofa parallel actions





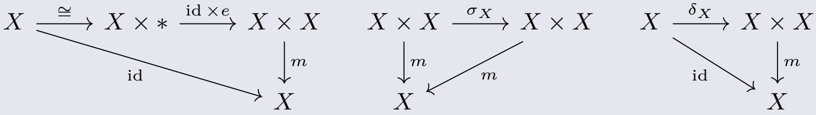
Algebraic View on Finite Powerset

Finite powerset as free algebra

- A (join-)semilattice (JSL) $(P, 0, \vee)$ is a commutative, idempotent monoid ($x \vee x = x$)
- There is an adjunction $L : \mathbf{Set} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{JSL} : U$ between category of sets and JSLs
- One can chose $LX = (\mathcal{P}_\omega X, \emptyset, \cup)$
- In other words, the finite powerset is the free join-semilattice

Simplicial JSL

- Write $\times$ for the product and $*$ for the terminal object in $\mathbf{sSet}$
- A JSL in $\mathbf{sSet}$ is a triple (X, e, m) of a simplicial set X with maps $e \colon * \to X$ and $m \colon X \times X \to X$
- …and the JSL equations expressed as commuting diagrams, e.g.



Free Simplical Join-Semilattices

Abstract non-sense proof

- Have category of simplicial JSL $\mathbf{sJSL}$ with homomorphisms
- It is isomorphic to category $[\Delta^{\mathrm{op}}, \mathbf{JSL}]$ of functors $\Delta^{\mathrm{op}} \to \mathbf{JSL}$
- Post-composition yields an adjunction $[\Delta^{\mathrm{op}}, L] : \mathbf{sSet} \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} [\Delta^{\mathrm{op}}, \mathbf{JSL}] : [\Delta^{\mathrm{op}}, U]$
- And composing with the ismorphism thus an adjunction $L_* : \mathbf{sSet} \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} \mathbf{sJSL} : U_*$
- We get an analogue of the finite powerset $P \colon \mathbf{sSet} \to \mathbf{sSet}$ by $P = U_* \circ L_*$

A bit more concrete

- $PX = \mathcal{P}_\omega \circ X$
- $L_* X = (\mathcal{P}_\omega \circ X, e, m)$ with e and m level-wise empty set and union
- Plus coherence with boundaries and degeneracies

Raising Dimension

Outgoing paths

- Starting an action corresponds to going from an n-cell to an $(n+1)$-cell
- Analogy: space of paths is one dimension higher

Adjoining a tip

- Write $U^{\triangleright}$ for the poset with a new maximal element added (join of U with $[0]$)
- Have $[n]^{\triangleright} \cong [n+1]$
- Get a functor $-^{\triangleright} \colon \Delta \to \Delta$ with $f^{\triangleright}$
- And thus a functor $\uparrow \colon \mathbf{sSet} \to \mathbf{sSet}$ by pre-composing: $\uparrow X = X \circ (-)^{\triangleright}$
- A map $X \to \uparrow X$ sends an n-cell to an $(n+1)$-cell

Simplicial coalgebras for true concurrency

- Define a functor $B\colon \mathbf{sSet} \to \mathbf{sSet}$ by

$$B^1 X = N\mathcal{L} \times P(X + {\uparrow}X)$$

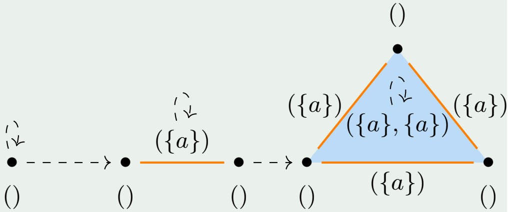
- A coalgebra $c\colon X \to B^1 X$ labels cells with parallel actions, and sends an n-cell to a finite set of n- and $(n+1)$-cells (starting zero or one action)

Example (Process replication, finitely presented[6])

- The terminal object $*$ is $\mathbf{sSet}$ is given by $*_n = \{\bullet\}$
- Define $c\colon * \to B^1 *$ by

$$c_n(\bullet) = (\{a\}^n, \{\varkappa_1(\bullet), \varkappa_2(\bullet)\})$$

- $\{a\}^n = (\underbrace{\{a\}, \ldots, \{a\}}_{n \text{ times}})$
- $\varkappa_1 : X \to X + {\uparrow}X$ and $\varkappa_2 : {\uparrow}X \to X + {\uparrow}X$ coprojections (coproduct inclusions)



Dashed edges are transitions

[6] Henning Basold, Thomas Baronner, and Márton Hablicsek. *Irrationality of Process Replication for Higher-Dimensional Automata*. submitted. 2023. arXiv: 2305.06428. URL: hda-process-repl-irrat.pdf. preprint.

Behaviour of Coalgebras

- Behaviour of coalgebra c by recursively expanding observations into a sequence

$$X \xrightarrow{c} FX \xrightarrow{Fc} F(FX) \xrightarrow{F(Fc)} \cdots$$

- Gives in the limit a total view on behaviour of c[7], if that exists
- Traces and logical formulas are partial view on this sequence
- Coalgebra homomorphisms relate the behaviour of systems

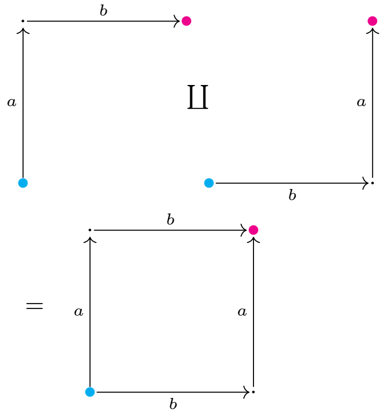
$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle d} \\ FX & \xrightarrow{Ff} & FY \end{array}$$

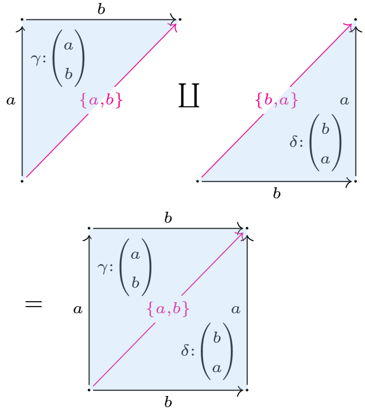
- Coalgebra homomorphisms preserve and reflect the behaviour
- Behaviour of the image f in d is equal to that of c
- Often coincide with bisimilarity[8], but we want homotopic behaviour

[7] Michael Barr. "Terminal Coalgebras in Well-Founded Set Theory". In: *TCS* 114.2 (1993), pp. 299–315. DOI: 10.1016/0304-3975(93)90076-6.

[8] Sam Staton. "Relating Coalgebraic Notions of Bisimulation". In: *LMCS* 7.1 (2011), pp. 1–21. DOI: 10.2168/LMCS-7(1:13)2011.

Homotopy for (i)Pomset Languages





No relation

$$a \bullet b \not\sim b \bullet a$$

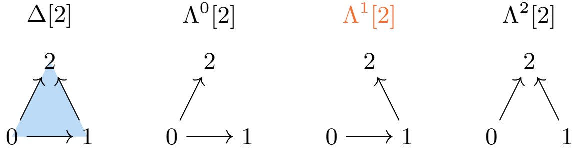
Homotopy via 2-cells

$$a \bullet b \overset{\gamma}{\sim} \{a, b\} = \{b, a\} \overset{\delta}{\sim} b \bullet a$$

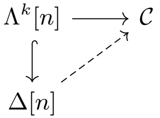
Higher Coalgebra

Quasi-Categories

- Model of $(\infty, 1)$-categories: n-cells in all arbitrary dimensions, invertible above 1
- Standard n-simplex $\Delta[n]$ is given by hom-functor: $\Delta[n] = \Delta(-, [n]) \colon \Delta^{\mathrm{op}} \to \mathbf{Set}$
- Horn: $\Lambda^k[n]$ is $\Delta[n]$ with interior and boundary opposite k left out

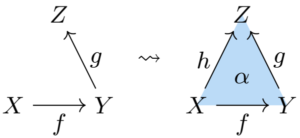


- Inner horn: $\Lambda^k[n]$ with $0 < k < n$
- Simplicial set $\mathcal{C}$ is quasi-category if every inner horn can be filled:



Intuition for Quasi-Categories

- 0-cells can be seen as objects
- 1-cells as morphisms
- higher cells as homotopies[9]
- inner horn filling:



- h is a composition of g and f
- α witnesses this
- Nerve of monoid $(\mathcal{P}_\omega(A), \emptyset, \cup)$ is a quasi-category with one object

[9] Jacob Lurie. *Higher Topos Theory*. Annals of Mathematics Studies 170. Princeton University Press, 2009. ISBN: 978-0-691-14049-0. arXiv: math/0608040.

Higher Coalgebra

Commutativity up to homotopy

$$\begin{array}{ccc} X & \xrightarrow{c} & FX \\ {\scriptstyle f}\downarrow & \sim & \downarrow{\scriptstyle Ff} \\ Y & \xrightarrow[d]{} & FY \end{array} \qquad \begin{array}{ccccccc} X & \xrightarrow{c} & FX & \xrightarrow{Fc} & F(FX) & \xrightarrow{F(Fc)} & \cdots \\ \downarrow{\scriptstyle f} & \sim & \downarrow{\scriptstyle Ff} & \sim & \downarrow{\scriptstyle F(Ff)} & \sim & \\ Y & \xrightarrow{d} & FY & \xrightarrow{Fd} & F(FY) & \xrightarrow{F(Fd)} & \cdots \end{array}$$

Long-term: homotopy theory of systems as higher coalgebra theory

- inspired by coalgebra[10] and higher algebra[11]
- $\mathcal{C}$ quasi-category, F simplicial map, c 1-cell in $\mathcal{C}$
- homotopy defined in terms of 2-cells
- homotopy (co)limits, obstruction theory via (co)homology, homotopy-invariant modal logic, ...

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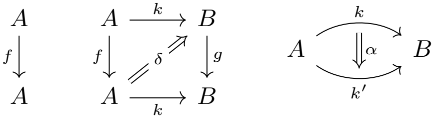
[10] Jan Rutten. "Universal Coalgebra: A Theory of Systems". In: *TCS* 249.1 (2000), pp. 3–80. ISSN: 0304-3975. DOI: 10.1016/S0304-3975(00)00056-6.

[11] Jacob Lurie. *Higher Algebra*. Sept. 2017. URL: https://www.math.ias.edu/~lurie/papers/HA.pdf.

Formal Coalgebra in 2-Categories

- Work in 2-category $\mathcal{C}$: $\mathbf{Cat}$, $\mathcal{V}$-$\mathbf{Cat}$, $\mathbf{Fib}$, $\mathbf{qCat_2}$ (homotopy 2-category of quasi-categories)[12], $\mathfrak{h}\mathcal{K}$ (homotopy 2-category of ∞-cosmos $\mathcal{K}$)[13]
- Define coalgebra objects (special 2-limits, inserters[14])
- Define 2-category $\mathrm{Endo}(\mathcal{C})$ of endomorphisms, distributive laws and distributive law morphisms with forgetful 2-functor $U\colon \mathrm{Endo}(\mathcal{C}) \to \mathcal{C}$



Theorem

If the 2-category $\mathcal{C}$ has a choice of coalgebra objects for all endomorphisms, then there is a product-preserving 2-functor $\mathrm{CoAlg}\colon \mathrm{Endo}(\mathcal{C}) \to \mathcal{C}$ with a 2-natural transformation $p\colon \mathrm{CoAlg} \to U$.

[12] Emily Riehl. *Categorical Homotopy Theory*. New Mathematical Monographs 24. Cambridge University Press, 2014. ISBN: 978-1-107-04845-4. URL: https://math.jhu.edu/~eriehl/cathtpy/.

[13] Emily Riehl and Dominic Verity. *Elements of ∞-Category Theory*. Cambridge University Press (CUP), 2022. ISBN: 978-1-108-93688-0. DOI: 10.1017/9781108936880.

[14] Claudio Hermida and Bart Jacobs. “Structural Induction and Coinduction in a Fibrational Setting”. In: *Information and Computation* 145 (1997), pp. 107–152. DOI: 10.1006/inco.1998.2725.

What's the point?

Many known results are instances of 2-functoriality

- transport of adjunctions
- monoidal structure on coalgebras
- determinisation
- adequacy of coalgebraic modal logic

For an appropriate 2-categorical definition of colimit we get a known result in general:

Theorem

If $\mathcal{C}$ is Cartesian closed, then $p\colon \mathrm{CoAlg} \to U$ creates colimits.

Instance: homotopy colimits in quasi-categories

Direction 1

Develop coalgebra further in higher categories, including enriched for good computation methods

Homotopy-Invariant Modal Logic

Modal Logic on HDA

Show modalities and homotopy axiom[15]

$$\varphi ::= p \mid \bot \mid \varphi \to \varphi \mid \Diamond^{\uparrow}\varphi \mid \Diamond^{\downarrow}\varphi$$

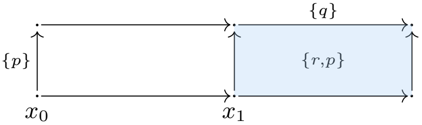
- $\Diamond^{\uparrow}\varphi$ holds if some action can be started and φ holds during execution
- $\Diamond^{\downarrow}\varphi$ holds if some action can be ended and φ holds afterwards

Interpretation over an HDA with cubes X

$$\begin{aligned}
[\![\Diamond^{\uparrow}\varphi]\!]_n &= \{x \in X_n \mid \exists x' \in X_{n+1}.\, x \text{ is a boundary cell of } x' \text{ and } x' \in [\![\varphi]\!]_{n+1}\} \\
[\![\Diamond^{\downarrow}\varphi]\!]_{n+1} &= \{x \in X_{n+1} \mid \exists x' \in X_n.\, x' \text{ is a boundary cell of } x \text{ and } x' \in [\![\varphi]\!]_n\} \\
x \vDash \varphi &\iff \exists n.\, x \in [\![\varphi]\!]_n
\end{aligned}$$

[15] Cristian Prisacariu. "Modal Logic over Higher Dimensional Automata". In: *Proc. of CONCUR 2010*. 2010, pp. 494–508. DOI: 10.1007/978-3-642-15375-4_34.

Homotopy-Invariance for HDA Logic



Example

$$x_0 \models \Diamond^{\uparrow} p \qquad x_1 \models \Diamond^{\uparrow}\Diamond^{\uparrow}\Diamond^{\downarrow} q$$

$$x_1 \models \Diamond^{\uparrow}\Diamond^{\uparrow} r \wedge p \qquad x_1 \models \Diamond^{\uparrow}\Diamond^{\downarrow}\Diamond^{\uparrow} q$$

Interchange Axioms[16]

$$\Diamond^{\uparrow}\Diamond^{\uparrow}\Diamond^{\downarrow}\varphi \to \Diamond^{\uparrow}\Diamond^{\downarrow}\Diamond^{\uparrow}\varphi \tag{A10}$$

$$\Diamond^{\uparrow}\Diamond^{\downarrow}\Diamond^{\downarrow}\varphi \to \Diamond^{\downarrow}\Diamond^{\uparrow}\Diamond^{\downarrow}\varphi \tag{A10'}$$

[16] Cristian Prisacariu. *Higher Dimensional Modal Logic*. 2014. arXiv: 1405.4100. URL: http://arxiv.org/abs/1405.4100. preprint.

Coalgebraic Modal Logic

One view based on dual adjunctions, so-called logical connections[17]

$$F \circlearrowright \mathcal{C} \underset{Q}{\overset{P}{\rightleftarrows}} \mathcal{D}^{\mathrm{op}} \circlearrowleft L^{\mathrm{op}} \quad (P \dashv Q) \quad \text{and} \quad \varrho \colon PF \to L^{\mathrm{op}} P \quad \text{and} \quad \alpha \colon L\Phi \to \Phi$$

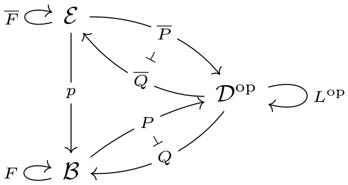
Components

- $\mathcal{C}$ category for "states" in coalgebras
- F behaviour functor to get coalgebras $X \to FX$
- $\mathcal{D}$ typically category of algebras for logical operators
- L specifies modal operators
- initial algebra α for syntax
- distributive law $\varrho \colon LP \to PF$ to give semantics of formulas in a coalgebra
- $P \dashv Q$ is often concrete duality by mapping into dualising object

[17] Dusko Pavlovic, Michael W. Mislove, and James Worrell. "Testing Semantics: Connecting Processes and Process Logics". In: *Proceedings of Algebraic Methodology and Software Technology, 11th International Conference, AMAST 2006*. Ed. by Michael Johnson and Varmo Vene. Vol. 4019. Lecture Notes in Computer Science. Springer, 2006, pp. 308–322. DOI: 10.1007/11784180_24; Toby Wilkinson. "Enriched Coalgebraic Modal Logic". PhD thesis. 2013. URL: http://eprints.soton.ac.uk/354112/.

Modal Logic for General Coinductive Predicates

Previous picture is restricted to logic for behavioural equivalence/bisimilarity!



Components[18]

- $p \colon \mathcal{E} \to \mathcal{B}$ fibration
- coalgebras for $\overline{F}$ are proofs of coinductive predicates
- final coalgebras in fibres are typically called coinductive predicates
- soundness (adequacy) and completeness (expressiveness) results provable in this setting

[18] Clemens Kupke and Jurriaan Rot. "Expressive Logics for Coinductive Predicates". In: *Logical Methods in Computer Science* Volume 17, Issue 4 (Dec. 15, 2021). DOI: 10.46298/lmcs-17(4:19)2021.

Higher Coalgebraic Modal Logic

- Theorem from earlier gives adequacy in categories
- Reason is that 2-categorically defined Cartesian fibrations are the right thing
- In quasi-categories this fails
- Needs some work directly with quasi-categories[19]

Direction 2

Develop coalgebraic modal logic further in higher categories

[19] Emily Riehl and Dominic Verity. *Elements of ∞-Category Theory*. Cambridge University Press (CUP), 2022. ISBN: 978-1-108-93688-0. DOI: 10.1017/9781108936880.

Wrapping Up

Outlook

1. Model concurrency adequately with repeated actions: replace finite powerset with multisets
2. Languages of concurrent systems via monoidal quasi-category of ipomsets
3. Model directed spaces as coalgebras, which requires Vietories-like functor on $\mathbf{Top}$
4. Reconciliation with directed approaches[20]
5. Integration with homotopical/model categories (so-called enriched homotopy theory)
6. General theory of coalgebraic modal logic, type theory and obstruction theory via (co)homology

Direction 3

Systematic development of tools to detect obstructions, like (co)homology.

7. Integration with type theory (synthetic $(\infty, 1)$-categories, possibly directed)

[20] Jérémy Dubut, Eric Goubault, and Jean Goubault-Larrecq. "The Directed Homotopy Hypothesis". In: *25th EACSL Annual Conference on Computer Science Logic (CSL 2016)*. Ed. by Jean-Marc Talbot and Laurent Regnier. Vol. 62. LIPIcs. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2016, 9:1–9:16. ISBN: 978-3-95977-022-4. DOI: 10.4230/LIPIcs.CSL.2016.9.

Thank you for your attention!